On the notion of pseudocategory internal to a category with a 2-cell structure

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Abstract

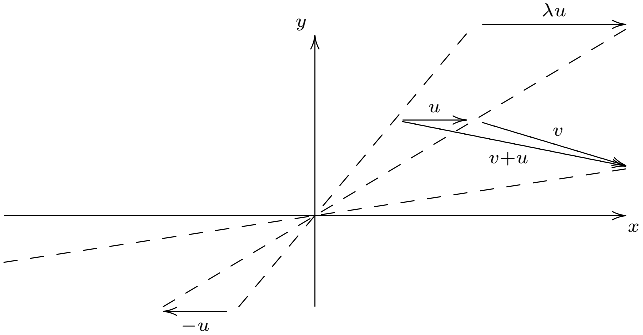
The notion of pseudocategory is extended from the context of a 2-category to the more general one of a sesquicategory, which is considered as a category equipped with a 2-cell structure. Some particular examples of 2-cells arising from internal transformations in internal categories, conjugations in groups, derivations in crossed-modules or homotopies in abelian chain complexes are studied in this context, namely their behaviour as abstract 2-cells in a 2-cell structure. Issues such as naturality of a 2-cell structure are investigated. This article is intended as a preliminary starting work towards the study of the geometric aspects of the 2-cell structures from an algebraic point of view.

2010 Mathematics Subject Classification. **18D05 18D35**. 18E05.
Keywords. Sesquicategory, 2-cell structure, cartesian 2-cell structure, natural 2-cell structure, pseudocategory.

1 Introduction

The purpose of this work is to extend the notion of pseudocategory internal to a 2-category [11] to the more general context of a pseudocategory internal to a category with a 2-cell structure (or sesquicategory).

In this article we are using a different notation for the vertical composition of 2-cells: instead of the usual dot '·' or bullet '•' we use plus '+'. To support this change of notation we present an analogy which compares the geometric vectors in the plane with the 2-cells between morphisms in a category.

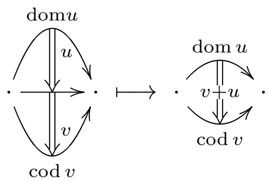


Two geometric vectors on the plane can be added only if the end point of the second one (u as in the picture above) is the starting point of the first one (v as in the picture); in that case the

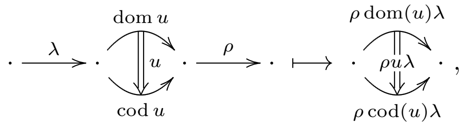
Tbilisi Mathematical Journal 8(1) (2015), pp. 107–141.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 08 October 2014.
Accepted for publication: 13 March 2015.

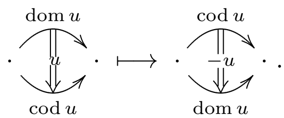
resulting vector, which is their sum, goes from the starting point of the second one to the end point of the first one. The same happens for 2-cells in a category:



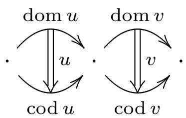
In some sense the analogy can be extended for scalar multiplication



and for inverses (in the case they exist, and in a much lesser degree of analogy)



Concerning horizontal composition, there is still an analogy with some relevance: it is, in some sense, analogous to the cross product of vectors — in the sense that it raises in dimension (see the introduction of [3] and its references for further discussion on this). Given 2-cells, u and v, displayed as,



the horizontal composition, usually denoted by $v \circ u$, is expected to be an isomorphic 3-cell, from the 2-cell

$$\operatorname{cod}(v)\, u + v \operatorname{dom}(u) \tag{1.1}$$

to the 2-cell

$$v \operatorname{cod}(u) + \operatorname{dom}(v)\, u. \tag{1.2}$$

In some cases (1.1) and (1.2) coincide (as it happens in a 2-category) and this is the reason why we may consider a horizontal composition.

As mentioned at the beginning of the introduction, our purpose is to enlarge the context in which an internal pseudocategory can be defined to the one of a category with a 2-cell structure. In this case, (1.1) and (1.2) are not necessarily equal anymore.

To do this we have to consider the horizontal composition as a relation, written as $v \circ u$, saying that the 2-cell v is natural with respect to the 2-cell u, which is defined by

$$v \circ u \Longleftrightarrow (1.1) = (1.2)\,.$$

With this regard, the horizontal composition is only defined for those pairs (v, u) that are in relation $v \circ u$, with the composite being given, in that case, by either (1.1) or (1.2).

This is a geometric intuition. An algebraic intuition is provided by Proposition 3.3. It is also possible to put the geometric analogy into a formal context. It is obtained by taking the category whose objects are the straight lines passing through the origin while the morphisms are the points in the plane except the origin. The domain and codomain associate to each non-null vector its direction; the composition of two vectors with the same direction is the vector which lies on the shared direction and whose magnitude is obtained as the product of the signed magnitudes of the two given vectors as real numbers; the identity morphism of a straight line is the vector of unitary length with that same direction. The geometric vectors in the plane appear in this situation as the codiscrete 2-cell structure.

This article is organized as follows. In Section 2 we recall the notions of internal (pre)category, internal (pre)functor and internal (natural) transformation. This section is also used to introduce some notation. In Section 3 we consider an arbitrary fixed category $\mathbf{C}$ and define a 2-cell structure over it, as to make it a sesquicategory. We give a characterization of that structure as a family of sets, together with maps and actions, satisfying some conditions (Proposition 3.3). This characterization generalizes the description of 2-Ab-categories as families of abelian groups, together with group homomorphisms and composition laws, which may be found in [10] and [12], except that, now, the (strong) condition

$$D\,(x)\,y = xD\,(y) \tag{1.3}$$

is no longer a requirement. A useful consequence of this is the possibility of considering the example of chain complexes, say of order 2, which is treated in 5.13. Condition (1.3) above is equivalent to the naturality condition, and since the results obtained in [10] and [12] rely on this assumption, we have to be careful in removing it. With that regard we introduce and study the concept of a 2-cell being natural with respect to another 2-cell (Definition 4.2), and the concept of a natural 2-cell (Definition 4.3), which is natural with respect to all the possible 2-cells whose horizontal composition with it makes sense. In Section 4 we work towards the definition of pseudocategory internal to an arbitrary category equipped with a 2-cell structure, which may be found in Section 6. We will need the notion of cartesian square with respect to a specified 2-cell structure (Definition 4.1), and the notions of natural and invertible 2-cell structures. We compare the abstract notions of naturality, which are being introduced, when $\mathbf{C}$ is a category $\mathsf{Cat}\,(\mathbf{B})$ of internal categories in some category $\mathbf{B}$. As expected, when the 2-cell structure consists of the internal transformations (not necessarily natural) then every natural transformation is a natural 2-cell (Corollary 5.2). Section 5 is entirely dedicated to examples. In Section 6 we extend the notion of pseudocategory from the context of a 2-category to the more general context of a category with a 2-cell structure (sesquicategory). As an example we consider the sesquicategory of abelian chain complexes with homotopies as 2-cells and study pseudocategories in there. We also give some results extracted from [14] concerning (weakly) Mal'tsev sesquicategories.

All the notions defined in [11]: pseudofunctor, natural and pseudo-natural transformation and modification, can be extended in this way.

However, these considerations are to be developed in a future work. This paper is the starting point for a systematic study of internal categorical structures in a category with a given 2-cell structure, and also to investigate how these categorical structures are changed when the given 2-cell structure over the (fixed) base category varies. For example, in a category with the discrete 2-cell structure, a pseudocategory is an internal category, while it is just a precategory if the 2-cell structure is the codiscrete one.

This work is a revised version of [15].

2 Internal precategories

A usual assumption on a category **C**, which is frequently asked when working with categories internal to **C**, is the existence of pullbacks. Indeed, a minimal requirement is the existence of pullbacks of split epimorphisms along split epimorphisms. In this work we are interested in the notion of internal (pre)category without assuming the existence of pullbacks on the ambient category **C**. This means that we have to consider an internal (pre)category as a structure where some conditions are satisfied, including, in particular, the requirement that some squares have the property of being pullback squares. This approach is useful, for example, in the study of (pre)categories internal to arbitrary ambient categories. Later on, when considering the notion of pseudocategory internal to a category with a 2-cell structure, we will also consider pullback squares that share their universal property with morphisms and 2-cells; they will be called cartesian squares.

There are currently several slightly different notions of precategory in the literature, mainly with respect to what concerns axiom (PC3) below (with the main reference on this topic being [7]). Here we consider an alternative notion, which best fits our setting.

Let **C** be an arbitrary category. A precategory internal to **C** is a diagram of the shape

$$A_2 \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} \!\!\xrightarrow{m}\! A_1 \underset{c}{\overset{d}{\rightrightarrows}} \!\!\xleftarrow{e}\! A_0$$

satisfying the following conditions:

PC1 $de = 1_{A_0} = ce$,

PC2 $dm = d\pi_2$ and $cm = c\pi_1$,

PC3 the square

$$\begin{array}{ccc} A_2 & \xrightarrow{\pi_2} & A_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle c} \\ A_1 & \xrightarrow{d} & A_0 \end{array}$$

is a pullback square.

A precategory A, internal to **C**, will be represented as a nine-tuple

$$(A_0, A_1, A_2, d, c, e, m, \pi_1, \pi_2).$$

In some cases though, and in order to simplify notation, when all pullbacks exist in **C** or when it is not important to specify the pullback A_2 and its projections, we will refer to a precategory A simply as a six-tuple

$$A = (A_0, A_1, d, c, e, m).$$

If $A = (A_0, A_1, d, c, e, m)$ and $B = (B_0, B_1, d', c', e', m')$ are two internal precategories, then a morphism between them is called a prefunctor and it consists of a pair (f_0, f_1) of morphisms, $f_i \colon A_i \to B_i$, $i = 0, 1$, such that $d' f_1 = f_0 d$ and $c' f_1 = f_0 c$. Observe that we do not ask preservation of e and m (i.e. the equalities $f_1 e = e' f_0$ and $f_1 m = m' f_2$, as presented below, do not need to be satisfied) in order to include the codiscrete 2-cell structure as an example, see the end of this section.

A transformation between two internal prefunctors $f = (f_0, f_1)$ and $g = (g_0, g_1)$, both from a precategory

$$A = (A_0, A_1, A_2, d, c, e, m, \pi_1, \pi_2)$$

to a precategory

$$B = (B_0, B_1, B_2, d', c', e', m', \pi_1', \pi_2'),$$

is a morphism

$$t_1 \colon A_0 \to B_1$$

such that $d' t_1 = f_0$ and $c' t_1 = g_0$. We will also consider two morphisms $t_2, t_3 \colon A_1 \to B_2$ such that

$$\begin{array}{ccc} \pi_1' t_2 = t_1 c & , & \pi_2' t_2 = f_1 \\ \pi_1' t_3 = g_1 & , & \pi_2' t_3 = t_1 d, \end{array}$$

which, by the universal property of the pullback B_2, are determined as $t_2 = \langle t_1 c, f_1 \rangle$ and $t_3 = \langle g_1, t_1 d \rangle$. We will sometimes refer to a natural transformation as a triple $t = (t_1, t_2, t_3)$.

An internal category is a precategory $(A_0, A_1, A_2, d, c, e, m, \pi_1, \pi_2)$ in which the following conditions are satisfied:

PC4 $m\langle ec, 1_{A_1} \rangle = 1_{A_1} = m\langle 1_{A_1}, ed \rangle$,

PC5 there exists a span

$$A_2 \xleftarrow{p_1} A_3 \xrightarrow{p_2} A_2$$

such that the square

$$\begin{array}{ccc} A_3 & \xrightarrow{p_2} & A_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ A_2 & \xrightarrow{\pi_2} & A_1 \end{array}$$

is a pullback square,

PC6 $m(1_{A_1} \times_{1_{A_0}} m) = m(m \times_{1_{A_0}} 1_{A_1})$.

An internal functor is a prefunctor $f \colon A \to B$ of internal precategories (that are internal categories) which, in addition to $d' f_1 = f_0 d$ and $c' f_1 = f_0 c$, satisfies $f_1 e = e' f_0$ and $f_1 m = m' f_2$, with $f_2 = f_1 \times_{f_0} f_1$. More specifically, the morphism f_2 is induced by the universal property of the pullbacks of d and c and of d' and c' as in the following diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{d} & A_0 & \xleftarrow{c} & A_1 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_1 & \xrightarrow{d'} & B_0 & \xleftarrow{c'} & B_1. \end{array}$$

A transformation $t = (t_1, t_2, t_3): f \to g$ is a natural transformation if, in addition to the conditions above, $m' t_2 = m' t_3$.

The purpose of this work is to consider a setting which is appropriate to the handling of internal categories and internal precategories as two extreme cases of the more general notion of pseudocategory. As it will be clear from the next section, every category can be equipped with several 2-cell structures. Any of them will give a different notion of a pseudocategory. In particular it is always possible to define two trivial 2-cell structures on the same ambient category: the discrete and the codiscrete ones. An internal category is a pseudocategory with respect to the discrete 2-cell structure, while a precategory is a pseudocategory with respect to the codiscrete 2-cell structure. Our interest will be focused on the notions which may arise as intermediate cases.

3 Categories with 2-cell structures or sesquicategories

As already mentioned in the introduction, this work considers the notion of a category with a 2-cell structure, which is the same thing as a sesquicategory. We prefer to call it a category with a 2-cell structure in order to emphasise the possibility of specifying different 2-cell structures over the same base category. The example which motivates this approach is the category of (left and right) R-modules for some unitary ring R, which also suggests the additive notation that is being used throughout the text.

Let $\mathbf{C}$ be a category. Its *hom* functor

$$\hom_{\mathbf{C}}: \mathbf{C}^{op} \times \mathbf{C} \to \mathsf{Set}$$

will be simply referred to as hom.

A 2-cell structure on $\mathbf{C}$, or over $\mathbf{C}$, is a category (C_0, C_1, d, c, e, m) internal to the functor category $\mathsf{Set}^{\mathbf{C}^{op} \times \mathbf{C}}$ whose object of objects C_0 is the functor hom.

Definition 3.1 (2-cell structure)**.** A 2-cell structure on $\mathbf{C}$ is a system

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

with

$$H: \mathbf{C}^{op} \times \mathbf{C} \to \mathsf{Set}$$

a functor, $\mathrm{dom}, \mathrm{cod}, 0, +$ natural transformations, displayed as,

$$H \times_{\hom} H \xrightarrow{+} H \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{0}}}}} \hom$$

and such that the six-tuple

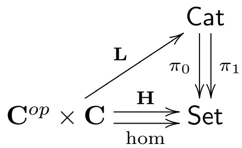
$$(\hom, H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

is an internal category in the functor category $\mathsf{Set}^{\mathbf{C}^{op} \times \mathbf{C}}$.

A category $\mathbf{C}$ equipped with a 2-cell structure $\mathbf{H}$, represented as a pair $(\mathbf{C}, \mathbf{H})$, is a sesquicategory (see for example [17, 18, 19]):

Proposition 3.2. Every category equipped with a 2-cell structure is a sesquicategory. Every sesquicategory determines a 2-cell structure over its underlying category.

Proof. A sesquicategory is a category $\mathbf{C}$ with a functor $\mathbf{L}\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathsf{Cat}$ into Cat, such that its restriction π_0 to Set (by forgetting the arrows) gives $\hom_{\mathbf{C}}$ and its restriction π_1 to Set (by forgetting the objects) gives $\mathbf{H}$, as illustrated.



Q.E.D.

The notation introduced in the following proposition will be used throughout the text. It gives a detailed description of the whole information which is needed to equip a category $\mathbf{C}$ with a 2-cell structure $\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)$. This notation is borrowed from the motivating example of left and right modules over a ring.

Proposition 3.3. Giving a 2-cell structure over a category $\mathbf{C}$ is to give, for every pair (A, B) of objects in $\mathbf{C}$, a set $H(A, B)$, together with maps

$$H(A,B) \times_{\hom(A,B)} H(A,B) \xrightarrow{+} H(A,B) \;\; \begin{matrix} \xrightarrow{\mathrm{dom}} \\ \xleftarrow{0} \\ \xrightarrow[\mathrm{cod}]{} \end{matrix} \;\; \hom(A,B), \tag{1.4}$$

and *actions*

$$\begin{array}{ccc} H(B,C) \times \hom(A,B) & \longrightarrow & H(A,C) \\ (x,f) & \longmapsto & xf \\ \hom(B,C) \times H(A,B) & \longrightarrow & H(A,C) \\ (g,y) & \longmapsto & gy \end{array}$$

satisfying the following conditions

$$\begin{gathered} g'(gy) = (g'g)\,y\ ,\ \ (xf)\,f' = x\,(ff') \\ g'(xf) = (g'x)\,f \\ 1_C x = x = x 1_B \end{gathered} \tag{1.5}$$

$$\begin{gathered} \mathrm{dom}(gy) = g\,\mathrm{dom}(y)\ \ ,\ \ \mathrm{dom}(xf) = \mathrm{dom}(x)\,f \\ \mathrm{cod}(gy) = g\,\mathrm{cod}(y)\ \ ,\ \ \mathrm{cod}(xf) = \mathrm{cod}(x)\,f \\ g0_f = 0_{gf} = 0_g f \\ (x + x')\,f = xf + x'f\ ,\ \ g\,(y + y') = gy + gy' \end{gathered} \tag{1.6}$$

$$\begin{gathered}\mathrm{dom}\,(0_f) = f = \mathrm{cod}\,(0_f) \\ \mathrm{dom}\,(x+x') = \mathrm{dom}(x')\ ,\ \mathrm{cod}\,(x+x') = \mathrm{cod}(x) \\ 0_{\mathrm{cod}\,x} + x = x = x + 0_{\mathrm{dom}\,x} \\ x + (x' + x'') = (x + x') + x''.\end{gathered} \tag{1.7}$$

Here, 0_f means $0_{A,B}(f)$ if $f \in \hom(A,B)$.

Proof. For every $f : A' \longrightarrow A$, $g : B \longrightarrow B'$ and $x \in H\,(A,B)$, write

$$H\,(f,g)\,(x) = gxf$$

it is then clear that the set of conditions (1.5) asserts the functoriality of H; the set of conditions (1.6) asserts the naturality of dom, cod, 0 and +; the set of conditions (1.7) asserts the axioms for an internal category. Q.E.D.

On a category **C** with a 2-cell structure **H**, in general there is no horizontal composition (or pasting) for 2-cells, that being the case only when the pair $(\mathbf{C},\mathbf{H})$ is a 2-category.

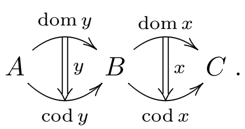
Proposition 3.4. A category **C** with a 2-cell structure

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)\,,$$

is a 2-category if and only if the *naturality condition*

$$\mathrm{cod}\,(x)\,y + x\,\mathrm{dom}\,(y) = x\mathrm{cod}\,(y) + \mathrm{dom}\,(x)\,y \tag{1.8}$$

holds for every triple of objects (A,B,C) in **C**, every $x \in H\,(B,C)$ and every $y \in H\,(A,B)$, as illustrated in the following diagram



Proof. If $(\mathbf{C},\mathbf{H})$ is a 2-category, the naturality condition follows from the horizontal composition of 2-cells and, conversely, given a 2-cell structure **H** over **C**, in order to make it a 2-category one has to define a horizontal composition which is given by

$$x \circ y = \mathrm{cod}\,(x)\,y + x\,\mathrm{dom}\,(y)$$

or

$$x \circ y = x\,\mathrm{cod}\,(y) + \mathrm{dom}\,(x)\,y,$$

provided the naturality condition is satisfied for every appropriate x and y. The middle interchange law follows from the naturality condition. Q.E.D.

4 Morphisms between 2-cell structures, naturally invertible 2-cell structures and cartesian squares

For a fixed category $\mathbf{C}$, we consider the category $\mathrm{Cell}(\mathbf{C})$ of all the possible 2-cell structures over $\mathbf{C}$. It has an initial object (the discrete 2-cell structure) and a terminal object (the codiscrete 2-cell structure). In many cases (Example 5.7, 5.9, 5.11, 5.13, etc.) a *canonical non-trivial* 2-cell structure is also present. One particular case of our interest is $\mathbf{C} = \mathsf{Cat}(\mathbf{B})$.

4.1 The category of 2-cell structures over a fixed category C

The category $\mathrm{Cell}(\mathbf{C})$ has as objects the 2-cell structures over $\mathbf{C}$. If

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

and

$$\mathbf{H}' = (H', \mathrm{dom}', \mathrm{cod}', 0', +')$$

are two 2-cell structures over $\mathbf{C}$, then a morphism $\varphi : \mathbf{H} \longrightarrow \mathbf{H}'$ is a natural transformation

$$\varphi : H \longrightarrow H'$$

such that

$$\begin{aligned} \mathrm{dom}' \varphi &= \mathrm{dom} \\ \mathrm{cod}' \varphi &= \mathrm{cod} \\ \varphi 0 &= 0' \\ \varphi + &= +' (\varphi \times \varphi) . \end{aligned}$$

The reason for describing $\mathrm{Cell}(\mathbf{C})$ is the study of pseudocategories internal to $\mathbf{C}$. As we will see in Section 6, a pseudocategory internal to $\mathbf{C}$ depends on the 2-cell structure which is considered over $\mathbf{C}$. For example, a pseudocategory in $\mathbf{C}$ with the codiscrete 2-cell structure is a precategory, while for $\mathbf{C}$ with the discrete 2-cell structure it is an internal category. It seems to be interesting to study of the variation of the notion of pseudocategory when the 2-cell structure over $\mathbf{C}$ varies. Another seemingly important aspect of $\mathrm{Cell}(\mathbf{C})$ is the following. Every morphism

$$\varphi : \mathbf{H} \longrightarrow \mathbf{H}' \tag{1.9}$$

in $\mathrm{Cell}(\mathbf{C})$ will induce a functor

$$PsCat\,(\mathbf{C}, \mathbf{H}) \longrightarrow PsCat\,(\mathbf{C}, \mathbf{H}') \tag{1.10}$$

from pseudocategories in $\mathbf{C}$, relative to the 2-cell structure $\mathbf{H}$, to pseudocategories in $\mathbf{C}$, relative to the 2-cell structure $\mathbf{H}'$. We reserve for a future work the study of equivalent 2-cell structures, by identifying $\mathbf{H}$ and $\mathbf{H}'$, via (1.9), whenever (1.10) is an equivalence of categories, relating it with homotopy theory.

The notion of a pseudocategory, as defined in [11] (see also [10, 12]), rests on the existence of some induced 2-cells between certain pullbacks. This means that those pullbacks have to share their universal property with morphisms and with 2-cells, thus the following definition.

4.2 Cartesian H-squares

We continue to assume that $\mathbf{C}$ is an arbitrary category and now consider a functor into Set, $H\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathsf{Set}$, for which we write $H(f,g)(x)$ as gxf.

Definition 4.1 (cartesian H-square)**.** A commutative square

$$\begin{array}{ccc} D & \xrightarrow{\pi_2} & C \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & B \end{array}$$

in $\mathbf{C}$ is said to be H-cartesian if for every object Z in $\mathbf{C}$, for every $x \in H(Z,A)$ and $y \in H(Z,C)$ with $fx = gy$, there is a unique element $w \in H(Z,D)$ such that $\pi_1 w = x$ and $\pi_2 w = y$. In that case we write w as $\langle x, y\rangle$.

An immediate observation is that, when $H = \hom_{\mathbf{C}}$, a square is H-cartesian if and only if it is a pullback square. It is also not difficult to observe that, when $H(D,-)$ preserves pullbacks for every object D in $\mathbf{C}$, then every pullback square is H-cartesian. For that reason we will be interested in considering 2-cell structures for which the functor $H(D,-) : \mathbf{C} \longrightarrow \mathsf{Set}$ preserves pullbacks for every object D in $\mathbf{C}$. This means that the functor

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathsf{Set},$$

giving a 2-cell structure to a category $\mathbf{C}$, has the property that

$$H(D, A \times_C B) \overset{\varphi}{\cong} \{(x,y) \in H(D,A) \times H(D,B) \mid fx = gy\}$$

for every object D in $\mathbf{C}$ and pullback square

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{g} & C, \end{array}$$

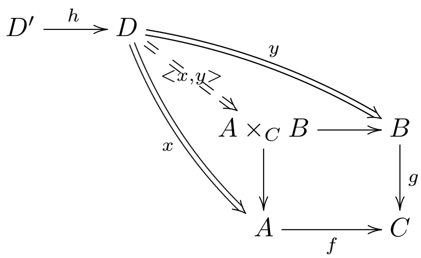
with φ a natural isomorphism, that is, for every $h : D \longrightarrow D'$, the following square commutes

$$\begin{array}{ccc} H(D, A \times_C B) & \xrightarrow{\cong\varphi} & \{(x,y) \mid fx = gy\} \\ {\scriptstyle H(h,1)}\downarrow & & \downarrow \\ H(D', A \times_C B) & \xrightarrow{\cong\varphi} & \{(x',y') \mid fx' = gy'\} \end{array}$$

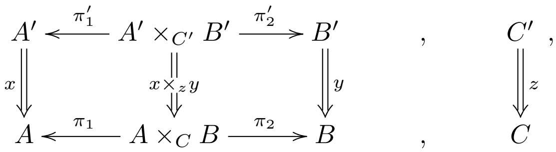
or, in other words, that

$$\langle x, y\rangle h = \langle xh, yh\rangle$$

as displayed.



In particular, for $D = A' \times_{C'} B'$, and appropriate x, y, z as in



the element $x \times_z y$ appears as the unique 2-cell in $H(A' \times_{C'} B', A \times_C B)$ such that

$$\begin{aligned} \pi_2 (x \times_z y) &= y\pi_2' \\ \pi_1 (x \times_z y) &= x\pi_1'. \end{aligned}$$

These observations will be used in Section 6 to define a pseudocategory internal to a category with a 2-cell structure.

4.3 Natural and invertible 2-cells on a 2-cell structure

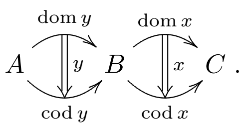
Recall from Proposition 3.4 that a category $\mathbf{C}$ with a 2-cell structure

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +),$$

is a 2-category if and only if the naturality condition (1.8)

$$\mathrm{cod}(x)\, y + x\, \mathrm{dom}(y) = x\mathrm{cod}(y) + \mathrm{dom}(x)\, y$$

holds for every triple of objects (A, B, C) in $\mathbf{C}$ and for every $x \in H(B, C)$, $y \in H(A, B)$ as displayed



It may happen that the naturality condition does not hold for all the possible x and y, but only for a few (as it is the case in Examples 5.6, 5.7, 5.8). Thus the following definitions sprung.

Let $\mathbf{C}$ be a category and $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$ a 2-cell structure over it.

Definition 4.2. A 2-cell $y \in H(A,B)$ is said to be *natural with respect to* a 2-cell $z \in H(X,A)$ if

$$\operatorname{cod}(y)\, z + y \operatorname{dom}(z) = y \operatorname{cod}(z) + \operatorname{dom}(y)\, z.$$

This means that the usual horizontal composition (or pasting) $y \circ z$ is well defined and it is given by each one of the formulas involved in the naturality condition.

Definition 4.3. A 2-cell $x \in H(A,B)$ is natural when

$$\operatorname{cod}(x)\, y + x \operatorname{dom}(y) = x \operatorname{cod}(y) + \operatorname{dom}(x)\, y \tag{1.11}$$

for every object X in $\mathbf{C}$ and for every element $y \in H(X,A)$.

In other words, a 2-cell x is natural when it is natural with respect to all the possible 2-cells that are horizontally composable with it.

Definition 4.4. A 2-cell $x \in H(A,B)$ is invertible when there is a (necessarily unique) element

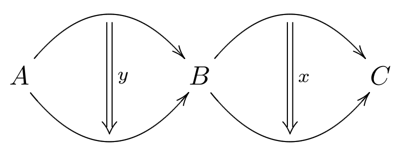
$$-x \in H(A,B)$$

such that $\operatorname{dom}(x) = \operatorname{cod}(-x)$, $\operatorname{cod}(x) = \operatorname{dom}(-x)$ and

$$x + (-x) = 0_{\operatorname{cod}(x)} \ , \ (-x) + x = 0_{\operatorname{dom}(x)}.$$

In the case when both the 2-cells are invertible, the notion of a natural 2-cell with respect to another 2-cell can be conveniently translated to the notion of a binary commutator, which is borrowed from the usual notion of commutator, in the sense of Group Theory.

Definition 4.5. Let $x \in H(B,C)$ and $y \in H(A,B)$ be two invertible 2-cells in a 2-cell structure. The commutator 2-cell of x and y, denoted by $[x,y] \in H(A,C)$



is given by the formula

$$\begin{aligned} [x,y] &= (c_1 + d_2 - d_1 - c_2)(x,y) \\ &= c_1(x,y) + d_2(x,y) - d_1(x,y) - c_2(x,y) \end{aligned}$$

with

$$\begin{aligned} c_1(x,y) &= \operatorname{cod}(x)\, y \ , \ c_2(x,y) = x \operatorname{cod}(y) \\ d_1(x,y) &= \operatorname{dom}(x)\, y \ , \ d_2(x,y) = x \operatorname{dom}(y). \end{aligned}$$

The commutator of x and y vanishes,

$$[x, y] = 0_{\mathrm{cod}(x)\,\mathrm{cod}(y)},$$

if and only if x is natural with respect to y.

We may thus consider the full subcategories of Cell($\mathbf{C}$) by taking the 2-cell structures where all the 2-cells are natural, or invertible, or both, denoted, respectively, by NatCell($\mathbf{C}$), InvCell($\mathbf{C}$) and NatInvCell($\mathbf{C}$).

In the simplest case, when $\mathbf{C}$=$\mathbf{1}$, considered in Example 5.1, we have that Cell($\mathbf{C}$) is the category of monoids, NatCell($\mathbf{C}$) the category of commutative monoids, InvCell($\mathbf{C}$) the category of groups, and NatInvCell($\mathbf{C}$) the category of abelian groups.

This observation suggests the study of a generalization for the well known reflection

$$\mathsf{Grp} \xrightarrow{I} \mathsf{Ab},$$

namely

$$I\colon \mathrm{Cell}(\mathbf{C}) \to \mathrm{NatCell}(\mathbf{C})$$

from the category of 2-cell structures over $\mathbf{C}$ into the subcategory of natural 2-cell structures over $\mathbf{C}$, associating to each 2-cell structure its *naturalization*. This study is postponed for a future work. A simple observation may however be given in the case of $\mathbf{C}$ being an Ab-category (see [10],[12] and Example 5.12). In this case the notion of commutator reduces to

$$[x, y] = D(x)\,y - xD(y)\,.$$

Another interesting particular case where we can see how the notions of naturality are related with the chosen 2-cell structure is when $\mathbf{C}$ is of the form $\mathsf{Cat}(\mathbf{B})$ for some category $\mathbf{B}$. This will be analysed in the Example 5.7. The entire following section is dedicated to examples which will be used further on.

5 Examples

In order to illustrate the notion of a category $\mathbf{C}$ with a 2-cell structure $\mathbf{H}$, we consider some special examples of categories on which we can define certain kinds of 2-cell structures.

5.1 The terminal category

In this case $\mathbf{C} = \mathbf{1}$ and $\hom(1,1)$ is a singleton, which means that dom and cod are uniquely determined constant maps. Hence $H = H(1,1)$ is just a set and $\mathbf{H} = (H, 0, +)$ is simply a monoid. Indeed all the conditions (1.6) are trivial, the conditions (1.5) force the actions to be trivial, while the last two conditions (1.7) ensure that $(H, 0, +)$ is a monoid.

5.2 A discrete category

In this case $\hom(A, B)$ is either empty or a singleton, respectively if $A \neq B$ or $A = B$. This means that a 2-cell structure on $\mathbf{C}$ is a collection of monoids $\mathbf{H}_A = (H, 0, +)_A$, one for each object A in $\mathbf{C}$.

5.3 A codiscrete category

If $\mathbf{C}$ is a codiscrete category, that is, $\hom(A,B)$ is a singleton for each pair of objects, then to give a 2-cell structure is to specify, for each pair of objects (A,B) in $\mathbf{C}$, a monoid $(H(A,B),0,+)$ and, for every four-tuple of objects (A,B,C,D) in $\mathbf{C}$, a homomorphism of monoids

$$\mu(A,B,C,D)\colon H(B,C)\to H(A,D)$$

such that, for every A',A,B,C,D,D' objects in $\mathbf{C}$,

$$\mu(A',B,C,D')=\mu(A',A,D,D')\circ\mu(A,B,C,D)$$

and

$$\mu(A,A,B,B)=1_{H(A,B)}.$$

In other words, we have to specify a functor $H\colon \mathbf{C}^{op}\times\mathbf{C}\to\mathsf{Mon}$ into the category Mon of monoids.

5.4 A preorder considered as a category

When $\mathbf{C}$ is a preorder, considered as a category, similarly to the previous example, to specify a 2-cell structure over it is to give, for every two objects A,B in $\mathbf{C}$ with $A\leq B$, a monoid $(H(A,B),0,+)$ and, for every four objects A,B,C,D in $\mathbf{C}$ with $A\leq B\leq C\leq D$, a homomorphism of monoids

$$\mu(A,B,C,D)\colon H(B,C)\to H(A,D)$$

such that, for every $A'\leq A\leq B\leq C\leq D\leq D'$,

$$\mu(A',B,C,D')=\mu(A',A,D,D')\circ\mu(A,B,C,D)$$

and

$$\mu(A,A,B,B)=1_{H(A,B)}.$$

In other words it is a functor $H\colon \mathbf{C}^{op}\times\mathbf{C}\to\mathsf{Mon}$.

5.5 A monoid considered as a one object category

When $\mathbf{C}=(M,1,\cdot)$ is a monoid, considered as a one object category, the description given in Proposition 3.3, for a 2-cell structure over $\mathbf{C}$, cannot be simplified much further. Nevertheless, we may consider some special classes of 2-cell structures that may be defined over $\mathbf{C}$.

Every monoid $(H,0,+)$, on which M acts on the left and on the right, together with a map

$$D\colon H\times M\to M$$

such that

$$\begin{aligned} D(0,f) &= f\\ D(x'+x,f) &= D(x',D(x,f))\\ gD(x,f)h &= D(gxf,gfh)\end{aligned}$$

for every $x,x'\in H$ and $f,g,h\in M$, induces a 2-cell structure over $\mathbf{C}$. Diagram (1.4) is obtained as

$$H\times H\times M \xrightarrow{\;m\;} H\times M \underset{D}{\overset{\pi_M}{\rightrightarrows}} M \quad (\langle 0,1\rangle\colon M\to H\times M)$$

with $m(x,y,f)=(x+y,f)$; in other words m is the morphism $+\times 1_M$. The needed action of M on $H\times M$ is given by the action of M on H considered by hypothesis and by the monoid operation in M.

5.6 A group considered as a one object groupoid

In the same fashion of the previous example, here we have $\mathbf{C}=(G,1,\cdot)$, which is now a group, and $(H,0,+)$, another group, on which G acts on the left and on the right, together with a map $D\colon H\times G\to G$ satisfying the three conditions above. This induces, as before, a 2-cell structure over $\mathbf{C}$. This example is important because here all the 2-cells are invertible (Definition 4.4).

The notion of commutator (Definition 4.5) for appropriate 2-cells, which in this case is always defined, is given by the formula

$$[(x,f),(y,g)]=D\,(x,f)\,y+xg-fy-xD\,(y,g)\,.$$

The commutator vanishes if and only if the 2-cells x and y can be composed horizontally (or pasted together).

5.7 A category of the form $\mathsf{Cat}(\mathbf{B})$ for some category $\mathbf{B}$

When $\mathbf{C}$ is a category of internal categories in some category $\mathbf{B}$, we can always consider the 2-cell structure given by the internal transformations in the sense of Section 2.

If $A=(A_0,A_1,A_2,d,c,e,m,\pi_1,\pi_2)$ and $B=(B_0,B_1,B_2,d',c',e',m',\pi_1',\pi_2')$ are two internal categories in $\mathbf{B}$, then we define

$$H\,(A,B)\quad=\quad\{(t_1,t_2,t_3)\mid t_1\colon A_0\to B_1,t_2,t_3\colon A_1\to B_2,\pi_1't_2=t_1c,\pi_2't_3=t_1d\}$$

and

$$\begin{aligned}\mathrm{dom}(A,B)\,(t)\quad&=\quad(d't_1,\pi_2't_2)\\ \mathrm{cod}(A,B)\,(t)\quad&=\quad(c't_1,\pi_1't_3).\end{aligned}$$

If $f=(f_0,f_1)\colon A'\to A$ and $g=(g_0,g_1)\colon B\to B'$ are internal functors, we write g_2 to abbreviate $g_1\times_{g_0}g_1$ and then define

$$H\,(f,g)\,(t)=H((f_0,f_1),(g_0,g_1))\,(t_1,t_2,t_3)=(g_1t_1f_0,g_2t_2f_1,g_2t_3f_1)\,.$$

To say what are the identity 2-cells and how the vertical composition is defined, we first consider the set $H(A,B)$ as the set

$$L(A,B)=\{(t_1,h_1,k_1)\mid t_1\colon A_0\to B_1,h_1,k_1\colon A_1\to B_1,(h_1,d't_1),(k_1,c't_1)\in\hom(A,B)\},$$

which is in bijection with $H(A,B)$ as follows:

$$\begin{aligned}(t_1,t_2,t_3)\quad&\mapsto\quad(t_1,\pi_2't_2,\pi_1't_3),\\ (t_1,h_1,k_1)\quad&\mapsto\quad(t_1,\langle t_1c,h_1\rangle,\langle k_1,t_1d\rangle).\end{aligned}$$

So if $h = (h_0, h_1)\colon A \to B$ is an internal functor, given any $t, s \in H(A,B) \cong L(A,B)$ such that $\mathrm{dom}(t) = h = \mathrm{cod}(s)$, we define

$$0_h \cong (e' h_0, h, h) \tag{1.12}$$
$$t +_h s \cong (m\langle t_1, s_1\rangle, \pi_2' s_2, \pi_1' t_3). \tag{1.13}$$

We are now going to see that, in this context, every internal natural transformation is a natural 2-cell (in the sense of Definition 4.3).

Before stating the result we refer to Section 2 observing that, for every internal category $A = (A_0, A_1, d, c, e, m)$, there exists an internal category

$$A^{\to} = (A_1, A_1, 1, 1, 1, 1)\,,$$

which is in some sense the category of arrows of A, together with two internal functors

$$d^{\to} = (ed, d) : A^{\to} \longrightarrow A$$

and

$$c^{\to} = (ec, c) : A^{\to} \longrightarrow A,$$

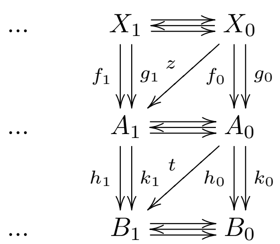
of which we may think of as the domain and codomain functors.

We are now going to prove that an internal transformation is natural if and only if it is natural with respect to the *arrow* transformation, and then, as a consequence, we can show that every internal natural transformation is a natural 2-cell.

Proposition 5.1. Let $\mathbf{B}$ be a category and take $\mathbf{C} = \mathsf{Cat}(\mathbf{B})$ with its 2-cell structure of internal transformations. A 2-cell $t \in H(A, B)$ is natural if and only if it is natural with respect to the 2-cell

$$(1_{A_1}, ed, ec) \in L(A^{\to}, A) \cong H(A^{\to}, A)\,.$$

Proof. Let us first observe what it means for $\mathbf{t} = (t, h_1, k_1) \in L(A,B) \cong H(A,B)$ to be natural with respect to an arbitrary appropriate 2-cell $\mathbf{z} = (z, f_1, g_1) \in L(X,A) \cong H(X,A)$:



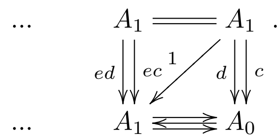
by definition of the relation of horizontal composition (see Definition 4.2), through some calculations we have

$$\begin{aligned} \mathbf{t} \circ \mathbf{z} \;&\Leftrightarrow\; (k_1 z, k_1 f_1, k_1 g_1) + (t f_0, h_1 f_1, k_1 f_1) = (t g_0, h_1 g_1, k_1 g_1) + (h_1 z, h_1 f_1, h_1 g_1) \\ &\Leftrightarrow\; (m\,\langle k_1 z, t f_0\rangle\,, h_1 f_1, k_1 g_1) = (m\,\langle t g_0, h_1 z\rangle\,, h_1 f_1, k_1 g_1) \\ &\Leftrightarrow\; m\,\langle k_1 z, t f_0\rangle = m\,\langle t g_0, h_1 z\rangle \end{aligned} \tag{1.14}$$

and, also by definition, t is an internal natural transformation when

$$m\langle k_1, td\rangle = m\langle tc, h_1\rangle \tag{1.15}$$

which is equivalent to saying that (t, h_1, k_1) is natural relative to $(1_{A_1}, ed, ec)$, as displayed below



Q.E.D.

Corollary 5.2. Every internal natural transformation in $\mathbf{C} = \mathsf{Cat}(\mathbf{B})$ is a natural 2-cell.

Proof. Simply observe that

$$(1.15) \Longrightarrow (1.14)$$

since

$$\begin{aligned} m\langle k_1, td\rangle z &= m\langle tc, h_1\rangle z \\ m\langle k_1 z, tdz\rangle &= m\langle tcz, h_1 z\rangle \\ m\langle k_1 z, tf_0\rangle &= m\langle tg_0, h_1 z\rangle . \end{aligned}$$

Q.E.D.

In this case there is a simple criterion which detects whether a 2-cell is natural or not without the need of comparing it with all the possible 2-cells. This seems to be an intrinsic phenomenon for this particular 2-cell structure in this type of category.

5.8 Abstract 2-cells, and conjugations

This example is motivated by Examples 5.5, 5.6 and 5.9 (see below).

Let $\mathbf{C}$ be a category and consider

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathsf{Mon}$$

a functor into Mon, together with a natural transformation

$$D : UH \times \hom_{\mathbf{C}} \longrightarrow \hom_{\mathbf{C}}$$

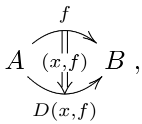
(with $U : \mathsf{Mon} \longrightarrow \mathsf{Set}$ denoting the usual forgetful functor) satisfying

$$\begin{aligned} D(0, f) &= f \\ D(x' + x, f) &= D(x', D(x, f)) \end{aligned}$$

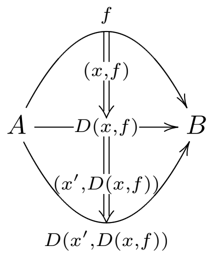
for all $f : A \longrightarrow B$ in $\mathbf{C}$ and $x', x \in H(A, B)$, where 0 denotes the zero element in the monoid $H(A, B)$, which is written in additive notation although it is not necessarily commutative.
With these data, we can define a 2-cell structure over $\mathbf{C}$:

$$UH \times UH \times \hom_{\mathbf{C}} \xrightarrow{+\times 1} UH \times \hom_{\mathbf{C}} \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{\langle 0,1\rangle} \\ \xrightarrow[D]{} \end{array} \hom_{\mathbf{C}}$$

specifically, the 2-cells are displayed as

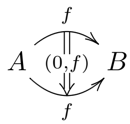


with vertical composition given as illustrated

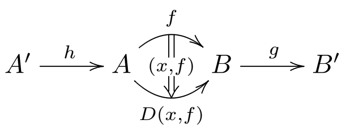


$$(x', D(x,f)) + (x,f) = (x'+x, f)\,, \tag{1.16}$$

which is well defined because $D(x'+x,f) = D(x', D(x,f))$. The identity 2-cells are of the form



which are well defined because $D(0,f) = f$. The left and right actions of the morphisms in the 2-cells,



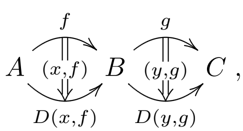
are given by the formulas

$$g(x,f)h = (gxh, gfh) = (H(h,g)(x), gfh)\,.$$

If in addition

$$D(y,g)x + yf = yD(x,f) + gx, \tag{1.17}$$

for all x, y, f, g as pictured



then the 2-cell structure is natural and the result is a 2-category. Observe that condition (1.17) is equivalent to

$$D(y,g)(x,f) + (y,g)f = (y,g)D(x,f) + g(x,f)$$

via the definition (1.16) of $+$ and the properties identified in Proposition 3.3.

5.9 The category of groups and group homomorphisms

When $\mathbf{C}$ is the category Grp of groups and group homomorphisms, we may consider the canonical 2-cells obtained by considering each group as a one object groupoid and each group homomorphism as a functor. In that case, as it is well known, for homomorphisms $f, g : A \to B$, a 2-cell

$$t : f \longrightarrow g$$

is an element $t \in B$ such that

$$tf(x) = g(x)t \quad , \quad \text{for all } x \in A.$$

Since, for given t and f, the homomorphism g is uniquely determined as

$$g(x) = tf(x)t^{-1} = {}^{t}f(x),$$

this particular 2-cell structure over Grp is just an instance of Example 5.8 with Grp instead of Mon.

To see it we consider the functor H that projects the second argument

$$H : \mathsf{Grp}^{op} \times \mathsf{Grp} \longrightarrow \mathsf{Grp}$$
$$(A,B) \longmapsto B$$

together with

$$D : B \times \hom(A,B) \longrightarrow \hom(A,B).$$
$$(t,f) \longmapsto {}^{t}f$$

It is a straightforward calculation to check that

$$\begin{aligned} D(0,f) &= f \\ D(t+t',f) &= D(t, D(t',f)). \end{aligned}$$

Moreover, since condition (1.17) is satisfied, the 2-cell structure is natural.

5.10 Abstract 2-cells, and derivations

This example is motivated by the example of crossed modules with derivations, as described in 5.11.

Here $\mathbf{C}$ has to be a category for which the functor

$$\hom_{\mathbf{C}} : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathsf{Set}$$

can be extended into Mon, that is, there is a functor (denoted by map — we are thinking of the underlying map of a homomorphism)

$$\mathbf{C}^{op} \times \mathbf{C} \xrightarrow{\text{map}} \mathsf{Mon} \xrightarrow{U} \mathsf{Set}$$

with $\hom \subseteq U\,\mathrm{map}$, in the sense that $\hom(A,B) \subseteq U(\mathrm{map}(A,B))$, naturally for every $A, B \in \mathbf{C}$.

When this is the case, any functor

$$K : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathsf{Mon}$$

together with a natural transformation

$$D : K \longrightarrow \mathrm{map},$$

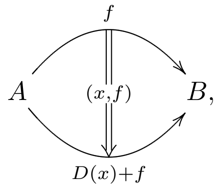
determines a 2-cell structure over $\mathbf{C}$ as follows. The functor $H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathsf{Set}$ is given by

$$H(A,B) = \{(x,f) \in K(A,B) \times \hom(A,B) \mid D(x) + f \in \hom(A,B)\}$$

and

$$H(h,g)(x,f) = (K(h,g)(x), gfh) = (gxh, gfh),$$

while $\mathrm{dom}, \mathrm{cod}, 0, +$, are given as illustrated



with $(x,f) \in H(A,B)$; the formula for the vertical composition is

$$(x', D(x) + f) + (x,f) = (x' + x, f);$$

the identity cells are of the form

$$(0,f);$$

while the left and right actions are computed as

$$g(x,f)h = (gxh, gfh).$$

We can represent the 2-cell structure via the diagram

$$L \xrightarrow{\;+\;} H \;\overset{\pi_2}{\underset{D+1}{\substack{\longrightarrow \\ \xleftarrow{\langle 0,1\rangle} \\ \longrightarrow}}}\; \hom_{\mathbf{C}}$$

with $L(A,B) = \{(x', x, f) \in K(A,B)^2 \times \hom(A,B) \mid D(x' + x) + f \in \hom(A,B)\}$. Observe that the functor H is well defined because

$$D(gxh) + gfh = gD(x)h + gfh = g(D(x) + f)h$$

is a morphism in $\mathbf{C}$ whenever $D(x) + f$ is in $\hom(A,B)$, and similarly for L.

If, in addition, the property

$$D(y)x + gx + yf = yD(x) + yf + gx \tag{1.18}$$

is satisfied, for all $(x,f) \in H(A,B)$ and $(y,g) \in H(A,C)$, then the resulting structure is a 2-category (compare with Equation (1.17)).

5.11 The case of crossed modules

In the case $\mathbf{C}$ is the category X-Mod of crossed modules of groups, we have the canonical 2-cell structure given by derivations, which is an instance of the above construction with Grp instead of Mon:

The objects in X-Mod are of the form

$$A=\left(X\xrightarrow{d}B,\varphi:B\longrightarrow Aut\left(X\right)\right)$$

in which $d:X\longrightarrow B$ is a group homomorphism, φ is a group action of B on X, denoted by $\varphi(b)(x)=b\cdot x$, and satisfying

$$\begin{aligned}d\left(b\cdot x\right)&=bd\left(x\right)b^{-1}\\ d\left(x\right)\cdot x'&=x+x'-x.\end{aligned}$$

A morphism $f:A\longrightarrow A'$ in X-Mod is of the form

$$f=\left(f_1,f_0\right)$$

with $f_1:X\longrightarrow X'$ and $f_0:B\longrightarrow B'$ group homomorphisms such that

$$f_0d=d'f_1$$

and

$$f_1\left(b\cdot x\right)=f_0\left(b\right)\cdot f_1\left(x\right).$$

Clearly there are functors

$$\text{map}:\mathbf{C}^{op}\times\mathbf{C}\longrightarrow\mathsf{Grp}$$

sending (A,A') to the group of pairs (f_1,f_0) of maps (not necessarily homomorphisms) $f_1:UX\longrightarrow UX'$ and $f_0:UB\longrightarrow UB'$ such that

$$f_0d=d'f_1,$$

with the group operation defined componentwise

$$\left(f_1,f_0\right)+\left(g_1,g_0\right)=\left(f_1+g_1,f_0+g_0\right).$$

Moreover, there is a functor

$$M:\mathbf{C}^{op}\times\mathbf{C}\longrightarrow\mathsf{Grp}$$

sending (A,A') to the group $M\left(A,A'\right)=\{t\mid t:UB\longrightarrow UX'\text{ is a map}\}$, and a natural transformation

$$D:M\longrightarrow\text{map}$$

defined by

$$D\left(A,A'\right)\left(t\right)=\left(td,dt\right).$$

With this data we define $H\left(A,A'\right)$ as

$$\left\{\left(t,f\right)\mid t\in M\left(A,A'\right)\ ,\ f=\left(f_1,f_0\right):A\longrightarrow A'\ ,\ \left(td+f_1,dt+f_0\right)\in\hom\left(A,A'\right)\right\}.$$

The condition $(td + f_1, dt + f_0) \in \hom(A, A')$ asserts that the pair $(td + f_1, dt + f_0)$ is a morphism of crossed modules

$$\begin{array}{ccc} X & \xrightarrow{d} & B \\ {\scriptstyle td+f_1}\downarrow & & \downarrow{\scriptstyle dt+f_0} \\ X' & \xrightarrow{d} & B' \end{array} \tag{1.19}$$

and it is equivalent to

- $dt + f_0$ is a homomorphism of groups
$$dt(bb') = d(t(b) + f_0(b) \cdot t(b'))$$
- $td + f_1$ is a homomorphism of groups
$$t(d(x)\,d(x')) = t(dx) + f_0 d(x) \cdot td(x')$$
- the square (1.19) commutes, which is trivial because $(f_1, f_0) \in \hom(A, A')$
- $(td + f_1)$ preserves the action of $(dt + f_0)$
$$t\left(bd(x)\,b^{-1}\right) = t(b) + f_0(b) \cdot t(d(x)) + f_0\left(bd(x)\,b^{-1}\right) \cdot (-t(b)).$$

This same condition implies that t is a derivation, which means that

$$t(bb') = t(b) + f_0(b) \cdot t(b') \quad , \quad \text{for all } b, b' \in B.$$

Thus, this is an instance of the example presented in 5.10. This particular example is further explored in [16] where the description of pseudocategories given in Theorem 6.2 is detailed.

5.12 Abstract 2-cells and homotopies

A particular (but important) case, which is obtained from the more general example presented in 5.10, is when **C** is an Ab-category. A 2-Ab-category, as defined in [10] and [12], is obtained in this way. The functor hom coincides with map and moreover it is a functor into the category Ab of abelian groups.

$$\hom : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathsf{Ab}$$

This means that giving a 2-cell structure on **C** is to give a functor

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathsf{Ab},$$

which is usually required to be an Ab-functor, together with a natural transformation $D : H \longrightarrow \hom$. The 2-cell structure thus obtained makes **C** into a 2-category (in fact a 2-Ab-category) if and only if the condition $D(y)\,x = yD(x)$ is satisfied for every appropriate x and y. This condition is just how the condition (1.18) simplifies in this context, see [10] and [12] for more details. Furthermore, as proved in [10], every 2-cell structure (if enriched in Ab) is obtained in this way.

Many of these considerations are still valid for an arbitrary monoidal category **V** instead of Ab, but in general not all 2-cell structures are obtained in this way.

5.13 The case of Abelian Chain Complexes

We now consider an example of a category with a *canonical* 2-cell structure in which not every 2-cell is a natural 2-cell (see Definition 4.3).

The example of abelian chain complexes (say of order 2 to simplify notation) is self explanatory (see also [1, 2] and references there). We have objects, morphisms and 2-cells (homotopies) as displayed

$$\begin{array}{ccccc} A_2 & \xrightarrow{d} & A_1 & \xrightarrow{d} & A_0 \\ f_2 \downarrow\downarrow g_2 & \overset{t_2}{\swarrow} & f_1 \downarrow\downarrow g_1 & \overset{t_1}{\swarrow} & f_0 \downarrow\downarrow g_0 \\ A'_2 & \xrightarrow{d} & A'_1 & \xrightarrow{d} & A'_0 \end{array} \tag{1.20}$$

with

$$\begin{aligned} dt_1 &= g_0 - f_0 \\ t_1 d + dt_2 &= g_1 - f_1 \\ t_2 d &= g_2 - f_2 \end{aligned}$$

or equivalently

$$\begin{aligned} g_0 &= dt_1 + f_0 \\ g_1 &= t_1 d + dt_2 + f_1 \\ g_2 &= t_2 d + f_2 \end{aligned}$$

and hence we have, for $\mathbf{C}$ =2-Ch(Ab), the functor

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathsf{Ab}$$

sending the pair of objects (A, A') to the abelian group of pairs (t_2, t_1); and the natural transformation

$$D : H \longrightarrow \hom$$

sending a pair (t_2, t_1) as above to the triple $(t_2 d, t_1 d + dt_2, dt_1)$ displayed as follows

$$\begin{array}{ccccc} A_2 & \xrightarrow{d} & A_1 & \xrightarrow{d} & A_2 \\ {\scriptstyle t_2 d}\downarrow & & \downarrow{\scriptstyle t_1 d + dt_2} & & \downarrow{\scriptstyle dt_1} \\ A'_2 & \xrightarrow{d} & A'_1 & \xrightarrow{d} & A'_0. \end{array}$$

This is clearly an instance of the construction used in Examples 5.10 and 5.12; however condition

$$D(x)y = xD(y)$$

is not always satisfied, since it becomes, for $x = (t_2, t_1)$ and $y = (s_2, s_1)$

$$(t_2 ds_2, dt_2 s_1 + t_1 ds_1) = (t_2 ds_2 + t_2 s_1 d, t_1 ds_1)$$

which holds if $t_2 s_1 = 0$, but not in general. Note that, as for homotopies, the formula gxh is defined as follows

$$(g_2, g_1, g_0)(t_2, t_1)(h_2, h_1, h_0) = (g_2 t_2 f_1, g_1 t_1 f_0).$$

The commutator (see Definition 4.5) in this case is given by the formula

$$[x, y] = (-t_2 s_1 d, d t_2 s_1)\,,$$

whenever it is defined.

5.14 A slightly more general example

This is still a slight generalization of Example 5.10.

Suppose $\mathbf{A}$ is a category admitting a *forgetful* functor $U : \mathbf{A} \longrightarrow \mathsf{Set}$, and let us assume the existence of a functor

$$\mathrm{map} : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathbf{A}$$

with a natural inclusion

$$\hom_C (A, B) \subseteq U(\mathrm{map}\,(A, B)) \tag{1.21}$$

as detailed in 5.10 but with a generic $\mathbf{A}$ instead of Mon.

Once having such data we may be interested in considering the 2-cell structures over $\mathbf{C}$ that are *loosely enriched* in $\mathbf{A}$ in the same way as $\mathbf{C}$ is. To do that we consider an internal category in $\mathbf{A}^{\mathbf{C}^{op} \times \mathbf{C}}$, say

$$M \times_{\mathrm{map}} M \xrightarrow{+} M \;\begin{array}{c}\xrightarrow{\mathrm{dom}}\\ \xleftarrow{0}\\ \xrightarrow[\mathrm{cod}]{}\end{array}\; \mathrm{map}, \tag{1.22}$$

and construct a 2-cell structure on $\mathbf{C}$ as follows:

$$H\,(A, B) = \{x \in UM\,(A, B) \mid U(\mathrm{dom}_{A,B})(x), U(\mathrm{cod}_{A,B})(x) \in \hom\,(A, B)\}$$

the (left and right) action gxh of the morphism $g \colon B \to B'$ and the morphism $h \colon A' \to A$ on the element $x \in H(A, B)$ is defined by the formula $UM(h, g)(x)$. The functor H, which on morphisms is given by $H(h, g)(x) = gxh$, is well defined because (1.21) is required to be a natural inclusion. In other words, the 2-cell structure thus constructed is obtained as a restriction of (1.22), after applying U, as illustrated in the following diagram

$$\begin{array}{ccccc}
UM \times_{U\,\mathrm{map}} UM & \xrightarrow{+} & UM & \begin{array}{c}\xrightarrow{\mathrm{dom}}\\ \xleftarrow{0}\\ \xrightarrow[\mathrm{cod}]{}\end{array} & U\,\mathrm{map} \\
\uparrow \subseteq & & \uparrow \subseteq & & \uparrow \subseteq \\
L & \longrightarrow & H & \begin{array}{c}\xrightarrow{d}\\ \longleftarrow\\ \xrightarrow[c]{}\end{array} & \hom
\end{array}$$

with $L(A, B) = \{(x, y) \in UM(A, B)^2 \mid U(\mathrm{dom}_{A,B})(x) = U(\mathrm{cod}_{A,B})(y)\}$.

It is now interesting to observe that the case $\mathbf{A} = \mathsf{Mon}$ is precisely the construction of Example 5.10. Another interesting case is when $\mathbf{A} = \mathsf{Grp}$. If $\mathbf{A} = \mathsf{Ab}$ and if we also ask that M has to be an Ab-functor, then the result is a 2-Ab-category whenever the condition

$$D\,(x)\,y = xD\,(y)\,,$$

is satisfied for all appropriate x and y (see also Example 5.12). Here the natural transformation D is obtained as $c \circ \ker(d)$ with c and d as in the diagram above (this is similar to the procedure that gives the categorical equivalence between morphisms and internal categories in abelian groups).

When $\mathbf{A}$ is a monoidal category and $\mathbf{C}$ a category enriched in $\mathbf{A}$ then we can always find the functor map in the conditions above by putting map = hom.

5.15 Topological Abelian Groups

When $\mathbf{C} = \mathsf{Ab}(\mathsf{Top})$, is the category of topological abelian groups, we can always consider the 2-cell structure given as follows: for every topological abelian groups X and Y, $H(X, Y)$ is the quotient set

$$\{\alpha : I \times X \longrightarrow Y \mid \alpha \text{ is continuous, } \alpha(0, \_) = 0 \text{ , } \alpha(t, \_) \text{ is a homomorphism}\} / \sim$$

in which I denotes the unit interval and the equivalence $\sim$ identifies

$$\alpha \sim \beta$$

if and only if:

1. $\alpha(1, \_) = \beta(1, \_)$, which is referred to as h in item 2. just below,
2. there exists $\Phi : I \times I \times X \longrightarrow Y$, continuous and such that
$$\begin{aligned} \Phi(0, \_, \_) &= \alpha \\ \Phi(1, \_, \_) &= \beta \\ \Phi(s, 0, \_) &= 0 \\ \Phi(s, 1, \_) &= h \end{aligned}$$
3. and $\Phi(s, t, \_)$ is a homomorphism.

The natural transformation $D : H \longrightarrow \text{hom}$ is given by

$$D([\alpha]) = \alpha(1, \_)$$

with

$$(g[\alpha]f)(t, x) = g(\alpha(t, f(x))) .$$

It is clear that the condition

$$[\alpha]D([\beta]) = D([\alpha])[\beta]$$

holds because

$$\alpha D(\beta) \sim D(\alpha)\beta \Leftrightarrow \alpha(t, \beta(1, x)) \sim \alpha(1, \beta(t, x))$$

and there exists

$$\Phi(s, t, x) = \alpha\left(t^{(1-s)}, \beta(t^s, x)\right) .$$

This means that we may use the previous information to build on $\mathsf{Ab}(\mathsf{Top})$ a 2-cell structure, as in Example 5.14, knowing that the result is a 2-category.

Finally we observe that the notion of a category with a 2-cell structure, besides giving a simple characterization of a 2-category as

"2-category" = "sesquicategory" + "naturality condition",

it also gives a useful tool in generating examples in arbitrary situations.

6 Pseudocategories

The original notion of internal pseudocategory considered in [11] was only defined internal to a 2-category. Here we extend it to the more general context of a category with a 2-cell structure (or sesquicategory).

In any category $\mathbf{C}$ it is always possible to consider two trivial 2-cell structures (see Definition 3.1), namely the discrete one, which is obtained when $H = \hom$ and $\mathrm{dom}, \mathrm{cod}, 0, +$ are all identities, and the codiscrete one, which is obtained when $H = \hom \times \hom$, dom is the second projection, cod is the first projection, 0 is the diagonal and + is given by $(f, g, h) \mapsto (f, h)$. As already observed, in the former case a pseudocategory is an internal category to $\mathbf{C}$, while in the later it is simply a precategory in $\mathbf{C}$.

When $\mathbf{C} = \mathsf{Cat}(\mathsf{Set})$, and choosing for the 2-cell structure the natural transformations, then a pseudocategory is a pseudo-double-category in the sense of Grandis and Paré (see [4, 5]), which is at the same time a generalization of a double-category and a bicategory.

At this level of generality, it seems that there is no particular reason why to prefer a specific 2-cell structure to another. For instance, the category of topological spaces is usually considered with the 2-cell structure which is obtained from the homotopy classes of homotopies, but there are perhaps others which could have been considered as well.

We are now going to extend the notion of pseudocategory to its full generality.

Let $\mathbf{C}$ be a category with a 2-cell structure $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$. Recall from Section 2 that an internal precategory in $\mathbf{C}$ is a system

$$(C_0, C_1, C_2, d, c, e, m, \pi_1, \pi_2)$$

with C_0, C_1, C_2 objects and d, c, e, m, π_1, π_2 morphisms in $\mathbf{C}$, displayed as

$$C_2 \overset{\pi_2}{\underset{\pi_1}{\substack{\longrightarrow \\ \xrightarrow{m} \\ \longrightarrow}}} C_1 \overset{d}{\underset{d}{\substack{\longrightarrow \\ \xleftarrow{e} \\ \longrightarrow}}} C_0 \ ,$$

such that the conditions (PC1), (PC2) and (PC3) are satisfied. We now need a reformulation of (PC3), denoted by (PC3*), to be used in the context of a 2-cell structure. We will say that a commutative square $d\pi_1 = c\pi_2$ such as the leftmost one in the diagram (1.23) below (compare with (PC3)) satisfies (PC3*) if there are spans

$$C_2 \xleftarrow{p_1} C_3 \xrightarrow{p_2} C2\ , \qquad C_3 \xleftarrow{p_1'} C_4 \xrightarrow{p_2'} C3$$

such that all the squares

$$\begin{array}{ccc} C_2 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle c} \\ C_1 & \xrightarrow{d} & C_0 \end{array} \qquad \begin{array}{ccc} C_3 & \xrightarrow{p_2} & C_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ C_2 & \xrightarrow{\pi_2} & C_1 \end{array} \qquad \begin{array}{ccc} C_4 & \xrightarrow{p_2'} & C_3 \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle p_1} \\ C_3 & \xrightarrow{p_2} & C_2 \end{array} \tag{1.23}$$

are H-cartesian pullback squares (see Definition 4.1). In that case we have induced morphisms

(indeed every morphism can be identified with its identity 2-cell)

$$\begin{array}{rcl} e_1 & = & \langle 1, ed\rangle : C_1 \longrightarrow C_2 \\ e_2 & = & \langle ec, 1\rangle : C_1 \longrightarrow C_2 \\ m_1 & = & 1\times m : C_3 \longrightarrow C_2 \\ m_2 & = & m\times 1 : C_3 \longrightarrow C_2 \\ i_0 & = & e_1\times e_2 : C_2 \longrightarrow C_3 \\ m_3 & = & 1\times m\times 1\colon C_4 \to C_3 \\ m_4 & = & m\times 1\times 1\colon C_4 \to C_3 \\ m_5 & = & 1\times 1\times m\colon C_4 \to C_3, \end{array}$$

and for appropriate 2-cells $\alpha \in H\,(C_3, C_1)$ and $\lambda, \rho \in H\,(C_1, C_1)$ we have (see Definition 4.1) induced 2-cells

$$(\alpha \times_{0_{1_{C_0}}} 0_{1_{C_1}}), (0_{1_{C_1}} \times_{0_{1_{C_0}}} \alpha) \in H(C_4, C_2)$$

and

$$(\rho \times_{0_{1_{C_0}}} 0_{1_{C_1}}), (0_{1_{C_1}} \times_{0_{1_{C_0}}} \lambda) \in H(C_2, C_2).$$

Explicitly $(\alpha \times_{0_{1_{C_0}}} 0_{1_{C_1}})$ is defined via the diagram

$$\begin{array}{ccccc} C_3 & \xrightarrow{d\pi_2 p_2} & C_0 & \xleftarrow{c} & C_1 \\ {\scriptstyle\alpha}\Downarrow & & \Downarrow{\scriptstyle 0_{1_{C_0}}} & & \Downarrow{\scriptstyle 0_{1_{C_1}}} \\ C_1 & \xrightarrow{d} & C_0 & \xleftarrow{c} & C_1; \end{array}$$

indeed C_4 can be seen also as the pullback of $d\pi_2 p_2$ and c. The other 2-cells are defined similarly.

Definition 6.1. A *pseudocategory* internal to a category $\mathbf{C}$, with respect to a 2-cell structure $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$, is a system

$$(C_0, C_1, C_2, d, c, e, m, \pi_1, \pi_2, \alpha, \lambda, \rho)$$

in which C_0, C_1, C_2 are objects in $\mathbf{C}$, d, c, e, m, π_1, π_2 are morphisms in $\mathbf{C}$, which display as

$$C_2 \overset{\pi_2}{\underset{\pi_1}{\overrightarrow{\underrightarrow{\;-m\to\;}}}} C_1 \overset{d}{\underset{c}{\overrightarrow{\underrightarrow{\;\leftarrow e-\;}}}} C_0\ ,$$

satisfying the conditions (PC1), (PC2) and (PC3*) (see (1.23) above). Moreover α, λ, ρ are natural and invertible 2-cells

$$\alpha \in H\,(C_3, C_1) \text{ and } \lambda, \rho \in H\,(C_1, C_1)$$

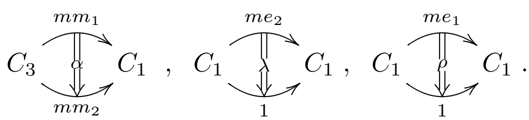
with

$$\begin{array}{rcl} \mathrm{dom}\,(\alpha) & = & mm_1\ ,\ \mathrm{cod}\,(\alpha) = mm_2 \\ \mathrm{dom}\,(\lambda) & = & me_2\ ,\ \mathrm{dom}\,(\rho) = me_1\ ,\ \mathrm{cod}\,(\lambda) = 1_{C_1} = \mathrm{cod}\,(\rho) \end{array}$$

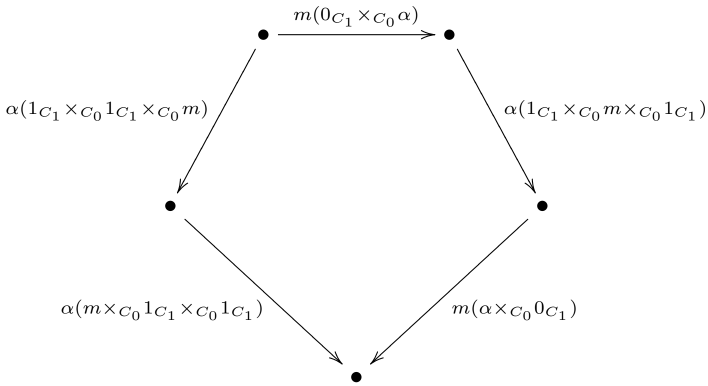
and such that the following conditions hold true

$$\begin{aligned} d\lambda = \quad & 0_d \quad = d\rho \\ c\lambda = \quad & 0_c \quad = c\rho \\ d\alpha = 0_{d\pi_2 p_2} \quad & , \quad c\alpha = 0_{c\pi_1 p_1} \\ \lambda e \quad & = \quad \rho e \\ \alpha m_4 + \alpha m_5 \quad & = \quad m\,(\alpha \times 0_1) + \alpha m_3 + m\,(0_1 \times \alpha) \qquad (1.24) \\ m\,(0_1 \times \lambda) \quad & = \quad m\,(\rho \times 0_1) + \alpha i_0. \qquad (1.25) \end{aligned}$$

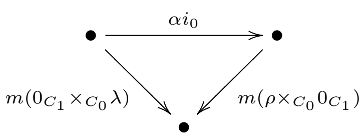
The 2-cells α, λ, ρ are usually presented as



Equations (1.24) and (1.25) correspond respectively to the internal versions of the Pentagon and Middle Triangle in Mac Lane's Coherence Theorem [9], presented diagrammatically as

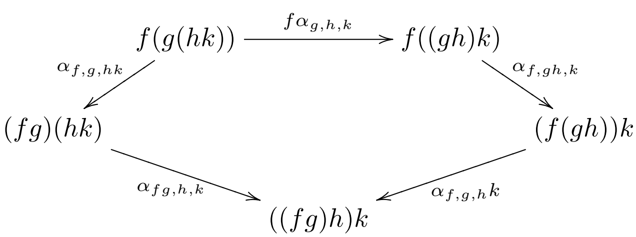


(1.26)



(1.27)

and restated in terms of generalized elements as



(pentagon)

$$\begin{array}{ccc} f(1g) & \xrightarrow{\alpha_{f,1,g}} & (f1)g \\ & {\scriptstyle f\lambda_g}\searrow \quad \swarrow{\scriptstyle \rho_f g} & \\ & fg & \end{array}$$ (middle triangle)

with $fg = m\langle f, g\rangle$ representing the *composition* of generalized elements which are nothing but two morphisms $f, g\colon X \to C_1$ such that $df = cg$; this means that $\langle f, g\rangle$ is the corresponding induced morphism into the pullback object C_2.

We conclude this exposition with three applications. The first example is an instance of 5.7, the second one is an application of 5.8, while the third one is from 5.12.

6.1 Pseudocategories internal to Cat(B) with B weakly Mal'tsev

In the setting of Section 2, let **B** be a weakly Mal'tsev category [13]. This means that **B** has pullbacks of split epimorphisms along split epimorphisms, and, for every two split epimorphisms $f\colon A \to B$ and $g\colon C \to B$ with sections r and s, respectively, the two induced morphisms $\langle 1_A, sf\rangle$ and $\langle rg, 1_C\rangle$ into the pullback of f and g are jointly epimorphic. Consider $\mathbf{C}$ =Cat(**B**) equipped with the 2-cell structure

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

as in 5.7.

As proved in [14], the pair $(\mathbf{C}, \mathbf{H})$ is a weakly Mal'tsev sesquicategory in which every pullback square is H-cartesian. The following theorem is also proved there (Chapter 9, Theorem 105 and following).

Theorem 6.2. Let **B** be a weakly Mal'tsev category and consider $\mathbf{C} = \mathsf{Cat}(\mathbf{B})$ with the 2-cell structure $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$ as in 5.7. A pseudocategory internal to **C** and relative to the given 2-cell structure, satisfying the additional condition

$$\lambda e = 0_e = \rho e \tag{1.28}$$

is completely determined by a reflexive graph in **C**

$$C_1 \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} C_0 \quad , \quad de = 1_{C_0} = ce$$

together with 2-cells

$$\lambda, \rho \in H(C_1, C_1)$$

satisfying the following conditions,

$$\mathrm{cod}(\lambda) = 1_{C_1} = \mathrm{cod}(\rho)$$

$$\begin{aligned} d\lambda &= 0_d = d\rho \\ c\lambda &= 0_c = c\rho \\ \lambda e &= 0_e = \rho e \end{aligned}$$

and furthermore it is equipped with a morphism

$$m : C_2 \longrightarrow C_1$$

uniquely determined by

$$me_1 = u \ \ , \ \ me_2 = v,$$

with $v = \mathrm{dom}\,(\lambda)$ and $u = \mathrm{dom}\,(\rho)$, together with a 2-cell $\alpha \in H\,(C_3, C_1)$, uniquely determined by

$$\alpha i_1 e_1 = -\rho u \ , \ \alpha i_2 e_1 = -u\lambda + \lambda u \ , \ \alpha i_2 e_2 = \lambda v.$$

6.2 Pseudocategories in groups

In this second example we describe pseudocategories internal to groups. It gives a *relaxed* notion of a crossed module (X, B, ∂) in which we have the freedom to choose an element δ in the centre of X.

It is well known that internal categories in the category of groups are equivalent to crossed modules (see [9] p. 285 for an explicit description of the equivalence). In a similar way a pseudo-category internal to the category of groups, with the 2-cell structure given as in 5.9, is completely determined by a group homomorphism

$$X \xrightarrow{\partial} B,$$

an action of B on X (denoted by $b \cdot x$) and a distinguished element δ in X satisfying the following conditions

$$\begin{aligned} \partial\delta &= 0 \\ x &= \delta + x - \delta \\ \partial\,(b \cdot x) &= b\partial\,(x)\,b^{-1} \\ \partial\,(x) \cdot \bar{x} &= x + \bar{x} - x. \end{aligned}$$

In this case it is not difficult to describe the objects and the pseudomorphisms from the internal pseudocategory, with the respective isomorphisms α, λ and ρ. The objects are elements of B, the arrows are pairs

$$(x, b) : b \longrightarrow \partial x + b$$

and the composition of

$$(x', \partial x + b) : \partial x + b \longrightarrow \partial x' + \partial x + b$$

with

$$(x, b) : b \longrightarrow \partial x + b$$

is

$$(x' + x - \delta + b \cdot \delta, b) : b \longrightarrow \partial x' + \partial x + b.$$

The isomorphism between $(0, \partial x + b) \circ (x, b) = (x, b) \circ (0, b)$ and (x, b) is the element $(\delta, 0) \in X \rtimes B$. Associativity is satisfied, since

$$(x'', \partial x' + \partial x + b) \circ ((x', \partial x + b) \circ (x, b)) = ((x'', \partial x' + \partial x + b) \circ (x', \partial x + b)) \circ (x, b)\,.$$

6.3 The additive case

In the case when **A** is an additive category with kernels, equipped with a 2-cell structure which is given by an Ab-functor

$$H : \mathbf{A}^{op} \times \mathbf{A} \longrightarrow \mathsf{Ab}$$

and a natural transformation

$$D : H \longrightarrow \hom_{\mathbf{A}},$$

as in 5.12, 5.13, 5.14, a pseudocategory internal to it is completely determined by

$$A \xrightarrow{h} B$$

$$\begin{array}{rcl} \lambda, \rho & \in & H(A, A) \\ \eta & \in & H(B, A) \end{array}$$

$$\begin{array}{rcl} h\lambda & = & 0 \\ h\rho & = & 0 \\ h\eta & = & 0 \end{array}$$

with α uniquely determined. The pseudocategory thus determined is of the form (see [10])

$$A \oplus A \oplus B \xrightarrow{m} A \oplus B \underset{(h\ 1)}{\overset{(0\ 1)}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\binom{0}{1}}}}}} B$$

$$m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rcl} g & = & 1 - D(\lambda) \\ f & = & 1 - D(\rho) \\ h & = & -D(\eta) \end{array}$$

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{rcl} \alpha_1 & = & -f\rho \\ \alpha_2 & = & \lambda + g\rho - \rho - f\lambda \\ \alpha_3 & = & g\lambda - f\eta\delta \\ \alpha_0 & = & g\eta - f\eta \end{array}$$

$$\lambda = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \end{pmatrix} \qquad \rho = \begin{pmatrix} \rho & \eta \\ 0 & 0 \end{pmatrix},$$

with obvious abuse of notation for λ and ρ.

In particular the category of abelian chain complexes is of this form but there we have to specify that λ and ρ are natural 2-cells in the sense of Definition 4.3. See also the last remark in the conclusion of this article.

Another example of this form is the category $\mathsf{Ab}(\mathsf{Top})$ (see example 5.15). A pseudocategory in $\mathsf{Ab}(\mathsf{Top})$ (with the 2-cell structure as in 5.15) is completely determined by a morphism in $\mathsf{Ab}(\mathsf{Top})$

$$k : A \longrightarrow B,$$

together with

$$\lambda, \rho : I \times A \longrightarrow A$$

in $H(A, A)$ and also

$$\eta : I \times B \longrightarrow A$$

in $H(B, A)$ satisfying

$$k\rho(t, \_) = 0, k\lambda(t, \_) = 0, k\eta(t, \_) = 0.$$

The objects in the pseudocategory are the points in B while the pseudomorphisms are pairs (a, b) with domain b and codomain $k(a) + b$; the composition of

$$b \xrightarrow{(a,b)} b' \xrightarrow{(a',b')} b''$$

is given by the following formula

$$(a - \rho(1, a) + a' - \lambda(1, a') - \eta(1, b), b).$$

In particular, when A is the space of paths in B starting at zero

$$A = \{x : I \longrightarrow B \mid x \text{ continuous and } x(0) = 0\}$$

with

$$k(x) = x(1)$$

and choose representatives of λ, ρ and η as

$$\begin{aligned} \lambda(s, x)(t) &= \begin{cases} x(st) - x(2st) & \text{if} \quad t \leqslant \frac{1}{2} \\ x(st) - x(s) & \text{if} \quad t \geqslant \frac{1}{2} \end{cases} \\ \rho(s, x)(t) &= \begin{cases} x(st) & \text{if} \quad t \leqslant \frac{1}{2} \\ x(st) - x(2st - s) & \text{if} \quad t \geqslant \frac{1}{2} \end{cases} \\ \eta &= 0 \end{aligned}$$

we obtain the usual composition of paths

$$y + x = \begin{cases} x(2t) & \text{if} \quad t \leqslant \frac{1}{2} \\ y(2t - 1) + x(1) & \text{if} \quad t \geqslant \frac{1}{2} \end{cases}.$$

7 Conclusion

As follows from Definition 4.3, recognising whether a given 2-cell $x \in H(A, B)$ is a natural one is generally a complicated task: we have to analyse equation (1.11) for every possible y. However, removing naturality conditions for α, λ, ρ, we loose the Coherence Theorem [9] and we have no longer guaranteed that, for example, the diagrams

$$\begin{array}{ccc} 1(fg) & \xrightarrow{\alpha_{1,f,g}} & (1f)g \\ & {\scriptstyle \lambda_{fg}} \searrow \quad \swarrow {\scriptstyle \lambda_f g} & \\ & fg & \end{array} \tag{1.29}$$

$$\begin{array}{ccc} f(g1) & \xrightarrow{\alpha_{f,g,1}} & (fg)1 \\ & {\scriptstyle f\rho_g} \searrow \quad \swarrow {\scriptstyle \rho_{fg}} & \\ & fg & \end{array} \tag{1.30}$$

are commutative. These diagrams, when internalised, correspond respectively to the following equations

$$\begin{aligned} m(\lambda \times 0_{C_1}) + \alpha i_2 &= \lambda m, \\ \rho m + \alpha i_1 &= m(0_{C_1} \times \rho) \end{aligned}$$

and since the 2-cells are assumed to be invertible they can be presented as

$$\begin{aligned} \alpha i_2 &= -m(\lambda \times 0_{C_1}) + \lambda m, \\ \alpha i_1 &= -\rho m + m(0_{C_1} \times \rho). \end{aligned}$$

The examples illustrated in the previous section suggest an intermediate notion for an *unnatural* pseudocategory, where we do not ask for the 2-cells α, λ, ρ to be natural, but only to be natural with respect to each other. In fact in all those cases the 2-cell α is completely determined; hence (at least for a further study of those cases) we may require that only λ and ρ are natural with respect to each other, which in other words means that the horizontal compositions

$$\lambda \circ \lambda, \lambda \circ \rho, \rho \circ \rho, \rho \circ \lambda \tag{1.31}$$

are defined, or even that their commutators vanish (Definition 4.5). In order to have some control on the coherence aspects we observe that at least the conditions

$$\alpha i_2 = -m(\lambda \times 0_{C_1}) + \lambda m \tag{1.32}$$

$$\alpha i_1 = -\rho m + m(0_{C_1} \times \rho), \tag{1.33}$$

should be required.

Clearly when α, λ, ρ are natural 2-cells then these three conditions are redundant and we obtain Definition 6.1.

For instance, in Subsection 6.1, as also proved in [14], we have that

$$(1.32) + (1.33) + (1.31) \Rightarrow (1.34) + (1.35),$$

with (1.34), (1.35) the *pentagon* and the *middle triangle* identities restated as:

$$m(\alpha \times 0_1) + \alpha m_3 + m(0_1 \times \alpha) = \alpha m_4 + \alpha m_5 \quad (1.34)$$

$$\alpha i_0 = -m(\rho \times 0_1) + m(0_1 \times \lambda). \quad (1.35)$$

In the example of Subsection 6.3 and using the notation

$$[x, y] = D(x)\, y - xD(y)$$

we have that α is determined by (1.32)+(1.33) but (1.34) is no longer a trivial condition: it becomes equivalent to

$$(1 - D\rho)[\rho, \rho] = 0$$
$$D\lambda[\rho, \rho] = D\rho[\lambda, \rho] + (1 - D\rho)[\rho, \lambda]$$
$$(1 - D\lambda)[\lambda, \rho] + D\lambda[\rho, \lambda] = D\rho[\lambda, \lambda] + (1 - D\rho)[\rho, \eta]\, h$$
$$(1 - D\lambda)[\lambda, \lambda] = (1 - D\lambda)[\lambda, \eta]$$
$$(1 - D\rho - D\lambda)[\lambda, \eta] = (1 - D\rho - D\lambda)[\rho, \eta]$$

which, however, is trivial as soon as we introduce (1.31). For further details about this, see the calculations in [14, Chapter 5].

Acknowledgements

The author is grateful to Maria Manuel Clementino, George Janelidze and Andrea Montoli for useful comments about the paper. This work was supported by IPLeiria(ESTG/CDRSP) and FCT grant SFRH/BPD/43216/2008. Also by the FCT projects: PTDC/EME-CRO/120585/2010, PTDC/MAT/120222/2010 and PEst-OE/EME/ UI4044/2013.

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