Semirings which are distributive lattices of t-k-simple semirings

Tapas Kumar Mondal[1], Anjan Kumar Bhuniya[2]

[1]Department of Mathematics, Dr. Bhupendra Nath Dutta Smriti Mahavidyalaya, Hatgobindapur, Burdwan - 713407, West Bengal, India
[2]Department of Mathematics, Visva-Bharati University, Santiniketan, Bolpur - 731235, West Bengal, India
E-mail: [1]tapumondal@gmail.com, [2]anjankbhuniya@gmail.com

Abstract

A semiring S is said to be a t-k-simple semiring if it has no non-trivial proper left k-ideal and no non-trivial proper right k-ideal. We introduce the notion of t-k-simple semirings and characterize the semirings in $\mathbb{SL}^+$, the variety of all semirings with a semilattice additive reduct, which are distributive lattices of t-k-simple subsemirings. A semiring S is a distributive lattice of t-k-simple subsemirings if and only if every k-bi-ideal in S is completely semiprime k-ideal. Also the semirings for which every k-bi-ideal is completely prime has been characterized.

1 Introduction

The notion of the semirings, in its most generalized form e.g. a universal algebra with two associative binary operations $'+'$ and $'\cdot'$ which are connected by the ring like distributive laws, was introduced by Vandiver [22] in 1934. Though they appeared in Mathematics long before e.g. the semiring of all ideals of a ring, the semiring of all endomorphisms on a commutative semigroup, the positive cone in ordered ring, etc., they found their full place in mathematics recently e.g. idempotent analysis[15] which are being used in theoretical physics, optimization etc., various applications in theoretical computer science and algorithm theory[11], [14]. The underlying semirings used both in idempotent analysis and theoretical computer science is one in which the additive reduct is both idempotent and commutative.

From the algebraic point of view while studying the structure of semigroups, semilattice decomposition of semigroups was first defined and studied by A. H. Clifford[9]. The idea consists of decomposing a semigroup S into subsemigroups(components) through a congruence η on S such that S/η is the greatest semilattice homomorphic image of S and each η-class is a component subsemigroup. The idea of decomposing into Archimedean components has since been studied and generalized by M. S. Putcha, T. Tamura, N. Kimura, S. Bogdanović, M. Ćirić, F. Kmet ([8], [13], [18], [21]) and many others. Y. Cao[7] studied the structure of ordered semigroups and characterized the ordered semigroups which are semilattices(chains) of t-simple subsemigroups.

The present paper is a continuation of our study on the structure of semirings whose additive reduct is a semilattice ([2], [3], [4], [5], [16], [17]). Interestingly, the semiring of all finite subsets of a semigroup is a free model in the variety of such semirings and there are interesting connections between the subvarieties of semigroups and different subclasses of these semirings.

The preliminaries and prerequisites for this article are discussed in Section 2. In Section 3 we characterize semirings which are distributive lattices(chains) of t-k-simple semirings.

2   Preliminaries

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz.$$

Thus the semirings can be regarded as a common generalization of both rings and distributive lattices. By $\mathbb{SL}^+$ we denote the variety of all semirings $(S, +, \cdot)$ such that $(S, +)$ is a semilattice, i.e. a commutative and idempotent semigroup. Throughout this paper, unless otherwise stated, S is always a semiring in $\mathbb{SL}^+$.

Let A be a nonempty subset of S. The *k-closure* of A is defined by

$$\overline{A} = \{x \in S \mid x + a_1 = a_2 \text{ for some } a_1, a_2 \in A\}.$$

If $(A, +)$ is a subsemigroup of $(S, +)$, then $A \subseteq \overline{A}$,

$$\overline{A} = \{x \in S \mid x + a = a \text{ for some } a \in A\},$$

and $\overline{A} = \overline{\overline{A}}$ since $(S, +)$ is a semilattice. A is called a *k-set* if $\overline{A} \subseteq A$. An ideal(left, right) A of S is called a *k-ideal*(left, right) if it is a *k-set*, i.e. $\overline{A} = A$.

A semiring S is called a *k-regular semiring*[6] if for every $a \in S$, there exists an $s \in S$ such that $a + asa = asa$. For any semigroup F, the set $P_f(F)$ of all finite subsets of F is a semiring in $\mathbb{SL}^+$, where addition and multiplication are defined by the set union and usual product of subsets of a semigroup respectively. The semiring $P_f(F)$ is a *k-regular semiring* if and only if F is regular[20].

For $a \in S$, the principal left *k-ideal*(resp. principal right *k-ideal*, principal *k-ideal*)([1], [3], [4]) generated by a is the least left *k-ideal*(resp. least right *k-ideal*) of S containing a. Bhuniya and Jana[1] introduced *k-bi-ideals* in a semiring in $\mathbb{SL}^+$. A non-empty subset B of S is said to be a *k-bi-ideal* of S if $BSB \subseteq B$ and B is a *k-subsemiring* of S. The structures of the principal left *k-*ideal(resp. principal right *k-ideal*, principal *k-ideal* and principal *k-bi-ideal*) are given, respectively by

$$L_k(a) = \{x \in S \mid x + a + sa = a + sa, \text{ for some } s \in S\},$$

$$R_k(a) = \{x \in S \mid x + a + as = a + as, \text{ for some } s \in S\},$$

$$I_k(a) = \{x \in S \mid x + a + sa + as + sas = a + sa + as + sas, \text{ for some } s \in S\}$$

and

$$B_k(a) = \{x \in S \mid x + a + a^2 + asa = a + a^2 + asa, \text{ for some } s \in S\}.$$

Sen and Bhuniya [19] defined four equivalence relations namely $\overline{\mathcal{L}}$, $\overline{\mathcal{R}}$, $\overline{\mathcal{J}}$ and $\overline{\mathcal{H}}$ analogous to the Green's relations, on a *k-regular* semiring S in $\mathbb{SL}^+$. Bhuniya and Mondal[4] generalized the Green's relations $\overline{\mathcal{L}}$ and $\overline{\mathcal{J}}$ on a semiring S in $\mathbb{SL}^+$ and they are

$$\overline{\mathcal{L}} = \{(x, y) \in S \times S \mid L_k(x) = L_k(y)\}$$

and

$$\overline{\mathcal{J}} = \{(x, y) \in S \times S \mid I_k(x) = I_k(y)\}.$$

Here we define the Green's relations $\overline{\mathcal{R}}$ and $\overline{\mathcal{H}}$ on a semiring S in $\mathbb{SL}^+$ by:

$$\overline{\mathcal{R}} = \{(x, y) \in S \times S \mid R_k(x) = R_k(y)\}$$

and

$$\overline{\mathcal{H}} = \overline{\mathcal{L}} \cap \overline{\mathcal{R}}.$$

If $S \in \mathbb{SL}^+$, then both $\overline{\mathcal{L}}$ and $\overline{\mathcal{R}}$ are congruences on $(S, +)$ and $\overline{\mathcal{L}}$ is a right congruence and $\overline{\mathcal{R}}$ is a left congruence on $(S, \cdot)$.

A semiring S is said to be left k-simple(respectively right k-simple)[5] if it has no non-trivial proper left k-ideal(resp. right k-ideal). The description of the principal k-bi-ideal in S motivates us defining the following:

Definition 2.1. A semiring S is called t-k-simple if it is both left k-simple and right k-simple.

Example 2.2. Let $\mathbb{N}$ be the set of all natural numbers. Then $(\mathbb{N}, +, \cdot)$ is a semiring in $\mathbb{SL}^+$, where $+$ and $\cdot$ are defined by: $a + b = \max\{a, b\}$, $a \cdot b$ is the usual multiplication of a and b. Then $(\mathbb{N}, +, \cdot)$ is a t-k-simple semiring.

Example 2.3. (Maslov's dequantization semiring) Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}_+$, the semiring of all positive real numbers(with respect to the usual addition and product).

Consider the change of variables $x \longmapsto u = h \ln x$, where $x \in \mathbb{R}_+, h > 0$; thus $x = \exp(u/h)$ and this defines a natural map $D_h : \mathbb{R}_+ \longrightarrow A = \mathbb{R} \cup \{-\infty\}$.

Denote by A_h the set A equipped with the two operations $\oplus$(generalized addition) and $\odot$(generalized multiplication) borrowed from the usual addition and multiplication in $\mathbb{R}_+$ by the map D_h; thus

$$u \oplus v = h \ln(\exp(u/h) + \exp(v/h)), u \odot v = u + v.$$

Also $D_h(u + v) = D_h(u) \oplus D_h(v)$ and $D_h(uv) = D_h(u) \odot D_h(v)$. It is easy show that $u \oplus v = h \ln(\exp(u/h) + \exp(v/h)) \to \max\{u, v\}$ as $h \longrightarrow 0$.

Let us denote by $\mathbb{R}_{max}$ the set $A = \mathbb{R} \cup \{-\infty\}$ equipped with operations $\oplus = \max$ and $\odot = +$; we set $\oslash = -\infty$, $I = 0$. Then $(\mathbb{R}_{max}, \oplus, \odot)$ is a t-k-simple semiring.

Example 2.4. Consider the semiring $(\mathbb{N}, \max, \cdot)$ and set $S = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{N}\}$. Then $(S, +, \cdot)$ is a t-k-simple semiring, where $+$ is defined by: for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in S$
$A + B = \begin{pmatrix} max\{a, e\} & max\{b, f\} \\ max\{c, g\} & max\{d, h\} \end{pmatrix}$ and $\cdot$ the usual multiplication of A and B.

Example 2.5. Let $\mathbb{R}^+$ denotes the set of all positive real numbers, and consider the group $(\mathbb{R}^+, \cdot)$. Let $P_f(\mathbb{R}^+)$ be the set of all finite subsets of $\mathbb{R}^+$. Define $+$ and $\cdot$ on $P_f(\mathbb{R}^+)$ by: $A + B = A \cup B$ and $A \cdot B = \{ab \mid a \in A, b \in B\}$ for all $A, B \in P_f(\mathbb{R}^+)$. Then $(P_f(\mathbb{R}^+), +, \cdot)$ is a t-k-simple semiring.

Remark 2.6. It is interesting to observe that every t-k-simple semiring is a k-regular semiring but the left k-simplicity or right k-simplicity of a semiring does not give the same implication, in general. This is the fact that makes the importance of studying the t-k-simple semirings.

A semiring S is called a distributive lattice of t-k-simple semirings if there exists a congruence ρ on S such that S/ρ is a distributive lattice and each ρ-class is a t-k-simple semiring. Since $(S, +)$ is a semilattice the proof of the following lemma becomes easy.

Lemma 2.7. Let S be a semiring in $\mathbb{SL}^+$.
(a) If $a, b, c, d \in S$ are such that $c + xa + a = xa + a$ and $d + yb + b = yb + b$ for some $x, y \in S$, then there is $z \in S$ such that $c + za + a = za + a$ and $d + zb + b = zb + b$.
(b) If $a, b, c, d \in S$ are such that $c + ax + a = ax + a$ and $d + by + b = by + b$ for some $x, y \in S$, then there is $z \in S$ such that $c + az + a = az + a$ and $d + bz + b = bz + b$.
(c) If $a, b, c, d \in S$ are such that $c + a + a^2 + axa = a + a^2 + axa$ and $d + b + b^2 + byb = b + b^2 + byb$ for some $x, y \in S$, then there is $z \in S$ such that $c + a + a^2 + aza = a + a^2 + aza$ and $d + b + b^2 + bzb = b + b^2 + bzb$.

By the above lemma it is clear that $B_k(a) \subseteq L_k(a) \cap R_k(a) = H_k(a)$ for all $a \in S$, and $H_k(a)$ is also a k-bi-ideal of S containing a.

We define an equivalence relation $\overline{\mathcal{B}}$ on S by:

$$\overline{\mathcal{B}} = \{(x, y) \in S \times S \mid B_k(x) = B_k(y)\}.$$

Since $(S, +)$ is a semilattice, by the structure of $B_k(a)$ it is clear that $\overline{\mathcal{B}}$ is additive congruence.

A nonempty subset A of S is called *completely prime*(respectively *completely semiprime*) if for all $x, y \in S$, such that $xy \in A$ one has $x \in A$ or $y \in A$(respectively if for all $x \in S$ such that $x^2 \in A$ one has $x \in A$). Let F be a subsemiring of S. F is called a filter of S if: (i) for any $a, b \in S$, $ab \in F \Rightarrow a, b \in F$; and (ii) for any $a \in F$, $b \in S$, $a + b = b \Rightarrow b \in F$. The least filter of S containing a is denoted by $N(a)$. Let $\mathcal{N}$ be the equivalence relation on S, defined by $\mathcal{N} = \{(x, y) \in S \times S \mid N(x) = N(y)\}$.

For undefined concepts in semigroup theory we refer to [10], [12] for undefined concepts in semiring theory cf. [11].

The following lemma plays a crucial role in proving the main theorem of this article.

Lemma 2.8 ([5], 2.2)**.** Let S be a semiring in $\mathbb{SL}^+$. Then we have the following:
(i) For $a \in S, N(a) = N(a^2)$.
(ii) For $a, b \in S, N(ab) = N(ba)$.
(iii) For $a, b \in S, N(a + ab) = N(a)$.
(iv) For $a, b, c \in S, N(a) = N(b)$ implies $N(ac) = N(bc)$.
(v) For $a, b \in S, N(a) \cup N(b) \subseteq N(ab)$.

3   Distributive lattices of t-k-simple semirings

In this section we characterize the semirings that are distributive lattices of t-k-simple semirings. First we note that a semiring S is said to be left k-simple(resp. right k-simple, t-k-simple)iff $\overline{\mathcal{L}}$(resp. $\overline{\mathcal{R}}, \overline{\mathcal{H}}) = S \times S$. S is said to be $\overline{\mathcal{B}}$-simple iff $\overline{\mathcal{B}} = S \times S$ iff it has no non-trivial proper k-bi-ideal of S.

Before we go to the main theorem we have the following lemma:

Lemma 3.1. Let S be a semiring in $\mathbb{SL}^+$. Then the following conditions are equivalent:

1. S is t-k-simple;

    2. S is $\overline{\mathcal{H}}$-simple;

    3. S is $\overline{\mathcal{B}}$-simple.

Proof. $(1) \Rightarrow (2)$: Let $a, b \in S$. Since S is t-k-simple, $L_k(a) = L_k(b)$ and $R_k(a) = R_k(b)$. Then $H_k(a) = L_k(a) \cap R_k(a) = L_k(b) \cap R_k(b) = H_k(b)$. Hence S is $\overline{\mathcal{H}}$-simple.

$(2) \Rightarrow (3)$: For every $x, y \in S$, $x\overline{\mathcal{H}}y$, whence $x\overline{\mathcal{L}}y$ and $x\overline{\mathcal{R}}y$. Therefore $L_k(x) = L_k(y)$ and $R_k(x) = R_k(y)$. Let $a, b \in S$. Then there exists $s \in S$ such that $a + sb + b = sb + b$. Also $s \in R_k(s) = R_k(b)$, and so there exists $t \in S$ such that $s + bt + b = bt + b$. Then $a + sb + b = sb + b$ gives $a + btb + b^2 + b = btb + b^2 + b$. This yields $a \in B_k(b)$ so that $B_k(a) \subseteq B_k(b)$. Similarly, we can obtain $B_k(b) \subseteq B_k(a)$. Thus $B_k(a) = B_k(b)$, and so S is $\overline{\mathcal{B}}$-simple.

$(3) \Rightarrow (1)$: Let $a, b \in S$. Then $a \in B_k(b)$ and $b \in B_k(a)$. By Lemma 2.7, there exists $s \in S$ such that $a + b + b^2 + bsb = b + b^2 + bsb$ and $b + a + a^2 + asa = a + a^2 + asa$. Then $a + b + (b + bs)b = b + (b + bs)b$ and $b + a + (a + as)a = a + (a + as)a$ imply $a \in L_k(b)$ and $b \in L_k(a)$. These give $L_k(a) = L_k(b)$, and so S is $\overline{\mathcal{L}}$-simple. Similarly S is $\overline{\mathcal{R}}$-simple. Thus S is t-k-simple.   Q.E.D.

    Now we present the main theorem of this article.

Theorem 3.2. The following conditions are equivalent on a semiring S in $\mathbb{SL}^+$:

    1. S is a distributive lattice of t-k-simple subsemirings;

    2. for all $a, b \in S$, $ab, ba \in B_k(a)$ and $a \in B_k(a^2)$;

    3. for all $a \in S$, $B_k(a)$ is a completely semiprime k-ideal of S;

    4. every k-bi-ideal set of S is a completely semiprime k-ideal of S;

    5. for all $a, b \in S$, $B_k(ab) = B_k(a) \cap B_k(b)$;

    6. for all $a \in S$, $N(a) = \{x \in S \mid a \in B_k(x)\}$;

    7. for every nonempty family $\{B_\lambda \mid \lambda \in \Lambda\}$ of k-ideals of S, $\bigcap_{\lambda \in \Lambda} B_\lambda$ is a completely semiprime k-ideal of S;

    8. $\overline{\mathcal{B}} = \mathcal{N}$ is the least distributive lattice congruence on S such that each of its congruence classes is t-k-simple subsemiring.

Proof. Scheme of the proof: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7)$, $(6) \Rightarrow (8)$, $(7) \Rightarrow (4)$, $(8) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: Let S be a distributive lattice $\mathcal{D}$ of t-k-simple subsemirings S_α $(\alpha \in \mathcal{D})$, and let $a, b \in S$ be such that $a \in S_\alpha$, $b \in S_\beta$. Now $aba, ab, ba \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is t-k-simple, $aba\overline{\mathcal{B}}ab$ and $aba\overline{\mathcal{B}}ba$, by Lemma 3.1. Then we have $ba \in B_k(aba)$, which implies $ba \in B_k(a)$. Similarly, $ab \in B_k(a)$. Also $a\rho a^2$ implies $a, a^2 \in S_\alpha$, and so $a \in B_k(a^2)$.

$(2) \Rightarrow (3)$: Let $a \in S$. Then by hypothesis we are only to show that $B_k(a)$ is completely semiprime. For, let $u \in S$ be such that $u^2 \in B_k(a)$. Then $u \in B_k(u^2) \subseteq B_k(a)$. Thus $B_k(a)$ is a completely semiprime k-ideal of S.

$(3) \Rightarrow (4)$: Follows similarly.

$(4) \Rightarrow (5)$: Let $a, b \in S$. Then $ab \in B_k(a) \cap B_k(b)$ and $B_k(a) \cap B_k(b)$ is a k-ideal of S containing ab implies that $B_k(ab) \subseteq B_k(a) \cap B_k(b)$. Conversely, let $x \in B_k(a) \cap B_k(b)$. Then by Lemma 2.7 there is $s \in S$ such that $x + a + a^2 + asa = a + a^2 + asa$ and $x + b + b^2 + bsb = b + b^2 + bsb$. From these we have $x^2 + (a + a^2 + asa)(b + b^2 + bsb) = (a + a^2 + asa)(b + b^2 + bsb)$, and $(a + a^2 + asa)(b + b^2 + bsb) \in B_k(ab)$

implies $x^2 \in B_k(ab)$ so that $x \in B_k(ab)$. Thus $B_k(ab) = B_k(a) \cap B_k(b)$.

$(5) \Rightarrow (6):$ Let $F = \{x \in S \mid a \in B_k(x)\}$ and $x, y \in F$. Then $a \in B_k(x) \cap B_k(y)$ implies that there is $u \in S$ such that

$$a + x + x^2 + xsx = x + x^2 + xsx \text{ and } a + y + y^2 + ysy = y + y^2 + ysy.$$

Adding both sides of $a + x + x^2 + xsx = x + x^2 + xsx$ by $y + y^2 + xy + yx + xsy + ysx + ysy$ we get $a + (x+y) + (x+y)^2 + (x+y)s(x+y) = (x+y) + (x+y)^2 + (x+y)s(x+y)$. This implies $x+y \in F$. Again $a \in B_k(x) \cap B_k(y) = B_k(xy)$ implies that $xy \in F$. Hence F is a subsemiring of S. Let $x, y \in S$ such that $xy \in F$. Then $a \in B_k(xy) = B_k(x) \cap B_k(y)$, and so $x, y \in F$. Now let $x \in S, y \in F$ such that $y + x = x$. Then $y \in F$, and so there is $u \in S$ such that $a + y + y^2 + yuy = y + y^2 + yuy$. Then we have $a + (y+x) + (y+x)^2 + (y+x)u(y+x) = (y+x) + (y+x)^2 + (y+x)u(y+x)$ which implies $a + x + x^2 + xux = x + x^2 + xux$, and so $x \in F$. Thus F is a filter of S. Let T be a filter of S containing a and $u \in F$. Then there exists $s \in S$ such that $a + u + u^2 + usu = u + u^2 + usu$. Now $a \in T$, T is a filter, so $u + u^2 + usu \in T$. Then $a(u + u^2 + usu)a \in T$, i.e. $(a + au + aus)ua \in T$ so that $u \in T$. Therefore $F \subseteq T$. Thus $F = N(a)$.

$(6) \Rightarrow (7):$ Let $B = \bigcap_{\lambda \in \Lambda} B_\lambda$. Then B is a k-bi-ideal of S. Now let $a \in S$ be such that $a^2 \in B$. Then $a^2 \in N(a)$ implies that $a \in B_k(a^2) \subseteq B$. Thus B is a completely semiprime k-ideal of S.

$(6) \Rightarrow (8):$ For $x, y \in S, B_k(x) = B_k(y)$ implies $x \in N(y)$ and $y \in N(x)$, which yield $\overline{\mathcal{B}} = \mathcal{N}$. Then in view of the fact that $\overline{\mathcal{B}}$ is an additive congruence on S, by Lemma 2.8, it follows that $\overline{\mathcal{B}}$ is a distributive lattice congruence on S.

Consider an arbitrary distributive lattice congruence ξ on S. Let $a, b \in S$ such that $a\overline{\mathcal{B}}b$. Then $B_k(a) = B_k(b)$. So there exists $s \in S$ such that

$$a + b + b^2 + bsb = b + b^2 + bsb \text{ and } b + a + a^2 + asa = a + a^2 + asa.$$

Then we have $(a+b)\xi(a+b+b^2+bsb) = (b+b^2+bsb)\xi b$, i.e. $(a+b)\xi b$. Similarly, $(b+a)\xi a$ and these give $a\xi b$. Thus $\overline{\mathcal{B}} \subseteq \xi$, which proves the minimality of $\overline{\mathcal{B}}$.

Now we show that each $\overline{\mathcal{B}}$-class is t-k-simple subsemiring. For, let S_i be a $\overline{\mathcal{B}}$-class. Clearly S_i is a subsemiring of S. Let $a, b \in S$ be such that $a, b \in S_i$. Then $B_k(a) = B_k(b)$ and so there is $s \in S$ such that $a + b + b^2 + bsb = b + b^2 + bsb$ and $b + a + a^2 + asa = a + a^2 + asa$. Adding both sides of the latter by $a^2sa + asa^2 + (asa)^2 + (a+asa)a(a+asa)$ we get $b + (a+asa) + (a+asa)^2 + (a+asa)a(a+asa) = (a+asa) + (a+asa)^2 + (a+asa)a(a+asa)$, and so we have $b \in B_k(a+asa) = B_k(a)$ in S_i. Similarly, $a \in B_k(b)$ in S_i. Thus $a\overline{\mathcal{B}}b$ in S_i. Thus S_i is t-k-simple.

$(7) \Rightarrow (4):$ Obvious.

$(8) \Rightarrow (1):$ Obvious.   Q.E.D.

Example 3.3. Let $\mathbb{N}$ be the set of all natural numbers. Then $A = (\mathbb{N}, +, \cdot)$ is a semiring in $\mathbb{SL}^+$, where $+$ and $\cdot$ are defined by: $a + b = \max\{a, b\}$, $a \cdot b = \min\{a, b\}$ for all $a, b \in \mathbb{N}$. Also $B = (\mathbb{N}, +, \cdot)$ is a semiring in $\mathbb{SL}^+$, where $+$ and $\cdot$ are defined by: $a + b = \max\{a, b\}$, $a \cdot b$ the usual multiplication of a and b. Now we take the direct product of A and B. Then $S = (A \times B, +, \cdot)$ is a semiring in $\mathbb{SL}^+$.

Let $(m, n) \in S$. Then $B_k(m, n) = \{(a, b) \in S \mid a \leq m, \ b \leq n^2q \text{ for some } q \in \mathbb{N}\}$. For any $(a, b), (c, d) \in S, (a, b)(c, d) = (min\{a, c\}, bd) \in B_k(a, b)$, since $min\{a, c\} \leq a$ and $bd \leq b^2q$ for some suitable $q \in \mathbb{N}$. Similarly, $(c, d)(a, b) \in B_k(a, b)$. Also $(a, b) \in B_k[(a, b)^2]$, since $(a, b)^2 = (a, b^2)$ with $min\{a, a\} = a$ and $b \leq b^4p$ for some suitable choice $p \in \mathbb{N}$. Thus S is a distributive lattice of t-k-simple subsemirings.

Example 3.4. We take the direct product of the semirings A in Example 3.3 and S in Example 2.4. Then $T = (A \times S, +, \cdot)$ is a distributive lattice of t-k-simple subsemirings.

In the next theorem we characterize the semirings which are chains of t-k-simple semirings. Let $(T, +, \cdot)$ be a distributive lattice with the partial order defined by $a \leq b \Leftrightarrow a + b = b$ for all $a, b \in S$. It is well known that $(T, \leq)$ is a chain if and only if $ab = b$ or $ab = a$ for all $a, b \in T$.

Theorem 3.5. The following conditions are equivalent on a semiring S in $\mathbb{SL}^+$:

1. S is a chain of t-k-simple subsemirings;

2. for all $a, b \in S$, $ab, ba \in B_k(a)$ and $a \in B_k(ab)$ or $b \in B_k(ab)$;

3. for all $a \in S$, $B_k(a)$ is a completely prime k-ideal of S;

4. every k-bi-ideal set of S is a completely prime k-ideal of S;

5. for all $a, b \in S$, $B_k(ab) = B_k(a) \cap B_k(b)$, and $B_k(a) \subseteq B_k(b)$ or $B_k(b) \subseteq B_k(a)$;

6. for all $a, b \in S$, $N(a) = \{ x \in S \mid a \in B_k(x) \}$ and $N(ab) = N(a) \cup N(b)$;

7. for every non-empty family $\{ B_\lambda; \lambda \in \Lambda \}$ of k-bi-ideal sets of S, $\bigcap_{\lambda \in \Lambda} B_\lambda$ is a completely prime k-ideal of S;

8. $\overline{\mathcal{B}} = \mathcal{N}$ is the least chain congruence on S such that each of its congruence classes is t-k-simple subsemiring.

Proof. $(1) \Rightarrow (2)$: Let S be a chain $\mathcal{C}$ of t-k-simple subsemirings S_α ($\alpha \in \mathcal{C}$). Then the first part follows from Theorem 3.2. Let $a, b \in S$. Then there exist $\alpha, \beta \in \mathcal{C}$ such that $a \in S_\alpha, b \in S_\beta$. Since $\mathcal{C}$ is a chain either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. Then $aba, ab, ba \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Now $ab \in S_{\alpha\beta}$ implies that either $ab \in S_\alpha$ or $ab \in S_\beta$. If $ab \in S_\alpha$ then $a, ab \in S_\alpha$ implies $a\overline{\mathcal{B}}ab$ which yields $a \in B_k(ab)$. If $ab \in S_\beta$ then $b, ab \in S_\beta$, and so, similarly we have $b \in B_k(ab)$. Thus either $a \in B_k(ab)$ or $b \in B_k(ab)$.

$(2) \Rightarrow (3)$: Let $a, x, y \in S$ such that $xy \in B_k(a)$. Now $x \in B_k(xy)$ or $y \in B_k(xy)$ implies $x \in B_k(a)$ or $y \in B_k(a)$. Thus $B_k(a)$ is a completely prime k-ideal of S.

$(3) \Rightarrow (4)$: Let B be a k-bi-ideal of S and $x, y \in S$ such that $xy \in B$. Since $B_k(xy)$ is completely prime, $xy \in B_k(xy)$ implies either $x \in B_k(xy)$ or $y \in B_k(xy)$, i.e. $x \in B$ or $y \in B$. Thus B is completely prime.

$(4) \Rightarrow (5)$: Let $a, b \in S$. Then $B_k(a), B_k(b)$ and $B_k(ab)$ are completely prime k-ideals of S. Now $ab \in B_k(ab)$ implies $a \in B_k(ab)$ or $b \in B_k(ab)$. This gives $B_k(a) \subseteq B_k(ab)$ or $B_k(b) \subseteq B_k(ab)$ so that $B_k(a) \cap B_k(b) \subseteq B_k(ab)$. Since $B_k(a)$ and $B_k(b)$ are k-ideals, $ab \in B_k(b)$ and $ab \in B_k(a)$ implies $B_k(ab) \subseteq B_k(b)$ and $B_k(ab) \subseteq B_k(a)$, these yield $B_k(ab) \subseteq B_k(a) \cap B_k(b)$. Thus $B_k(ab) = B_k(a) \cap B_k(b)$. Again $B_k(a) \subseteq B_k(ab)$ or $B_k(b) \subseteq B_k(ab)$ implies $B_k(a) \subseteq B_k(ab) \subseteq B_k(b)$ or $B_k(b) \subseteq B_k(ab) \subseteq B_k(a)$, i.e. $B_k(a) \subseteq B_k(b)$ or $B_k(b) \subseteq B_k(a)$.

$(5) \Rightarrow (6)$: By Theorem 3.2, $N(a) = \{x \in S \mid a \in B_k(x)\}$. Let $a, b \in S$. Then $a, b \in N(ab)$ so that $N(a) \cup N(b) \subseteq N(ab)$. Let $x \in N(ab)$. Then $ab \in B_k(x)$. Now we have, $B_k(ab) = B_k(a)$ or $B_k(ab) = B_k(b)$ implies $B_k(a) \subseteq B_k(x)$ or $B_k(b) \subseteq B_k(x)$ so that $x \in N(a)$ or $x \in N(b)$, i.e. either $N(ab) \subseteq N(a)$ or $N(ab) \subseteq N(b)$. Therefore $N(ab) \subseteq N(a) \cup N(b)$. Thus $N(ab) = N(a) \cup N(b)$.

$(6) \Rightarrow (7)$: Let $B = \bigcap_{\lambda \in \Lambda} B_\lambda$. Then B is a k-bi-ideal of S. Let $xy \in B$ for some $x, y \in S$. Then $xy \in B_k(xy)$ implies $x \in B_k(xy) \subseteq B$ or $y \in B_k(xy) \subseteq B$ so that B is a completely prime.

$(6) \Rightarrow (8)$: By Theorem 3.2, we are only to show that $\mathcal{N}$ is a chain congruence on S. For, let

$a, b \in S$. Then $ab \in N(ab) = N(a) \cup N(b)$ implies $ab \in N(a)$ or $ab \in N(b)$, i.e. $N(ab) \subseteq N(a) \subseteq N(a) \cup N(b) = N(ab)$ or $Nab \subseteq N(b) \subseteq N(a) \cup N(b) = N(ab)$ so that $N(ab) = N(a)$ or $N(ab) = N(b)$. Thus $ab\mathcal{N}a$ or $ab\mathcal{N}b$.

$(8) \Rightarrow (1)$: Obvious.

$(7) \Rightarrow (4)$: Obvious.   Q.E.D.

References

[1] A.K. Bhuniya, K. Jana, *Bi-ideals in k-regular and intra k-regular semirings*, Discussiones Mathematicae-General Algebra and Applications **31**(2011) 5-25

[2] A.K. Bhuniya, T.K. Mondal, *On the least distributive lattice congruence on a semiring with a semilattice additive reduct*, Acta Mathematica Hungarica DOI: 10.1007/s10474-015-0526-5

[3] A.K. Bhuniya, T.K. Mondal, *Distributive lattice decompositions of semirings with a semilattice additive reduct*, Semigroup Forum **80**(2010), 293-301

[4] A.K. Bhuniya, T.K. Mondal, *Distributive lattices of left k-Archimedean semirings*, Semigroup Forum(to appear)

[5] A.K. Bhuniya, T.K. Mondal, *Semirings which are distributive lattices of k-simple semirings*, Southeast Asian Bulletin of Mathematics **36**(2012), 309-318

[6] S. Bourne, *The Jacobson radical of a semiring*, Proc. Nat. Acad. Sci. U.S.A. **37**(1951), 163-170

[7] Y. Cao, *Chain Decompositions of Ordered Semigroups*, Semigroup Forum **65**(2002), 83-106

[8] M. Ćirić, S. Bogdanović, *Semilattice decompositions of semigroups*, Semigroup Forum **52**(1996), 119-132

[9] A.H. Clifford, *Semigroups admitting relative inverses*, Ann. Math. **42**(1941), 1037-1049

[10] A.H. Clifford, G.B. Preston, *The algebraic theory of semigroups*, Vol. 1, Math. Surveys, no. 7, Amer. Math. Soc., Providence R. I., 1961, MR A2627

[11] U. Hebisch, H.J. Weinert, *Semirings: Algebraic theory and applications in computer science*, World Scientific, Singapore(1998)

[12] J.M. Howie, *Fundamentals in semigroup theory*, Clarendon, Oxford(1995)

[13] F. Kmet, *Radicals and their left ideal analogues in a semigroup*, Math. Slovaca **38**(1988), 139-145

[14] G.L. Litvinov, V.P. Maslov, *The Correspondence principle for idempotent calculus and some computer applications*, in Idempotency, J. Gunawardena(Editor), Cambridge Univ. Press, Cambridge, 1998, 420-443

[15] G.L. Litvinov, V.P. Maslov, G.B. Shpiz, *Idempotent functional analysis: An algebraic approach*, arXiv:math/0009128v2

[16] T.K. Mondal, A.K. Bhuniya, *On k-radicals of Green's relations in semirings with a semilattice additive reduct*, Discussiones Mathematicae-General Algebra and Applications **33** (2013), 85-93

[17] T.K. Mondal, *Distributive lattices of t-k-Archimedean semirings*, Discussiones Mathematicae-General Algebra and Applications **31**(2011), 147-158

[18] M.S. Putcha, *Semilattice decomposition of semigroups*, Semigroup Forum **6**(1973), 12-34

[19] M.K. Sen, A.K. Bhuniya, *On additive idempotent k-regular semirings*, Bull. Cal. Math. Soc. **93**(2001), 371-384

[20] M.K. Sen, A.K. Bhuniya, *On semirings whose additive reduct is a semilattice*, Semigroup forum **82**(2011), 131-140

[21] T. Tamura, *Another proof of a theorem concerning the greatest semilattice decomposition of a semigroup*, Proc. Japan Acad. **40**(1964), 777-780

[22] H.S. Vandiver, *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40**(1934), 914-920