Retraction Note

K.K.Dewan and A.Ahuja, Growth of polynomials with prescribed zeros, II,

Tbilisi Math. J. 8 (2) (2015), 69-74, De Gruyter Open,

DOI 10 1515/tmj - 2015 – 0011.

The authors have requested that their paper be withdrawn because main results of the text were published in Mathematick Vesnik,

S.M.Pukhta, Growth of polynomials with prescribed zeros, Journal of Mathematick Vesnik 65 (1) (2013), 137-142.

Growth of polynomials with prescribed zeros – II

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Abstract

In this paper we consider a class of polynomials $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, having all its zeros on $|z| = k$, $k \leq 1$. Using the notation $M(p,t) = \max_{|z|=t} |p(z)|$, we measure the growth of p by estimating $\left\{\dfrac{M(p,t)}{M(p,1)}\right\}^s$ from above for any $t \geq 1$, s being an arbitrary positive integer.

2010 Mathematics Subject Classification. **30A10**. 30C10, 30C15.
Keywords. Polynomials, maximum modulus, extremal problems.

1 Introduction and Statement of Results

For an arbitrary entire function $f(z)$, let $M(f,r) = \max_{|z|=r} |f(z)|$. As a consequence of maximum modulus principle [5, Vol. I, p. 137, Problem III, 269]) it is known that if $p(z)$ is a polynomial of degree n, then

$$M(p,R) \leq R^n M(p,1) \quad \text{for} \quad R \geq 1. \tag{1.1}$$

The result is best possible and equality holds for polynomials having zeros at the origin.

Ankeny and Rivlin [1] considered polynomials not vanishing in the interior of the unit circle and obtained refinement of inequality (1.1) by proving that if $p(z) \neq 0$ in $|z| < 1$, then

$$M(p,R) \leq \left(\frac{R^n + 1}{2}\right) M(p,1), \quad R \geq 1. \tag{1.2}$$

The result is sharp and equality in (1.2) holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in $|z| < k$, $k \leq 1$, recently the authors [2] proved the following result.

Theorem A. If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then for every positive integer s

$$\{M(p,R)\}^s \leq \left(\frac{k^{n-1}(1+k) + (R^{ns} - 1)}{k^{n-1} + k^n}\right) \{M(p,1)\}^s, \quad R \geq 1. \tag{1.3}$$

By involving the coefficients of $p(z)$, Dewan and Ahuja [2] in the same paper obtained the following refinement of Theorem A.

Tbilisi Mathematical Journal 8(2) (2015), pp. 69–74.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 11 February 2014.
Accepted for publication: 12 March 2015.

Theorem B. If $p(z)=\sum\limits_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then for every positive integer s

$$\{M(p,R)\}^s \le \frac{1}{k^n}\left(\frac{n|c_n|\{k^n(1+k^2)+k^2(R^{ns}-1)\}+|c_{n-1}|\{2k^n+R^{ns}-1\}}{2|c_{n-1}|+n|c_n|(1+k^2)}\right)\{M(p,1)\}^s,\ R\ge 1. \tag{1.4}$$

In this paper, we consider a class of polynomials $p(z)=c_n z^n+\sum\limits_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1\le \mu\le n$, having all its zeros on $|z|=k$, $k\le 1$ and generalize Theorem A and Theorem B. More precisely, we prove

Theorem 1. If $p(z)=c_n z^n+\sum\limits_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1\le \mu<n$, is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then for every positive integer s

$$\{M(p,R)\}^s \le \left(\frac{k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s, \quad R\ge 1. \tag{1.5}$$

Remark 1. If we choose $\mu=1$ in Theorem 1,then inequality (1.5) reduces to Theorem A.

For $s=1$ in Theorem 1, we get the following result.

Corollary 1. If $p(z)=c_n z^n+\sum\limits_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1\le \mu<n$, is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then

$$M(p,R)\le \left(\frac{k^{n-2\mu+1}+k^{n-\mu+1}+R^n-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)M(p,1), \quad R\ge 1. \tag{1.6}$$

The following corollary immediately follows from inequality (1.6) by taking $k=1$.

Corollary 2. If $p(z)=\sum\limits_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z|=1$, then

$$M(p,R)\le \left(\frac{R^n+1}{2}\right)M(p,1), \quad R\ge 1. \tag{1.7}$$

If we involve the coefficients of $p(z)$ also, then we are able to obtain a bound which is better than the bound of Theorem 1. In fact, we prove

Theorem 2. If $p(z)=c_n z^n+\sum\limits_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1\le \mu<n$, is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then for every positive integer s

$$\{M(p,R)\}^s \le \frac{1}{k^{n-\mu+1}}\left(\frac{\begin{array}{c} n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^{ns}-1)\} \\ +\mu|c_{n-\mu}|\{k^{n-\mu+1}(1+k^{\mu-1})+k^{\mu-1}(R^{ns}-1)\}\end{array}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right)\{M(p,1)\}^s,\ R\ge 1. \tag{1.8}$$

To prove that the bound obtained in the above theorem is, in general, better than the bound obtained in Theorem 1, we show that

$$\frac{1}{k^{n-\mu+1}}\frac{\begin{pmatrix} n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^{ns}-1)\} \\ +\mu|c_{n-\mu}|\{k^{n-\mu+1}(1+k^{\mu-1})+k^{\mu-1}(R^{ns}-1)\}\end{pmatrix}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}$$
$$\le \frac{k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}$$

which is equivalent to

$$\begin{aligned}&n|c_n|(k^{n-2\mu+1}+k^{n-\mu+1})(k^n+k^{n+\mu+1}+k^{2\mu}R^{ns}-k^{2\mu})\\&\quad+\mu|c_{n-\mu}|(k^{n-2\mu+1}+k^{n-\mu+1})(k^n+k^{n-\mu+1}+k^{\mu-1}R^{ns}-k^{\mu-1})\\&\qquad\le n|c_n|(k^n+k^{n+\mu+1})(k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1)\\&\qquad\quad+\mu|c_{n-\mu}|(k^n+k^{n-\mu+1})(k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1)\end{aligned}$$

which implies

$$\begin{aligned}&n|c_n|(k^{2n-2\mu+1}+k^{2n-\mu+2}+k^{n+1}R^{ns}-k^{n+1}+k^{2n-\mu+1}+k^{2n+2}+k^{n+\mu+1}R^{ns}-k^{n+\mu+1})\\&+\mu|c_{n-\mu}|(k^{2n-2\mu+1}+k^{2n-3\mu+2}+k^{n-\mu}R^{ns}-k^{n-\mu}+k^{2n-\mu+1}+k^{2n-2\mu+2}+k^nR^{ns}-k^n)\\&\quad\le n|c_n|(k^{2n-2\mu+1}+k^{2n-\mu+1}+k^nR^{ns}-k^n+k^{2n-\mu+2}+k^{2n+2}+k^{n+\mu+1}R^{ns}-k^{n+\mu+1})\\&\qquad+\mu|c_{n-\mu}|(k^{2n-2\mu+2}+k^{2n-3\mu+2}+k^{n-\mu+1}R^{ns}-k^{n-\mu+1}+k^{2n-2\mu+1}+k^{2n-\mu+1}+k^nR^{ns}-k^n),\end{aligned}$$

$$\mu|c_{n-\mu}|\{k^{n-\mu}(R^{ns}-1)-k^{n-\mu+1}(R^{ns}-1)\}\le n|c_n|\{k^n(R^{ns}-1)-k^{n+1}(R^{ns}-1)\},$$
$$\mu|c_{n-\mu}|k^{n-\mu}\le n|c_n|k^n,$$
$$\frac{\mu|c_{n-\mu}|}{n|c_n|}\le k^\mu,$$

which is always true (see Lemma 4).

If we choose $s=1$ in Theorem 2,we get the following result.

Corollary 3. If $p(z)=c_nz^n+\sum_{j=\mu}^{n}c_{n-j}z^{n-j}$, $1\le\mu<n$, is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$

$$M(p,R)\le\frac{1}{k^{n-\mu+1}}\left(\frac{\begin{matrix}n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^n-1)\}\\+\mu|c_{n-\mu}|\{k^{n-\mu+1}(1+k^{\mu-1})+k^{\mu-1}(R^n-1)\}\end{matrix}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right)M(p,1),R\ge 1. \tag{1.9}$$

Remark 2. (i) If we choose $\mu=1$ in Theorem 2,then inequality (1.8) reduces to Theorem B.
(ii) For $k=1$ in inequality (1.9), we get Corollary 2.

2 Lemmas

We need the following lemmas for the proof of these theorems. The first lemma is due to Dewan and Hans [3].

Lemma 1. If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)|. \tag{2.1}$$

Lemma 2. If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-\mu+1}} \frac{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1 + k^{\mu-1}) + n|c_n|k^{\mu-1}(1 + k^{\mu+1})} \max_{|z|=1} |p(z)|. \tag{2.2}$$

The above lemma is due to Dewan and Hans [4].

Lemma 3. Let $p(z) = c_0 + \sum_{v=\mu}^{n} c_v z^v$, $1 \le \mu \le n$, be a polynomial of degree n having no zero in the disk $|z| < k$, $k \ge 1$,

$$\frac{\mu}{n}\left|\frac{c_\mu}{c_0}\right| k^\mu \le 1\,. \tag{2.3}$$

The above lemma was given by Qazi [6, Remark 1].

Lemma 4. Let $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$,

$$\frac{\mu}{n}\left|\frac{c_{n-\mu}}{c_n}\right| \le k^\mu\,. \tag{2.4}$$

Proof of Lemma 4. If $p(z)$ has all its zeros on $|z| = k$, $k \le 1$, then $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, has all its zeros on $|z| = \frac{1}{k}$, $\frac{1}{k} \ge 1$. Now applying Lemma 3 to the polynomial $q(z)$, Lemma 4 follows.

3 Proof of the theorems

Proof of Theorem 1. Let $M(p,1) = \max_{|z|=1} |p(z)|$. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, therefore by Lemma 1, we have

$$|p'(z)| \le \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} M(p,1) \quad \text{for} \quad |z| = 1.$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore,it follows by (1.1) that for all $r \ge 1$ and $0 \le \theta < 2\pi$

$$|p'(re^{i\theta})| \le \frac{nr^{n-1}}{k^{n-2\mu+1} + k^{n-\mu+1}} M(p,1). \tag{3.1}$$

Also for each $\theta, 0 \le \theta < 2\pi$ and $R \ge 1$, we obtain

$$\begin{aligned}\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s &= \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt \\ &= \int_1^R s\{p(te^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta} dt.\end{aligned}$$

This implies

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \le s\int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})|dt,$$

which on combining with inequalities (3.1) and (1.1), we get

$$\begin{aligned}|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\le \frac{ns}{k^{n-2\mu+1}+k^{n-\mu+1}}\{M(p,1)\}^s\int_1^R t^{ns-1}dt, \\ &= \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s.\end{aligned}$$

Therefore,

$$\begin{aligned}|p(Re^{i\theta})|^s &\le |p(e^{i\theta})|^s + \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s, \\ &\le \{M(p,1)\}^s + \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s \qquad (3.2)\end{aligned}$$

Hence, from (3.2), we conclude that

$$\{M(p,R)\}^s \le \left(\frac{k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. The proof of Theorem 2 follows on the same lines as that of Theorem 1 by using Lemma 2 instead of Lemma 1. But for the sake of completeness we give a brief outline of the proof. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, therefore, by Lemma 2, we have

$$|p'(z)| \le \frac{n}{k^{n-\mu+1}}\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}(1+k^{\mu+1})+\mu|c_{n-\mu}|(1+k^{\mu-1})}\right)M(p,1) \quad \text{for} \quad |z| = 1.$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore, it follows by (1.1) that for all $r \ge 1$ and $0 \le \theta < 2\pi$

$$|p'(re^{i\theta})| \le \frac{nr^{n-1}}{k^{n-\mu+1}}\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}(1+k^{\mu+1})+\mu|c_{n-\mu}|(1+k^{\mu-1})}\right)M(p,1). \qquad (3.3)$$

Also for each θ, $0 \le \theta < 2\pi$ and $R \ge 1$, we have

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \le s\int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})|dt,$$

which on combining with inequalities (1.1) and (3.3), we get

$$
\begin{aligned}
&|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \\
&\quad \le \left(\frac{R^{ns}-1}{k^{n-\mu+1}}\right)\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}(1+k^{\mu+1})+\mu|c_{n-\mu}|(1+k^{\mu-1})}\right)\{M(p,1)\}^s,
\end{aligned}
$$

which implies

$$
\begin{aligned}
|p(Re^{i\theta})|^s \quad \le \quad & \{M(p,1)\}^s + \left(\frac{R^{ns}-1}{k^{n-\mu+1}}\right) \\
& \times\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}(1+k^{\mu+1})+\mu|c_{n-\mu}|(1+k^{\mu-1})}\right)\{M(p,1)\}^s,
\end{aligned}
$$

from which the proof of Theorem 2 follows.

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