On some Hermite-Hadamard-type integral inequalities for co-ordinated (α, QC)- and (α, CJ)-convex functions

Bo-Yan Xi[1], Jian Sun[2] and Shu-Ping Bai[3]

College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

E-mail: baoyintu78@qq.com[1], baoyintu68@sohu.com[1], baoyintu78@imun.edu.cn[1], sunjian91@qq.com[2], bsp0838@126.com[3]

Abstract

In the article, the authors introduce the new concepts "co-ordinated (α, QC)-, (α, JQC)-, (α, CJ)- and (α, J)-convex functions", establish some Hermite-Hadamard's type integral inequalities for the co-ordinated (α, QC)-, (α, JQC)-, (α, CJ)- and (α, J)-convex functions.

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1 Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is Jensen-convex(J), if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \tag{1.2}$$

holds for all $x, y \in I$.

Definition 1.3. ([5, 6, 8]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex(QC), if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\} \tag{1.3}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

In [5], the authors introduced the class of real functions of JQ type, defined as follows.

Definition 1.4. ([5]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is Jensen- or J-quasi-convex(JQC) if

$$f\left(\frac{x+y}{2}\right) \le \max\{f(x), f(y)\} \tag{1.4}$$

holds for all $x, y \in I$.

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In [5], Dragomir and Pearce proved the following theorem:

Theorem 1.1 ([5, Theorem 2.2])**.** Suppose $a, b \in I \subseteq \mathbb{R}$ and $a < b$. If $f \in JQC(I) \cap L_1([a, b])$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x + I(a,b), \tag{1.5}$$

where

$$I(a,b) = \int_0^1 |f(ta+(1-t)b) - f((1-t)a+tb)|\,\mathrm{d}\,t. \tag{1.6}$$

In [3, 4], S.S. Dragomir considered the convexity on the co-ordinated.

Definition 1.5 ([3, 4])**.** A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a < b$ and $c < d$ if the partial mappings

$$f_y : [a,b] \to \mathbb{R}, f_y(u) = f_y(u,y) \text{ and } f_x : [c,d] \to \mathbb{R}, f_x(v) = f_x(x,v) \tag{1.7}$$

are convex where defined for all $x \in (a, b), y \in (c, d)$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1.6. A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a < b$ and $c < d$ if the inequality

$$\begin{aligned} &f(tx+(1-t)z, \lambda y + (1-\lambda)w) \\ &\qquad \leq t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda)f(z,w) \end{aligned} \tag{1.8}$$

holds for all $t, \lambda \in [0, 1], (x, y), (z, w) \in \Delta$.

In [3, 4], S.S. Dragomir established the following theorem.

Theorem 1.2 ([3, Theorem 2.2])**.** Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be convex on the co-ordinates on Δ with $a < b$ and $c < d$. Then, one has the inequalities:

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq &\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)\mathrm{d}\,x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)\mathrm{d}\,y\right] \\ \leq &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}\,y\,\mathrm{d}\,x \\ \leq &\frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c)\,\mathrm{d}\,x + \int_a^b f(x,d)\,\mathrm{d}\,x\right) + \frac{1}{d-c}\left(\int_c^d f(a,y)\,\mathrm{d}\,y + \int_c^d f(b,y)\,\mathrm{d}\,y\right)\right] \\ \leq &\frac{1}{4}[f(a,c) + f(b,c) + f(a,d) + f(b,d)]. \end{aligned} \tag{1.9}$$

In this paper, we introduce the new concepts "(α, QC)-, (α, JQC)-, (α, CJ)- and (α, J)-convex functions on the co-ordinates on the rectangle of the $\mathbb{R}^2$" and we establish some new integral inequalities of Hermite-Hadamard type for the co-ordinated (α, QC)-, (α, JQC)-, (α, CJ)- and (α, J)-convex functions.

2 Some Definitions and Properties

We will start the following definition.

Definition 2.1. A mapping $f : [a,b] \times [c,d] \to \mathbb{R}$ will be called co-ordinated (α, QC)-convex on $[a,b] \times [c,d]$ with $a,\ b,\ c,\ d \in \mathbb{R}$ and $a < b,\ c < d$, if the following inequality:

$$\begin{aligned} &f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ &\qquad \le t^{\alpha} \max\{f(x,y), f(x,w)\} + (1-t^{\alpha}) \max\{f(z,y), f(z,w)\} \end{aligned} \tag{2.1}$$

holds for all $t, \lambda \in [0,1], (x,y), (z,w) \in [a,b] \times [c,d]$ and some $\alpha \in (0,1]$.

Now we introduce the new concept "(α, JQC)-convex functions on the co-ordinates on the rectangle of the $\mathbb{R}^2$".

Definition 2.2. A mapping $f : [a,b] \times [c,d] \to \mathbb{R}$ will be called co-ordinated (α, JQC)-convex on $[a,b] \times [c,d]$ with $a,\ b,\ c,\ d \in \mathbb{R}$ and $a < b,\ c < d$, if the following inequality:

$$f\left(tx + (1-t)z, \frac{y+w}{2}\right) \le t^{\alpha} \max\{f(x,y), f(x,w)\} + (1-t^{\alpha}) \max\{f(z,y), f(z,w)\} \tag{2.2}$$

holds for all $t \in [0,1], (x,y), (z,w) \in [a,b] \times [c,d]$ and some $\alpha \in (0,1]$.

We give the definitions of co-ordinated (α, CJ)- and (α, J)-convex functions.

Definition 2.3. For $\alpha \in (0,1]$, a function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said co-ordinated (α, CJ)-convex function on the co-ordinates on $[a,b] \times [c,d]$, if

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le t^{\alpha} \frac{f(x,y) + f(x,w)}{2} + (1-t^{\alpha}) \frac{f(z,y) + f(z,w)}{2} \tag{2.3}$$

holds for all $t, \lambda \in [0,1], (x,y), (z,w) \in [a,b] \times [c,d]$.

Definition 2.4. For $\alpha \in (0,1]$, a function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said co-ordinated (α, J)-convex function on the co-ordinates on $[a,b] \times [c,d]$, if

$$f\left(tx + (1-t)z, \frac{y+w}{2}\right) \le t^{\alpha} \frac{f(x,y) + f(x,w)}{2} + (1-t^{\alpha}) \frac{f(z,y) + f(z,w)}{2} \tag{2.4}$$

holds for all $t \in [0,1]$ and $(x,y), (z,w) \in [a,b] \times [c,d]$.

Theorem 2.1. *Let (α, QC), (α, JQC), (α, CJ) and (α, J) denote the class of (α, QC)-, (α, JQC)-, (α, CJ)- and (α, J)-convex functions on $[a,b] \times [c,d] \subseteq \mathbb{R}^2$ for some $\alpha \in (0,1]$, respectively. Then*

$$(\alpha, QC) \subseteq (\alpha, CJ) \text{ and } (\alpha, JQC) \subseteq (\alpha, J).$$

Proof. Since

$$\max\{u,v\} = \frac{u + v + |u-v|}{2} \ge \frac{u+v}{2}$$

for all $u,\ v \in \mathbb{R}$, then $(\alpha, \mathrm{QC}) \subseteq (\alpha, \mathrm{CJ})$ and $(\alpha, \mathrm{JQC}) \subseteq (\alpha, \mathrm{J})$. Theorem 2.1 is proved. Q.E.D.

Theorem 2.2. Let (α, QC), (α, JQC), (α, CJ) and (α, J) denote the class of (α, QC)-, (α, JQC)-, (α, CJ)- and (α, J)-convex functions on $[a,b]\times[c,d]\subseteq\mathbb{R}^2$ for some $\alpha\in(0,1]$, respectively. Then

$$(\alpha, QC)\subseteq(\alpha, JQC) \textit{ and } (\alpha, CJ)\subseteq(\alpha, J).$$

Proof. In (2.1) and (2.3), if $\lambda=\frac{1}{2}$, then (2.2) and (2.4) hold. So $(\alpha,\mathrm{QC})\subseteq(\alpha,\mathrm{JQC})$ and $(\alpha,\mathrm{CJ})\subseteq(\alpha,\mathrm{J})$. The proof of Theorem 2.2 is complete. Q.E.D.

Corollary 2.2.1. Under the conditions of Theorem 2.1 and Theorem 2.2, then

$$(\alpha, QC)\subseteq(\alpha, JQC)\subseteq(\alpha, J) \textit{ and } (\alpha, QC)\subseteq(\alpha, CJ)\subseteq(\alpha, J).$$

3 Some integral inequalities of Hermite-Hadamard type

In this section, we establish Hermite-Hadamard integral inequality for co-ordinated (α,QC)-, (α,JQC)-, (α,CJ)- and (α,J)-convex functions on rectangle from the $\mathbb{R}^2$.

Theorem 3.1. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a integrable on $[a,b]\times[c,d]$ with $a<b$ and$c<d$. If f is co-ordinated (α, J)-convex on $[a,b]\times[c,d]$ for some $\alpha\in(0,1]$, then

$$\begin{aligned}
&2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\le&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{2}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\
\le&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
\le&\frac{1}{(\alpha+1)(d-c)}\int_c^d\big[f(a,y)+\alpha f(b,y)\big]\,\mathrm{d}\,y.
\end{aligned}\tag{3.1}$$

Proof. From the (α,J)-convexity of f, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
=&\int_0^1 f\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\frac{(c+d)/2+(c+d)/2}{2}\right)\mathrm{d}\,t\\
\le&\frac{1}{2^{\alpha+1}}\int_0^1\left[f\left(ta+(1-t)b,\frac{c+d}{2}\right)+(2^\alpha-1)f\left((1-t)a+tb,\frac{c+d}{2}\right)\right]\mathrm{d}\,t\\
=&\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x.
\end{aligned}\tag{3.2}$$

By the (α, J)-convexity of $f($ with $t = \frac{1}{2}$ in (2.4)), and using the (3.2), give

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq & \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x = \frac{1}{2(b-a)} \int_0^1 \int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x \, \mathrm{d}\, \lambda \\
\leq & \frac{1}{4(b-a)} \int_0^1 \int_a^b [f(x, \lambda c + (1-\lambda) d) + f(x, (1-\lambda) c + \lambda d)] \, \mathrm{d}\, x \, \mathrm{d}\, \lambda \\
= & \frac{1}{2(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, \mathrm{d}\, x \, \mathrm{d}\, y
\end{aligned} \tag{3.3}$$

Similarly, we obtain

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq & \frac{1}{2} \int_0^1 \left[f\left(\frac{a+b}{2}, \lambda c + (1-\lambda) d\right) + f\left(\frac{a+b}{2}, (1-\lambda) c + \lambda d\right) \right] \mathrm{d}\, \lambda \\
= & \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y \\
\leq & \frac{1}{4(d-c)} \int_c^d \int_0^1 \big[f(ta + (1-t) b, y) + f((1-t) a + tb, y) \big] \, \mathrm{d}\, t \, \mathrm{d}\, y \\
= & \frac{1}{2(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, \mathrm{d}\, x \, \mathrm{d}\, y.
\end{aligned} \tag{3.4}$$

By addition (3.3) and (3.4), the first inequality in (3.1) is proved.

On the other hand, letting $x = ta + (1-t)b, \ 0 \leq t \leq 1$, by the (α, J)-convexity of f, then

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, \mathrm{d}\, x \, \mathrm{d}\, y = \frac{1}{d-c} \int_c^d \int_0^1 f(ta + (1-t) b, y) \, \mathrm{d}\, t \, \mathrm{d}\, y \\
\leq & \frac{1}{d-c} \int_c^d \int_0^1 \big[t^\alpha f(a, y) + (1 - t^\alpha) f(b, y) \big] \, \mathrm{d}\, t \, \mathrm{d}\, y \\
= & \frac{1}{(\alpha+1)(d-c)} \int_c^d \big[f(a, y) + \alpha f(b, y) \big] \, \mathrm{d}\, y.
\end{aligned} \tag{3.5}$$

The proof of Theorem 3.1 is complete. Q.E.D.

Corollary 3.1.1. Under the conditions of Theorem 3.1, if $\alpha = 1$, then

$$\begin{aligned}
&2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
\le&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)\mathrm{d}\,x + \frac{2}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)\mathrm{d}\,y\right]\\
\le&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
\le&\frac{1}{2(d-c)}\int_c^d\big[f(a,y)+f(b,y)\big]\,\mathrm{d}\,y.
\end{aligned}$$

Theorem 3.2. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a integrable on $[a,b]\times[c,d]$ with $a<b$ and$c<d$. If f is co-ordinated (α, CJ)-convex on $[a,b]\times[c,d]$ for some $\alpha\in(0,1]$, then

$$\begin{aligned}
&2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
\le&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)\mathrm{d}\,x + \frac{2}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)\mathrm{d}\,y\right]\\
\le&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
\le&\frac{1}{2}\left[\frac{1}{2(b-a)}\int_a^b\big[f(x,c)+f(x,d)\big]\,\mathrm{d}\,x + \frac{1}{(\alpha+1)(d-c)}\int_c^d\big[f(a,y)+\alpha f(b,y)\big]\,\mathrm{d}\,y\right]\\
\le&\frac{1}{2(\alpha+1)}\Big\{f(a,c)+f(a,d)+\alpha\big[f(b,c)+f(b,d)\big]\Big\}. \qquad (3.6)
\end{aligned}$$

Proof. Using the (α,CJ)-convexity of f, similarly to the proof of Theorem 3.1, we obtain first inequality in (3.6).

Putting $y=\lambda c+(1-\lambda)d,\ 0\le\lambda\le1$, by the (α,CJ)-convexity of f, then

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
\le&\frac{1}{(\alpha+1)(d-c)}\int_c^d\big[f(a,y)+\alpha f(b,y)\big]\,\mathrm{d}\,y\\
=&\frac{1}{\alpha+1}\int_0^1\Big\{[f(a,\lambda c+(1-\lambda)d)+\alpha f(b,\lambda c+(1-\lambda)d)\Big\}\,\mathrm{d}\,\lambda\\
\le&\frac{1}{2(\alpha+1)}\Big\{f(a,c)+f(a,d)+\alpha\big[f(b,c)+f(b,d)\big]\Big\} \qquad (3.7)
\end{aligned}$$

and setting $x = ta + (1-t)b,\ 0 \le t \le 1$, by the (α, CJ)-convexity of f, we get

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y = \frac{1}{b-a} \int_0^1 \int_a^b f(x, \lambda c + (1-\lambda)d)\,\mathrm{d}\,x\,\mathrm{d}\,\lambda \\
\le&\frac{1}{2(b-a)} \int_0^1 \int_a^b \big[f(x,c) + f(x,d)\big]\,\mathrm{d}\,x\,\mathrm{d}\,\lambda = \frac{1}{2(b-a)} \int_a^b \big[f(x,c) + f(x,d)\big]\,\mathrm{d}\,x \\
\le&\frac{1}{2} \int_0^1 \big[t^\alpha f(a,c) + (1-t^\alpha) f(b,c) + t^\alpha f(a,d) + (1-t^\alpha) f(b,d)\big]\,\mathrm{d}\,t \\
=&\frac{1}{2(\alpha+1)} \Big\{f(a,c) + f(a,d) + \alpha\big[f(b,c) + f(b,d)\big]\Big\}
\end{aligned} \tag{3.8}$$

The proof of Theorem 3.2 is complete. Q.E.D.

Corollary 3.2.1. In Theorem 3.2, if $\alpha = 1$, then

$$\begin{aligned}
&2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\le&\frac{1}{2}\left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right)\mathrm{d}\,x + \frac{2}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right)\mathrm{d}\,y\right] \\
\le&\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \\
\le&\frac{1}{4}\left[\frac{1}{b-a} \int_a^b \big[f(x,c) + f(x,d)\big]\,\mathrm{d}\,x + \frac{1}{d-c} \int_c^d \big[f(a,y) + f(b,y)\big]\,\mathrm{d}\,y\right] \\
\le&\frac{1}{4}\big[f(a,c) + f(a,d) + f(b,c) + f(b,d)\big].
\end{aligned}$$

Theorem 3.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a integrable on $[a,b] \times [c,d]$ with $a < b$ and$c < d$. If f is co-ordinated (α, JQC)-convex on $[a,b] \times [c,d]$ for some $\alpha \in (0,1]$, then

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\le&\frac{1}{2}\left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right)\mathrm{d}\,x + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right)\right] + \frac{1}{4} M_{a,b}(c,d) \\
\le&\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y + \frac{1}{4} M_{a,b}(c,d) + \frac{1}{4} D(a,b;c,d)
\end{aligned} \tag{3.9}$$

and

$$\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \le \frac{1}{(\alpha+1)(d-c)} \int_c^d \big[f(a,y) + \alpha f(b,y)\big]\,\mathrm{d}\,y, \tag{3.10}$$

where

$$M_{a,b}(c,d) = \frac{1}{d-c} \int_c^d \left| f\left(\frac{a+b}{2}, y\right) - f\left(\frac{a+b}{2}, c+d-y\right)\right| \mathrm{d}\,y, \tag{3.11}$$

$$D(a,b;c,d) = \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b |f(x,y) - f(x, c+d-y)|\,\mathrm{d}\,x\,\mathrm{d}\,y. \tag{3.12}$$

Proof. From the (α, JQC)-convexity of f, we have

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{2^\alpha}\left[f\left(ta+(1-t)b, \frac{c+d}{2}\right) + (2^\alpha - 1)f\left((1-t)a+tb, \frac{c+d}{2}\right)\right] \tag{3.13}$$

for all $t \in [0,1]$.

Integrating the inequality (3.13) on $[0,1]$ over t, we obtain

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\le&\frac{1}{2^\alpha}\int_0^1 \left[f\left(ta+(1-t)b, \frac{c+d}{2}\right) + (2^\alpha - 1)f\left((1-t)a+tb, \frac{c+d}{2}\right)\right] \mathrm{d}\, t \\
=&\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x = \frac{1}{b-a}\int_0^1\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x \,\mathrm{d}\,\lambda \\
\le&\frac{1}{b-a}\int_0^1\int_a^b \max\{f(x, \lambda c+(1-\lambda)d), f(x,(1-\lambda)c+\lambda d)\} \,\mathrm{d}\, x \,\mathrm{d}\,\lambda \\
=&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b \max\{f(x,y), f(x,c+d-y)\} \,\mathrm{d}\, x \,\mathrm{d}\, y \\
=&\frac{1}{2(b-a)(d-c)}\int_c^d\int_a^b \big[2f(x,y) + |f(x,y) - f(x,c+d-y)|\big] \,\mathrm{d}\, x \,\mathrm{d}\, y.
\end{aligned} \tag{3.14}$$

Similarly to the proof of (3.14), we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\le&\int_0^1 \max\left\{f\left(\frac{a+b}{2}, \lambda c+(1-\lambda)d\right), f\left(\frac{a+b}{2}, (1-\lambda)c+\lambda d\right)\right\} \mathrm{d}\,\lambda \\
=&\frac{1}{(d-c)}\int_c^d \max\left\{f\left(\frac{a+b}{2}, y\right), f\left(\frac{a+b}{2}, c+d-y\right)\right\} \mathrm{d}\, y \\
=&\frac{1}{2(d-c)}\int_c^d \left[2f\left(\frac{a+b}{2}, y\right) + \left|f\left(\frac{a+b}{2}, y\right) - f\left(\frac{a+b}{2}, c+d-y\right)\right|\right] \mathrm{d}\, y.
\end{aligned} \tag{3.15}$$

Here

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y = \frac{1}{d-c}\int_c^d\int_0^1 f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, t \,\mathrm{d}\, y \\
\le&\frac{1}{2^\alpha(d-c)}\int_c^d\int_0^1 \left[f\left(ta+(1-t)b, \frac{c+d}{2}\right) + (2^\alpha - 1)f\left((1-t)a+tb, \frac{c+d}{2}\right)\right] \mathrm{d}\, t \,\mathrm{d}\, y \\
=&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y) \,\mathrm{d}\, x \,\mathrm{d}\, y.
\end{aligned} \tag{3.16}$$

By the (3.16) into the inequality (3.15), then

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq&\frac{1}{2(d-c)}\int_c^d\left[2f\left(\frac{a+b}{2},y\right)+\left|f\left(\frac{a+b}{2},y\right)-f\left(\frac{a+b}{2},c+d-y\right)\right|\right]\mathrm{d}\,y\\
\leq&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&+\frac{1}{2(d-c)}\int_c^d\left|f\left(\frac{a+b}{2},y\right)-f\left(\frac{a+b}{2},c+d-y\right)\right|\mathrm{d}\,y. \qquad (3.17)
\end{aligned}$$

Choose $x=ta+(1-t)b$ for $0\leq t\leq 1$, by the (α, JQC)-convexity of f(with $0\leq t\leq 1$, $\lambda=\frac{1}{2}$ in (2.2)), we can write

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
=&\frac{1}{d-c}\int_c^d\int_0^1 f(ta+(1-t)b,y)\,\mathrm{d}\,t\,\mathrm{d}\,y\\
\leq&\frac{1}{d-c}\int_c^d\int_0^1\left[t^\alpha f(a,y)+(1-t^\alpha)f(b,y)\right]\mathrm{d}\,t\,\mathrm{d}\,y\\
=&\frac{1}{(\alpha+1)(d-c)}\int_c^d\left[f(a,y)+\alpha f(b,y)\right]\mathrm{d}\,y. \qquad (3.18)
\end{aligned}$$

The proof of Theorem 3.3 is complete. Q.E.D.

Corollary 3.3.1. Under the conditions of Theorem 3.3, if $f_x(y)=f_x(x,y)$ be symmetric to $\frac{c+d}{2}$ on $[c,d]$ for all $x\in[a,b]$, then

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\right]\\
\leq&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\leq\frac{1}{(\alpha+1)(d-c)}\int_c^d\left[f(a,y)+\alpha f(b,y)\right]\mathrm{d}\,y.
\end{aligned}$$

By the Theorem 2.2 and the Theorem 3.3, we have

Theorem 3.4. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a integrable on $[a,b]\times[c,d]$ with $a<b$ and$c<d$. If f is co-ordinated (α,QC)-convex on $[a,b]\times[c,d]$ for some $\alpha\in(0,1]$, then

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\right]+\frac{1}{4}M_{a,b}(c,d)\\
\leq&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y+\frac{1}{4}M_{a,b}(c,d)+\frac{1}{4}D(a,b;c,d), \qquad (3.19)
\end{aligned}$$

where $M_{a,b}(c,d)$ and $D(a,b;c,d)$ are given by (3.11) and (3.12).

Theorem 3.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a integrable on $[a, b] \times [c, d]$ with $a < b$ and$c < d$. If f is co-ordinated (α, QC)-convex on $[a, b] \times [c, d]$ for some $\alpha \in (0, 1]$, then

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \\
\le &\frac{1}{2}\left[\frac{1}{2(b-a)} \int_a^b \big[f(x,c)+f(x,d)\big]\,\mathrm{d}\,x \right.\\
&\left. + \frac{1}{(\alpha+1)(d-c)} \int_c^d \big[f(a,y)+\alpha f(b,y)\big]\,\mathrm{d}\,y\right] + \frac{1}{4} N_{c,d}(a,b) \\
\le &\frac{1}{2(\alpha+1)}\Big\{\big[f(a,c)+f(a,d)+\alpha\big[f(b,c)+f(b,d)\big]\Big\} + \frac{1}{4} N_{c,d}(a,b) \\
&+ \frac{1}{4(\alpha+1)}\big\{|f(a,c)-f(a,d)|+\alpha|f(b,c)-f(b,d)|\big\},
\end{aligned} \tag{3.20}$$

where

$$N_{c,d}(a,b) = \frac{1}{b-a} \int_a^b |f(x,c)-f(x,d)|\,\mathrm{d}\,x. \tag{3.21}$$

Proof. Similarly to the proof of (3.7) and (3.8), and using the (α, QC)-convexity of f, we obtain

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \\
\le &\frac{1}{2(b-a)} \int_a^b \big[f(x,c)+f(x,d)+|f(x,c)-f(x,d)|\big]\,\mathrm{d}\,x \\
\le &\frac{1}{2} \int_0^1 \Big\{t^\alpha\big[f(a,c)+f(a,d)+(1-t^\alpha)\big[f(b,c)+f(b,d)\big]\Big\}\,\mathrm{d}\,t + \frac{1}{2} J(c,d) \\
= &\frac{1}{2(\alpha+1)}\Big\{f(a,c)+f(a,d)+\alpha\big[f(b,c)+f(b,d)\big]\Big\} + \frac{1}{2} J(c,d).
\end{aligned} \tag{3.22}$$

By a similar argument and from (3.10), we observe that

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \\
\le &\frac{1}{(\alpha+1)(d-c)} \int_c^d \big[f(a,y)+\alpha f(b,y)\big]\,\mathrm{d}\,y \\
\le &\frac{1}{2(\alpha+1)}\Big\{f(a,c)+f(a,d)+|f(a,c)-f(a,d)| \\
&+ \alpha\big[f(b,c)+f(b,d)+|f(b,c)-f(b,d)|\big]\Big\}.
\end{aligned} \tag{3.23}$$

By (3.22) and (3.23), the inequality (3.20) is proved. Q.E.D.

Corollary 3.5.1. Under the conditions of Theorem 3.4 and Theorem 3.5, if $f_x(y) = f_x(x, y)$ is symmetric to $\frac{c+d}{2}$ on $[c, d]$ for all $x \in [a, b]$, then

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y\right] \\
\le &\frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x, y) \,\mathrm{d}\, x \,\mathrm{d}\, y \\
\le &\frac{1}{2}\left[\frac{1}{2(b-a)}\int_a^b \big[f(x, c) + f(x, d)\big] \mathrm{d}\, x + \frac{1}{(\alpha+1)(d-c)}\int_c^d \big[f(a, y) + \alpha f(b, y)\big] \mathrm{d}\, y\right] \\
\le &\frac{1}{2(\alpha+1)}\Big\{f(a, c) + f(a, d) + \alpha\big[f(b, c) + f(b, d)\big]\Big\}.
\end{aligned}$$

Furthermore, if $\alpha = 1$, then

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y\right] \\
\le &\frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x, y) \,\mathrm{d}\, x \,\mathrm{d}\, y \\
\le &\frac{1}{4}\left[\frac{1}{b-a}\int_a^b \big[f(x, c) + f(x, d)\big] \mathrm{d}\, x + \frac{1}{d-c}\int_c^d \big[f(a, y) + f(b, y)\big] \mathrm{d}\, y\right] \\
\le &\frac{1}{4}\big[f(a, c) + f(a, d) + f(b, c) + f(b, d)\big].
\end{aligned}$$

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