Wobbly double functors

Robert Paré

Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada, B3H 3J5
E-mail: pare@mathstat.dal.ca

Dedicated to Marco Grandis on his seventieth birthday

Abstract

We introduce a weakened notion of double functor, which we call *wobbly*, and which arises naturally in the study of double adjoints. We then show how horizontal invariance can be used to lift results about wobbly double functors to genuine ones.

2010 Mathematics Subject Classification. **18A30**. 18A25.
Keywords. Double functor, double adjoint, weak equivalence, horizontal invariance.

Introduction

There are many flavours of double functor (between weak double categories) which occur in practice. There are lax, colax, strong, normal and even strict ones, but they all have one thing in common: they all preserve domains and codomains on the nose. This is reasonnable as these "boundary conditions" lie at the most basic geometric level and serve as the foundation upon which the whole theory of double categories is built. Yet it often happens, in the course of doing double category theory, that we arrive at what would be lax (or colax or strong) double functors, were it not that domains and codomains are only preserved up to isomorphism. This happens in particular with adjoints to double functors. We discuss this situation in Section 1.

This leads to the introduction in Section 2 of *wobbly double functors* (either lax, colax, or strong). Their basic properties are discussed there. In Section 3 we consider weak equivalence double functors. We show that they are precisely the double functors with a wobbly pseudo-inverse. Finally, Section 4 revisits the notion of horizontal invariance from [1]. It is seen as precisely the tool needed to lift results on wobbly double functors to "steady" ones.

1 Adjoints recalled

In [2] we defined and studied adjoint double functors. In the general situation the left adjoint is colax and the right adjoint is lax. To formalize this we constructed the strict double category $\mathbb{D}\mathrm{bl}$ whose horizontal arrows are lax functors and whose vertical arrows are colax ones. We then defined adjointness as a conjoint pair (called orthogonal adjoint pair there) in $\mathbb{D}\mathrm{bl}$. We then gave the double versions of the usual characterizations of adjointness. For example ([2], 3.5), a colax functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is left adjoint to a lax $U : \mathbb{B} \longrightarrow \mathbb{A}$ iff there is an isomorphism, over $\mathbb{A} \times \mathbb{B}$, of double comma categories $(F \Downarrow \mathbb{B}) \cong (\mathbb{A} \Downarrow U)$. Another characterization is that each $F_i (i = 0, 1, 2)$ be left adjoint to U_i (see diagram (*) below) and that the colaxity transformations of F_i be the mates of the laxity transformations of U_i. This can be expressed in terms of hom sets if we like ([2], 3.4).

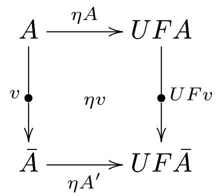
In the definition and the two characterizations mentioned, both the F and U are given. But in the case of ordinary adjoints it is often very useful to construct a left adjoint say, object by object

Tbilisi Mathematical Journal 8(1) (2015), pp. 63–84.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 22 August 2014.
Accepted for publication: 12 February 2015.

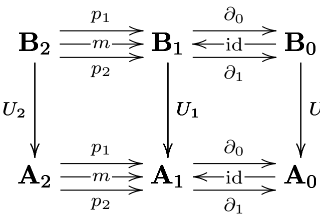
and use the universal property to extend it uniquely to a functor. This is the case with the adjoint functor theorem, e.g. There is a result along these lines in *loc. cit.*, Theorem 3.6, which says that U has a left adjoint F iff

(0) for every A there is a universal arrow $A \xrightarrow{\eta A} UFA$
(1) for every vertical arrow $v : A \longrightarrow\!\!\!\!\!\bullet\; \bar{A}$ there is a universal cell



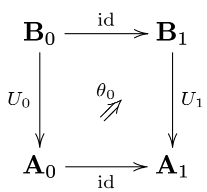
There is something here that may not be apparent at first glance. What (1) says is that there is a choice of universal arrow ηA in (0) and that there there are universal cells for each v whose vertical domains and codomains are those chosen arrows. This is a condition that has to be satisfied, which is usually easy to do.

Let $U : \mathbb{B} \longrightarrow \mathbb{A}$ be a lax functor represented by a diagram in $\mathcal{C}at$

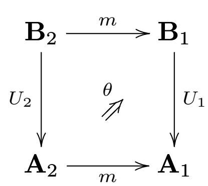


$(*)$

where $\mathbf{A}_2 = \mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1$ and $\mathbf{B}_2 = \mathbf{B}_1 \times_{\mathbf{B}_0} \mathbf{B}_1$ and p_i are the pullback projections. We have the commutativities $U_0 \partial_i = \partial_i U_1$, $U_1 p_j = p_j U_2$, which are at the centre of our discussion. They imply that $U_2 = U_1 \times_{U_0} U_1$. The laxity of U is given by natural transformations



and



satisfying the usual conditions. We would like to be able to say that U has a double adjoint if and only if U_0 and U_1 have left adjoints preserved by ∂_0 and ∂_1. For example, to say that $\mathbb{B}$ has pushouts if $\mathbf{B}_0$ and $\mathbf{B}_1$ have pushouts and ∂_0 and ∂_1 preserve them, is quick to state, easy to verify and true to the intention if not quite correct.

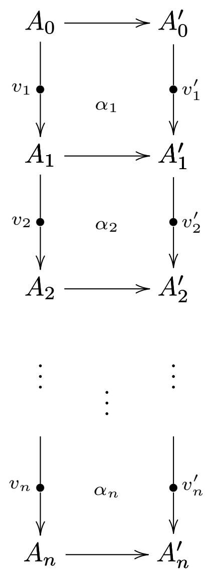
2 Wobbly double functors

Just having left adjoints, F_0 and F_1, to U_0 and U_1 is not enough to express the colaxity that a left adjoint F would have. For this we need $F_2 : \mathbf{A}_2 \longrightarrow \mathbf{B}_2$. If F_0 and F_1 had commuted with the ∂_i

then F_2 would come from the pullback property of $\mathbf{B}_2$. However, as F_0 and F_1 only commute with the ∂_i up to isomorphism, we can't infer the existence of F_2. The solution is to simply postulate its existence, i.e. that U_2 has a left adjoint commuting with the projections p_j up to isomorphism. In most practical cases this is just as easy as finding the left adjoints to U_0 and U_1. But in order to express the coherence of the laxity morphisms we also need an $F_3 : \mathbf{A}_3 \longrightarrow \mathbf{B}_3$, left adjoint to U_3. At this point we have four functors and lots of natural transformations. We may as well go all the way to a lax morphism of simplicial categories, a point of view favoured by Steve Lack [3].

As we shall be doing calculations, we should mention our conventions. For a weak double category $\mathbb{A} = (\mathbf{A}_2 \rightrightarrows \mathbf{A}_1 \leftrightarrows \mathbf{A}_0)$, the arrows $\mathbf{A}_0$ are called horizontal, the identities are denoted by 1_A and composition by juxtaposition. We use the functional (classical) order for the composite. The objects of $\mathbf{A}_1$ are called vertical arrows. There are vertical identities, denoted id_A, and a vertical composition, denoted $\bullet$, which is associative and unitary up to coherent isomorphism. Vertical composition is usually some kind of tensor product and the diagrammatic order, which we use, seems more natural here. Horizontal and vertical composition of cells use the same conventions.

A weak double category $\mathbb{A}$ has a "nerve" which is a pseudo-functor $\mathbf{A}_{(\ )} : \mathbf{\Delta}^{op} \longrightarrow \mathcal{C}at$ ($\mathbf{\Delta}$ is the category of non-zero finite ordinals). An object of $\mathbf{A}_n$ is a compatible path $\langle v_i \rangle$ of vertical arrows of length n, and a morphism a compatible path $\langle \alpha_i \rangle$ of cells



For $f : [m] \longrightarrow [n]$ in $\mathbf{\Delta}$ we have a functor $f^* : \mathbf{A}_n \longrightarrow \mathbf{A}_m$ as follows.

$$
\begin{array}{ccc}
A_0 & & A_{f(0)} \\
{\scriptstyle v_1}\downarrow\bullet & & \bullet\downarrow{\scriptstyle w_1} \\
A_1 & & A_{f(1)} \\
{\scriptstyle v_2}\downarrow\bullet & \overset{f^*}{\longmapsto} & \bullet\downarrow{\scriptstyle w_2} \\
A_2 & & A_{f(2)} \\
{\scriptstyle v_3}\downarrow\bullet & & \bullet\downarrow{\scriptstyle w_3} \\
\vdots & & \vdots \\
{\scriptstyle v_n}\downarrow\bullet & & \bullet\downarrow{\scriptstyle w_m} \\
A_n & & A_{f(m)}
\end{array}
$$

where w_j is the composite of the v's between $A_{f(j-1)}$ and $A_{f(j)}$ taken in a given order, chosen once and for all. Specifically, we take

$$
w_j = \begin{cases} (\cdots(v_{f(j-1)+1} \bullet v_{f(j-1)+2}) \bullet \cdots) \bullet v_{f(j)} & \text{if} \quad f(j) > f(j-1)+1 \\ v_{f(j)} & \text{if} \quad f(j) = f(j-1)+1 \\ \mathrm{id}_{A_{f(j)}} & \text{if} \quad f(j) = f(j-1) \end{cases}
$$

The same formula applies to morphisms.

Given $g : [n] \longrightarrow [p]$, the structural isomorphism

$$\mathfrak{a}_{f,g} : f^* g^* \longrightarrow (gf)^*$$

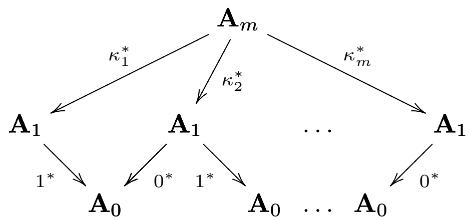
is an iteration of the associativity isomorphisms $\mathfrak{a}$ of $\mathbb{A}$. It comes from rearranging the parentheses to put the factors in the "standard" order. Degenerate cases also use the unit isomorphisms $\mathfrak{l}$ and $\mathfrak{r}$. Saying that this makes $\mathbf{A}_{(\ )}$ into a pseudo-functor is a global statement of $\mathbb{A}$'s coherence conditions.

Although we won't need this below, because we start with weak double categories, we mention that $\mathbf{A}_{(\ )} : \mathbf{\Delta}^{op} \longrightarrow \mathcal{C}at$ is not an arbitrary pseudo-functor. First of all, some of the $\mathfrak{a}_{f,g}$ are identities. Say that f and g are *in step* if for every $0 < i \le m$ and $f(i-1) < j \le f(i)$ we have $f(i) = f(i-1)+1$ or $g(j) = g(j-1)+1$. Then, if f and g are in step, we have that $\mathfrak{a}_{f,g}$ is equality. Further, the canonical isomorphisms $1_{\mathfrak{a}_n} \longrightarrow 1^*_{[n]}$ are also equalities.

For each $0 < i \le m$ we have the *increment function* $\kappa_i : [1] \longrightarrow [m]$ given by

$$\kappa_i(0) = i-1, \kappa_i(1) = i.$$

The final condition on a pseudo-functor $\mathbf{A}_{(\ )}$ for it to be the nerve of a weak double category is that for every $m \ge 2$



be a limit diagram. That is, $\mathbf{A}_m$ is the generalized pullback $\mathbf{A}_1 \times_{\mathbf{A}_0} \mathbf{A}_1 \times_{\mathbf{A}_0} \cdots \times_{\mathbf{A}_0} \mathbf{A}_1$ and κ_i^* is the projection onto the i^{th} factor. The 0 and 1 in the above are the functions $[0] \longrightarrow [1]$ which pick out the elements $0, 1 \in [1]$. For a weak double category, 0^* and 1^* are ∂_0, ∂_1, the vertical domain and codomain functors $\mathbf{A}_1 \longrightarrow \mathbf{A}_0$. Note that any pair of functions starting at [0] will be trivially in step so that we always have $0^* \kappa_i^* = 1^* \kappa_{i-1}^*$.

Before giving the definition of wobbly functor we need one more notion. A function $f : [m] \longrightarrow [n]$ is called a *translation* if $f(i) = f(i-1) + 1$ for all $0 < i \leq m$. Any function $[0] \longrightarrow [n]$ is trivially a translation, and the translations $[1] \longrightarrow [n]$ are the increments.

Definition 2.1. Let $\mathbb{A}$ and $\mathbb{B}$ be weak double categories and $\mathbf{A}_{(\ )}$ and $\mathbf{B}_{(\ )}$ the corresponding pseudo-functors. A *colax wobbly functor* $F : \mathbb{A} \rightsquigarrow \mathbb{B}$ is a colax transformation $F_{(\ )} : \mathbf{A}_{(\ )} \longrightarrow \mathbf{B}_{(\ )}$ such that the structural morphisms

$$
\begin{array}{ccc}
\mathbf{A}_n & \xrightarrow{F_n} & \mathbf{B}_n \\
{\scriptstyle f^*}\downarrow & \overset{\varphi_f}{\nearrow} & \downarrow{\scriptstyle f^*} \\
\mathbf{A}_m & \xrightarrow[F_m]{} & \mathbf{B}_m
\end{array}
$$

are isomorphisms for all translations $f : [m] \longrightarrow [n]$.

A *transformation* $t : F \longrightarrow F'$ of colax wobbly functors is a modification $t : F_{(\ )} \longrightarrow F'_{(\ )}$.

Remark: An ordinary colax double functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a wobbly one in which the φ_f are identities for all translations f.

General properties of colax transformations and modifications give the following.

Proposition 2.2. Weak double categories with colax wobbly functors between them and transformations of wobbly functors form a 2-category $\mathcal{C}ol\mathcal{W}ob$.

The following result, whose proof is easy, says that wobbly functors are stable under "degree-wise" isomorphism. This doesn't hold for double functors (lax, colax or strong).

Proposition 2.3. Let $F : \mathbb{A} \rightsquigarrow \mathbb{B}$ be a colax (resp. lax, strong) wobbly functor. Also let $G_n : \mathbf{A}_n \longrightarrow \mathbf{B}_n$ be functors for each n and $t_n : G_n \longrightarrow F_n$ natural isomorphisms. If we take

$$
\gamma_f \quad = \quad
\begin{array}{ccccccc}
\mathbf{A}_n & = & \mathbf{A}_n & \xrightarrow{f^*} & \mathbf{A}_m & = & \mathbf{A}_m \\
{\scriptstyle G_n}\downarrow & \overset{t_n^{-1}}{\swarrow} & \downarrow{\scriptstyle F_n} & \overset{\varphi_f}{\swarrow} & \downarrow{\scriptstyle F_m} & \overset{t_m}{\swarrow} & \downarrow{\scriptstyle G_m} \\
\mathbf{B}_n & = & \mathbf{B}_n & \xrightarrow[f^*]{} & \mathbf{B}_m & = & \mathbf{B}_m
\end{array}
$$

then the G_n equipped with these γ_f give a wobbly functor $G : \mathbb{A} \rightsquigarrow \mathbb{B}$ and $t : G \longrightarrow F$ is an isomorphism.

Although the above definition is better for proving general results about wobbly functors, a more combinatorial description in terms of generators and relations may be better in other situations. For example understanding the definitions and results is often best achieved by seeing what happens for $0, 1, 2$ and then mentally extrapolating. So we give a description which parallels the usual simplicial set calculus.

There are the usual *face functors*

$$d_0, \ldots, d_n : \mathbf{A}_n \longrightarrow \mathbf{A}_{n-1}$$

$$d_i\langle v_1, \ldots, v_n\rangle = \begin{cases} \langle v_2, \ldots, v_n\rangle & \text{if} \quad i = 0 \\ \langle v_1, \ldots, v_i \bullet v_{i+1}, \ldots, v_n\rangle & \text{if} \quad 0 < i < n \\ \langle v_1, \ldots, v_{n-1}\rangle & \text{if} \quad i = n \end{cases}$$

and *degeneracy functors*

$$s_0, \ldots, s_n : \mathbf{A}_n \longrightarrow \mathbf{A}_{n+1}$$

$$s_i\langle v_1, \ldots, v_n\rangle = \langle v_1, \ldots, v_i, \mathrm{id}_{A_i}, v_{i+1}, \ldots, v_n\rangle.$$

The simplicial identities now take the form

(1)

$$d_i d_j = d_{j-1} d_i \quad \text{if} \quad i < j-1 \quad \text{or} \quad (i, j) = (0, 1) \quad \text{or} \quad ((n-1), n).$$

(2)

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if} \quad i < j \\ 1_{\mathbf{A}_n} & \text{if} \quad (i, j) = (0, 0) \text{ or } (n, n) \\ s_j d_{i-1} & \text{if} \quad i > j+1 \end{cases}$$

(3)

$$s_i s_j = s_{j+1} s_i \quad \text{if} \quad i \leq j.$$

The missing identities are replaced with natural isomorphisms

(4)

$$\alpha_i : d_i d_{i+1} \longrightarrow d_i d_i \quad \text{if} \quad 0 < i < n-1$$

(5)

$$\begin{array}{ll} \lambda_I : & d_i s_i \longrightarrow 1_{\mathbf{A}_n} \quad 0 < i < n \\ & d_0 s_0 = 1_{\mathbf{A}_n} \\ & d_n s_n = 1_{\mathbf{A}_n}, \end{array}$$

(6)

$$\begin{array}{ll} \rho_i : & d_{i+1} \longrightarrow 1_{\mathbf{A}_n} \quad 0 < i < n \\ & d_1 s_0 = 1_{\mathbf{A}_n} \\ & d_{m+1} s_n = 1_{\mathbf{A}_n}. \end{array}$$

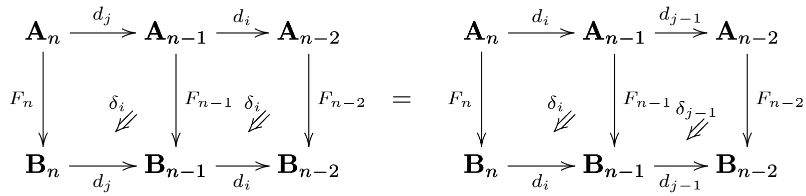
The α_i, λ_i, ρ_i have to satisfy the usual coherence conditions which we need not spell out here.

Definition 2.4. (Reformulated) A colax *wobbly functor* $F : \mathbb{A} \rightsquigarrow \mathbb{B}$ consists of functors $F_n : \mathbf{A}_n \longrightarrow \mathbf{B}_n$, $n \geq 0$, and natural transformations

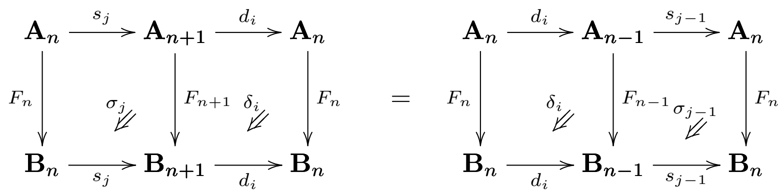
$$\begin{array}{ccc} \mathbf{A}_n & \xrightarrow{d_i} & \mathbf{A}_{n-1} \\ {\scriptstyle F_n}\downarrow & \overset{\delta_i}{\swarrow} & \downarrow{\scriptstyle F_{n-1}} \\ \mathbf{B}_n & \xrightarrow[d_i]{} & \mathbf{B}_{n-1} \end{array} \qquad \begin{array}{ccc} \mathbf{A}_n & \xrightarrow{s_i} & \mathbf{A}_{n+1} \\ {\scriptstyle F_n}\downarrow & \overset{\sigma_i}{\swarrow} & \downarrow{\scriptstyle F_{n+1}} \\ \mathbf{B}_n & \xrightarrow[s_i]{} & \mathbf{B}_{n+1} \end{array} \qquad 0 \leq i \leq n$$

satisfying equations corresponding to the simplicial identities, namely:

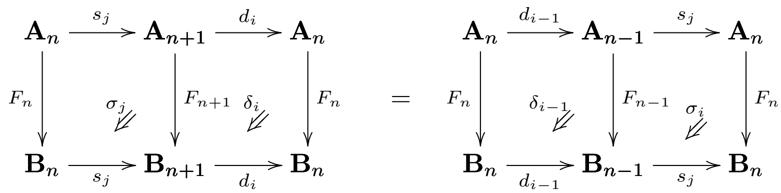
(1) If $i < j-1$ or $(i,j) = (0,1)$ or $(n-1,n)$



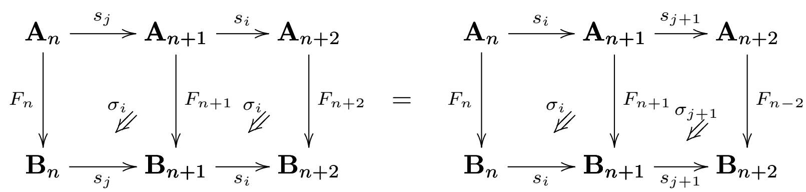
(2) If $i < j$



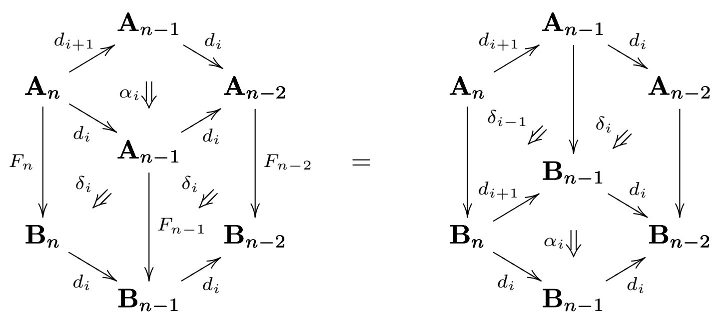
and if $i > j+1$



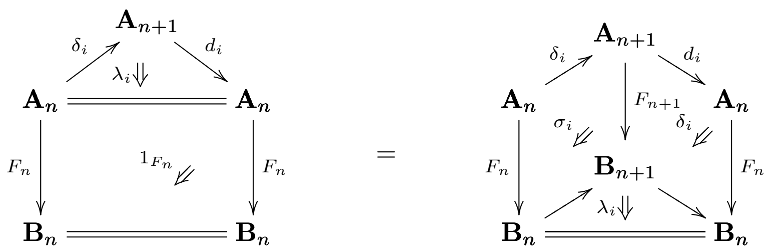
(3) If $i \leq j$



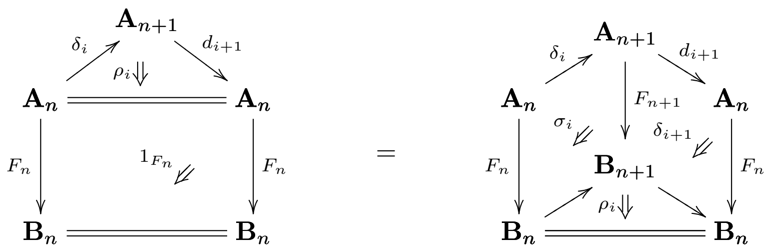
(4) If $0 < i < n-1$



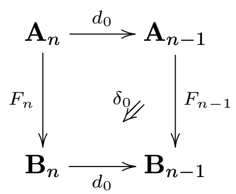
(5) If $0 < i < n$



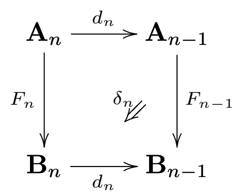
(6) If $0 < i < n$



(7) For each n



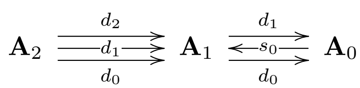
and



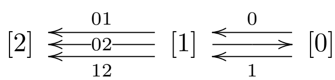
are isomorphisms.

If in (7) δ_0 and δ_n are identities, then all F_n, $n \geq 2$ are determined by F_0 and F_1, as are all the σ_i and δ_i with values in $\mathbf{B}_n$. That is, F is a colax functor $\mathbb{A} \longrightarrow \mathbb{B}$.

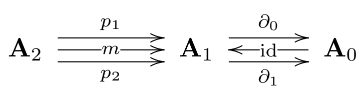
In degrees $0, 1, 2$ we have



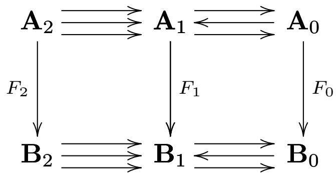
which correspond to the functions



for which we use the more usual notation



A wobbly functor F then has (among other things) functors F_0, F_1, F_2



and some natural transformations, some of which are isomorphisms. The crucial ones, as far as wobbly functors are concerned, are the isomorphisms

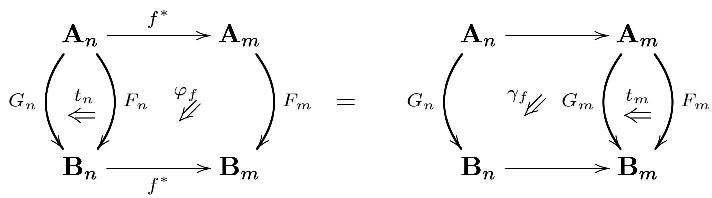
$$\begin{array}{ccc} \mathbf{A}_1 & \xrightarrow{\partial_i} & \mathbf{A}_0 \\ {\scriptstyle F_1}\downarrow & \overset{\delta_i}{\swarrow} & \downarrow{\scriptstyle F_0} \\ \mathbf{B}_1 & \xrightarrow[\partial_i]{} & \mathbf{B}_0 \end{array} \qquad i = 0, 1$$

That is, F takes objects to objects and horizontal arrows to horizontal arrows in a functorial way. It also takes vertical arrows to vertical arrows but vertical domains and codomains are not preserved. Thus for $v : A \longrightarrow\!\!\!\!\!\!\bullet\;\; \bar{A}$ we have isomorphisms $\delta_0 v$ and $\delta_1 v$

$$\begin{array}{ccc} FA & \xrightarrow{\delta_0 v} & \partial_0 Fv \\ & & \downarrow\bullet\, {\scriptstyle Fv} \\ F\bar{A} & \xrightarrow[\delta_1 v]{} & \partial_1 Fv \end{array}$$

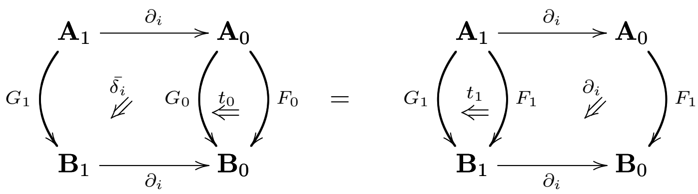
Because of this, F_2 is not determined by F_0 and F_1, and so it becomes part of the structure along with more isomorphisms and compatibilities with δ_0, δ_1. But the idea is clear.

A transformation of wobbly functors $t : F \longrightarrow G$ was defined to be a modification of the lax functors $F_{(\ )} \longrightarrow G_{(\ )}$. So it consists of a sequence of natural transformations $t_n : F_n \longrightarrow G_n$ such that for every $f : [m] \longrightarrow [n]$ we have



Transformations of wobbly functors are determined by their 0 and 1 components.

Theorem 2.5. Let $F, G : \mathbb{A} \rightsquigarrow \mathbb{B}$ be wobbly functors with $t_0 : F_0 \longrightarrow G_0$, $t_1 : F_1 \longrightarrow G_1$ such that for $i = 0, 1$ we have



then there exists a unique transformation $t : F \longrightarrow G$ extending t_0 and t_1.

Proof. Suppose there were such a $t : F \longrightarrow G$. Fix $n \geq 2$. For

$$A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \ldots \xrightarrow{v_n} A_n$$

then $F\langle v_i \rangle = \langle w_i \rangle$ and $G\langle v_i \rangle = \langle x_i \rangle$ and $t\langle v_i \rangle = \langle \beta_i \rangle$ as in

$$\begin{array}{ccc} B_0 & \longrightarrow & C_0 \\ {\scriptstyle w_1}\downarrow & \beta_1 & \downarrow{\scriptstyle x_1} \\ B_1 & \longrightarrow & C_1 \\ {\scriptstyle w_2}\downarrow & \beta_2 & \downarrow{\scriptstyle x_2} \\ \vdots & \vdots & \vdots \\ {\scriptstyle w_n}\downarrow & \beta_n & \downarrow{\scriptstyle x_n} \\ B_n & \longrightarrow & C_n \end{array}$$

For any $0 < k \leq n$, let $f : [1] \longrightarrow [n]$ be the increment κ_k. The compatibility of t with the structural morphisms says that

$$\begin{array}{ccc} F_1 f^* & \xrightarrow{\varphi_f} & f^* F_n \\ {\scriptstyle t_1 f^*}\downarrow & & \downarrow{\scriptstyle f^* t_n} \\ G_1 f^* & \xrightarrow[\gamma_f]{} & f^* G_n \end{array}$$

commutes. If we apply this to $\langle v_i \rangle$ we get

$$\begin{array}{ccc} F_1 v_k & \xrightarrow{\varphi_f \langle v_i \rangle} & w_k \\ {\scriptstyle t_1 v_k}\downarrow & & \downarrow{\scriptstyle \beta_k} \\ G_1 v_k & \xrightarrow[\gamma_f \langle v_i \rangle]{} & x_k \end{array}$$

As f is a translation, $\varphi_f\langle v_i\rangle$ and $\gamma_f\langle v_i\rangle$ are isomorphisms so β_k is determined uniquely

$$\beta_k = \gamma_f\langle v_i\rangle t_1(v_k)\varphi_f^{-1}\langle v_i\rangle.$$

In order to establish existence we have to show that the β_k defined by the above formula are compatible, i.e. that the vertical codomain of β_k is equal to the domain of β_{k+1}. Let $g : [1] \longrightarrow [n]$ be given by $g(0) = k$ and $g(1) = k + 1$. Then we have

$$\begin{array}{ccccc}
 & & [1] & & \\
 & \overset{1}{\nearrow} & & \overset{f}{\searrow} & \\
[0] & & \xrightarrow{\quad k \quad} & & [n] \\
 & \underset{0}{\searrow} & & \underset{g}{\nearrow} & \\
 & & [1] & &
\end{array}$$

We want to show that

$$\partial_1(\gamma_f\langle v_i\rangle t_1(v_k)\varphi_f^{-1}\langle v_1\rangle) = \partial_0(\gamma_g\langle v_i\rangle t_1(v_{k+1})\varphi_g^{-1}\langle v_i\rangle).$$

Consider the diagram, the top and bottom of which will give the left and right sides of this equality when evaluated at $\langle v_i\rangle$:

$$\begin{array}{ccccccc}
\partial_1 f^* F_n & \xrightarrow{\partial_1\varphi_f^{-1}} & \partial_1 F_1 f^* & \xrightarrow{\partial_1 t_1 f^*} & \partial_1 G_1 f^* & \xrightarrow{\partial_1\gamma_f} & \partial_1 f^* G_n \\
\| & & \uparrow{\scriptstyle \partial_1 f^*} & & \uparrow{\scriptstyle \bar{\partial}_1 f^*} & & \| \\
k^* F_n & \xrightarrow[\varphi_k^{-1}]{} & F_0 k^* & \xrightarrow{t_0 k^*} & G_0 k^* & \xrightarrow[\gamma_k]{} & k^* G_n \\
\| & & \downarrow{\scriptstyle \gamma_0 g^*} & & \downarrow{\scriptstyle \bar{\delta}_0 g^*} & & \| \\
\partial_0 g^* F_n & \xrightarrow[\partial_0\varphi_s^{-1}]{} & \partial_0 F_1 g^* & \xrightarrow[\partial_0 t_1 g^*]{} & \partial_0 G_1 g^* & \xrightarrow[\partial_0\gamma_g]{} & \partial_0 g^* G_n
\end{array}$$

The middle squares commute by hypothesis, the left and right by the colaxity conditions on F and G.

Thus we can define

$$t_n\langle v_i\rangle = \langle \gamma_f\langle v_i\rangle t_1(v_k)\varphi_f^{-1}\langle v_i\rangle\rangle_k$$

which is clearly natural. That it satisfies the compatibility conditions with the colaxity morphisms follows from the fact that the f as above are jointly faithful (for all k).

Q.E.D.

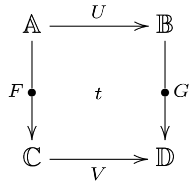
Corollary 2.6. If for F_0 and F_1 compatible with ∂_0 and ∂_1, there is a wobbly functor F extending F_0 and F_1. It is unique up to a unique isomorphism which is the identity on F_0 and F_1.

Corollary 2.7. The embedding

$$\mathrm{Colax}(\mathbb{A}, \mathbb{B}) \longrightarrow \mathrm{ColWob}(\mathbb{A}, \mathbb{B})$$

of colax functors into wobbly ones is full and faithful.

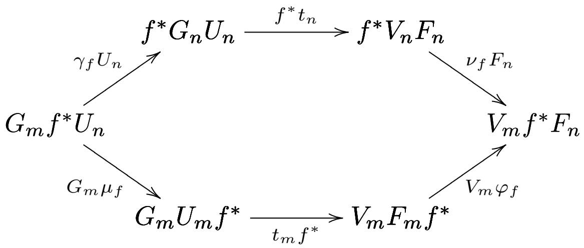
We also have lax wobbly functors which are the horizontal duals of the colax ones (the φ_f go in the opposite direction). They all fit together in a strict double category $\mathbb{W}$ob which generalizes our double category $\mathbb{D}$bl of [2]. The objects are weak double categories, the horizontal arrows are lax wobbly functors and the vertical arrows are colax wobbly functors. A cell t



is a sequence of natural transformations

$$t_n : G_n U_n \longrightarrow V_n F_n$$

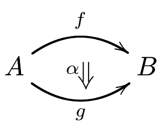
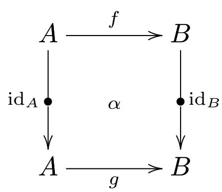
such that for every $f : [m] \longrightarrow [n]$ in $\mathbf{\Delta}$



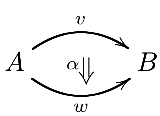
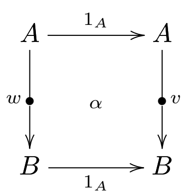
commutes. Showing that we get a double category with these cells is just an algebraic calculation which we omit.

Remarks: (1) The t has to go in the direction from GU to VF. The other direction produces a hexagon in which none of the arrows compose.

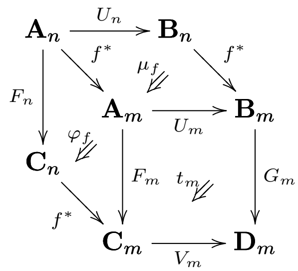
(2) If we take U and V to be identities, we get a colax transformation $t : G \longrightarrow F$. This direction is inevitable. It suggests that a weak double category $\mathbb{A}$ has a horizontal 2-category $\mathbb{H}$or$\mathbb{A}$ in which the 2-cells

 are special cells 

but has a vertical bicategory $\mathbb{V}$ert$\mathbb{A}$ in which the 2-cells

 are special cells 

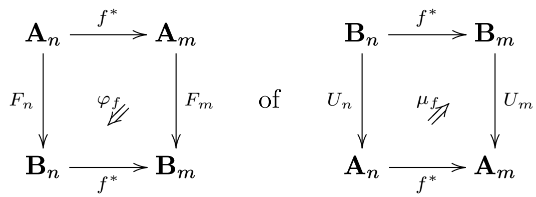
(3) Commutative hexagons set off warning bells about composability. Any such worries should be put to rest by noting that the condition on t is in fact commutativity of the cube



(The hidden faces can easily be guessed. They correspond to the top path in the hexagon.)

The notion of an adjoint pair of wobbly functors in which the right adjoint is lax and the left colax can now be formalized. It is a conjoint pair in $\mathbb{W}\mathrm{ob}$.

Theorem 2.8. A lax wobbly functor $U : \mathbb{B} \rightsquigarrow \mathbb{A}$ has a colax wobbly left adjoint F if and only if each $U_n : \mathbf{B}_n \longrightarrow \mathbf{A}_n$ has a left adjoint $F_n : \mathbf{A}_n \longrightarrow \mathbf{B}_n$ and for each translation $f : [m] \longrightarrow [n]$, the mate



is an isomorphism.

Proof. The mates of the laxity morphisms of U give the colaxity morphisms of F. That they satisfy the coherence conditions follow easily from those for U using the calculus of mates.

Q.E.D.

Thus the existence of wobbly adjoints is reduced to the notions of adjoints and their preservation in the usual categorical sense, i.e. up to canonical isomorphism. In fact it is not necessary to check preservation for all translations, only for increments $[1] \longrightarrow [n]$ and $[0] \longrightarrow [1]$. That is, if $F_n\langle v_n\rangle = \langle w_n\rangle$ then $F_1(v_k)$ is isomorphic to w_k, and also that domains and codomains are preserved.

3 Weak equivalences

In a 2-category (or a bicategory) there is the standard notion of equivalence 1-cells, namely $u : B \longrightarrow A$ is an equivalence if it has a pseudo-inverse, i.e. there is a 1-cell $f : A \longrightarrow B$ such that $fu \cong 1_B$ and $uf \cong 1_A$. Given the first isomorphism we can always choose the second to be the unit of an adjunction $f \dashv u$. These are adjoint equivalences. In more concrete 2-categories there is an easier and sometimes weaker notion of equivalence, namely full and faithful and representative, i.e. essentially surjective on objects.

That is the case for the 2-category of weak double categories and strong functors. A strong functor $U : \mathbb{B} \longrightarrow \mathbb{A}$ is said to be *full and faithful* if

(1) for every B, B' in $\mathbb{B}$ and horizontal $f : UB \longrightarrow UB'$ there is a unique horizontal $g : B \longrightarrow B'$ such that $f = Ug$,

(2) for every pair of vertical arrows $w : B \longrightarrow \bar{B}$ and $w' : B' \longrightarrow \bar{B}'$ and every cell $\alpha : Uw \longrightarrow Uw'$ there is a unique cell $\beta : w \longrightarrow w'$ such that $\alpha = U\beta$.

The conditions (1) and (2) are not independent. (2) implies (1). On the other hand, in the presence of (1), (2) can be weakened to the following: for every boundary

$$\begin{array}{ccc} B & \xrightarrow{\ b\ } & B' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle w'} \\ \bar{B} & \xrightarrow[\ \bar{b}\ ]{} & \bar{B}' \end{array}$$

every cell

$$\begin{array}{ccc} UB & \xrightarrow{\ Ub\ } & UB' \\ {\scriptstyle Uw}\downarrow & \alpha & \downarrow{\scriptstyle Uw'} \\ U\bar{B} & \xrightarrow[\ U\bar{b}\ ]{} & U\bar{B}' \end{array}$$

lifts to a unique β with the given boundary and $U\beta = \alpha$.

Put another way, U is full and faithful if and only if the functors $U_0 : \mathbf{B}_0 \longrightarrow \mathbf{A}_0$ and $U_1 : \mathbf{B}_1 \longrightarrow \mathbf{A}_1$ are in the usual sense.

The generalization of "essentially surjective on objects" is less straightforward. We say that U is *representative* if for every compatible path $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \ldots \xrightarrow{v_n} A_n$ of vertical arrows of $\mathbb{A}$ there is a compatible path $B_0 \xrightarrow{w_1} B_1 \xrightarrow{w_2} \ldots \xrightarrow{w_n} B_n$ and compatible morphisms

$$Uw_k \cong v_k,$$

i.e. isomorphisms α_i

$$\begin{array}{ccc} UB_0 & \longrightarrow & A_0 \\ {\scriptstyle Uw_1}\downarrow & \alpha_1 & \downarrow{\scriptstyle v_1} \\ UB_1 & \longrightarrow & A_1 \\ {\scriptstyle Uw_2}\downarrow & \alpha_2 & \downarrow{\scriptstyle v_2} \\ \vdots & \vdots & \vdots \\ {\scriptstyle Uw_n}\downarrow & \alpha_n & \downarrow{\scriptstyle v_n} \\ UB_n & \longrightarrow & A_n \end{array}$$

That is, U is representative if and only if each $U_n : \mathbf{B}_n \longrightarrow \mathbf{A}_n$ is essentially surjective on objects. This certainly holds for strong equivalences and does not follow from the corresponding conditions on U_0 and U_1. In fact no finite number of them suffice.

Consider the following example. Let $U : \mathbf{B} \longrightarrow \mathbf{A}$ be a functor such that $U_k : B_k \longrightarrow A_k$ is onto for all $k < n$ but $U_n : B_n \longrightarrow A_n$ (U on compatible paths of length k) is not. Examples of these exist. Let $\mathbf{A}$ be the totally ordered set $[n] = \{0, 1, \ldots, n\}$ and B the disjoint union of all full subcategories with only n objects, and $U : \mathbf{B} \longrightarrow \mathbf{A}$ given by the inclusions. Then any path of length $k < n$ is U of one in $\mathbf{B}$, i.e. $U_k : B_k \longrightarrow A_k$ is onto, but the maximal chain in $\mathbf{A}$ is not, i.e. U_n is not onto. Now from any such U we construct a double functor. $\mathbb{A}$ is $\mathbb{V}\mathrm{ert}\mathbf{A}$, i.e. the morphisms of $\mathbf{A}$ are vertical arrows of $\mathbb{A}$ and the only horizontal arrows and cells are horizontal identities. $\mathbb{B}$ is somewhat similar. Its objects are those of $\mathbf{B}$ and its vertical arrows are the morphisms of $\mathbf{B}$. There is a unique horizontal arrow $B \longrightarrow B'$ if $UB = UB'$ and none otherwise. There is a unique cell

$$\begin{array}{ccc} B & \longrightarrow & B' \\ {\scriptstyle v}\downarrow & \beta & \downarrow{\scriptstyle v'} \\ \bar{B} & \longrightarrow & \bar{B}' \end{array}$$

if and only if $Uv = Uv'$ and none otherwise. $U : \mathbb{B} \longrightarrow \mathbb{A}$ is given by U in the obvious way. As long as $n \geq 2$, U is a full and faithful double functor. The $U_k : \mathbf{B}_k \longrightarrow \mathbf{A}_k$ are essentially surjective on objects for all $k < n$ but not for $k = n$.

Theorem 3.1. The following conditions on a strong double functor $U : \mathbb{B} \longrightarrow \mathbb{A}$ are equivalent.
(1) U is full and faithful and representative.
(2) Each $U_n : \mathbf{B}_n \longrightarrow \mathbf{A}_n$ is an equivalence of categories.
(3) U has a wobbly pseudo-inverse.

Proof. The equivalence of (1) and (2) follows from the discussion above. That (3) implies (2) is clear. Now assume that each U_n is an equivalence. Then U_n has a left adjoint F_n and the mates of the structural isomorphisms for U are isomorphisms. So by Theorem 2.8 the F_n give a wobbly pseudo-functor F left adjoint to U. But the adjunctions are isomorphisms so F is a wobbly pseudo-inverse.

Q.E.D.

We call double functors satisfying the equivalent conditions of the theorem, *weak equivalences.*

Remark: We require weak equivalences to be strong double functors. It is possible to have a strictly lax or colax functor U with all of the U_n equivalences but such U are not really equivalences in any reasonable sense. Consider the following example of two monoidal structures on the same category for which the identity is colax. A monoidal category is (or can be viewed as) a double category with one object and one horizontal arrow. The category is $\mathbf{Set} \times \mathbf{Set}$ with tensors $(A, B) \otimes (C, D) = (A \times C, A \times D + B \times C)$ and $(A, B) \boxtimes (C, D) = (A \times C, A \times D + B)$. The unit I is $(1, 0)$ in each case. The identity functor

$$\mathrm{Id} : (\mathbf{Set} \times \mathbf{Set}, \otimes, I) \longrightarrow (\mathbf{Set} \times \mathbf{Set}, \boxtimes, I)$$

is colax with colaxity morphisms given by projection. The inverse

$$\mathrm{Id} : (\mathbf{Set} \times \mathbf{Set}, \boxtimes, I) \longrightarrow (\mathbf{Set} \times \mathbf{Set}, \otimes, I)$$

is lax. All the U_n here are identities. Yet these two monoidal categories are quite different. E.g. one is symmetric and the other not.

4 Horizontal invariance

We recall from [1] the notion of horizontal invariance of double categories.

Definition 4.1. ([1], 2.4) A weak double category $\mathbb{A}$ is *horizontally invariant* if for every vertical arrow $v : A \longrightarrow\!\!\bullet\!\!\rightarrow B$ and horizontal isomorphisms $f : A' \longrightarrow A$ and $g : B' \longrightarrow B$ there exists a vertical arrow $v' : A' \longrightarrow\!\!\bullet\!\!\rightarrow B'$ and a horizontally invertible double cell α

$$\begin{array}{ccc} A' & \xrightarrow{\ f\ } & A \\ {\scriptstyle v'}\big\downarrow\bullet & \alpha & \bullet\big\downarrow{\scriptstyle v} \\ B' & \xrightarrow[\ g\ ]{} & B \end{array}$$

This is a weak condition to impose on a double category and those that don't satisfy it are definitely strange. But it has some nice consequences.

First of all the lifting, if it exists, is unique up to special isomorphism: if $(\bar{v'}, \bar{\alpha})$ is another, then

$$\begin{array}{ccc} A' & = & A' \\ {\scriptstyle \bar{v'}}\big\downarrow\bullet & \alpha^{-1}\bar{\alpha} & \bullet\big\downarrow{\scriptstyle v'} \\ B' & = & B' \end{array}$$

is a horizontal isomorphism.

Recall the following results from [2]. It shows that horizontal invariance is more "functorial" than it looks.

Theorem 4.2. The following are equivalent for a weak double category $\mathbb{A}$
(1) $\mathbb{A}$ is horizontally invariant,
(2) Every horizontal isomorphism has a companion,
(3) Every horizontal isomorphism has a conjoint.

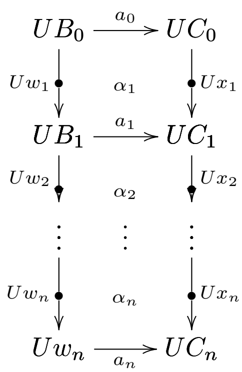
Another result from [2] is the following.

Proposition 4.3. If A is horizontally isomorphic to B then they are vertically equivalent. More precisely, if $f : A \longrightarrow B$ is an isomorphism then the companion $f_* : A \longrightarrow\!\!\bullet\!\!\rightarrow B$ and the conjoint $f^* : B \longrightarrow\!\!\bullet\!\!\rightarrow A$ give an adjoint equivalence.

The following result shows how horizontal invariance simplifies the definition of weak equivalence, especially the "representative" part.

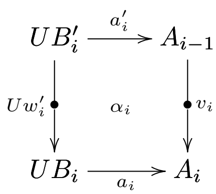
Proposition 4.4. Let $\mathbb{B}$ be horizontally invariant and $U : \mathbb{B} \longrightarrow \mathbb{A}$ a strong double functor. Then U is a weak equivalence if and only if U_0 and U_1 are equivalences.

Proof. We have to show that if U_0 and U_1 are equivalences, then so is $U_n : \mathbf{B}_n \longrightarrow \mathbf{A}_n$ for all n. We've already said that U_n is full and faithful but we indicate why, for completeness. Consider two objects $\langle w_i \rangle$ and $\langle x_i \rangle$ in $\mathbf{B}_n$ and a morphism $\langle \alpha_i \rangle : U_n\langle w_i \rangle \longrightarrow U_n\langle x_i \rangle$

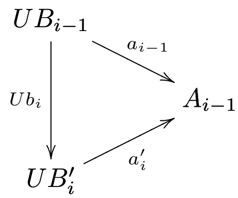


For each i there exists a unique $b_i : B_i \longrightarrow C_i$ such that $Ub_i = a_i$ and also for each $i > 0$ there exists a unique $\beta_i : w_i \longrightarrow x_i$ such that $U\beta_i = \alpha_i$. By faithfulness of U_0 the domain and codomain of β_i must be b_{i-1} and b_i respectively. So $\langle \beta_i \rangle$ is a morphism $\langle w_i \rangle \longrightarrow \langle x_i \rangle$ in $\mathbf{B}_n$, and $U\langle \beta_i \rangle = \langle \alpha_i \rangle$.

Now let $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \ldots \xrightarrow{v_n} A_n$ be an object of A_n. For each $i > 0$ there exists a w'_i and an isomorphism $\alpha_i : Uw'_i \longrightarrow v_i$



But the domains and codomains of the w'_i don't necessarily match up to give an object of $\mathbf{B}_n$, let alone an isomorphism in $\mathbf{A}_i$. For each $i > 0$, the isomorphism $(a'_i)^{-1}a_{i-1} : UB_{i-1} \longrightarrow UB'_i$ lifts to a unique isomorphism $b_i : B_{i-1} \longrightarrow B'_i$ such that



(We take $B_0 = B'_1$ and $a_0 = a'_1, b_1 = 1_{B'_1}$.) By horizontal invariance there exist w_i and an isomorphism β_i

$$\begin{array}{ccc} B_{i-1} & \xrightarrow{b_i} & B'_i \\ {\scriptstyle w_i}\downarrow\bullet & \beta_i & \bullet\downarrow{\scriptstyle w'_i} \\ B_i & = & B_i \end{array}$$

Then $\langle w_i \rangle$ is an object of $\mathbf{B}_n$ and $\langle \alpha_i U \beta_i \rangle$ an isomorphism $U\langle w_i \rangle \longrightarrow \langle v_i \rangle$.

Q.E.D.

The following result, the "steadying lemma" together with Theorem 4.8, are the main results of the paper.

Theorem 4.5. Suppose that $\mathbb{B}$ is horizontally invariant. Then every colax (resp. lax, strong) wobbly functor $F : \mathbb{A} \rightsquigarrow \mathbb{B}$ is isomorphic to a colax (resp. lax, strong) double functor $G : \mathbb{A} \longrightarrow \mathbb{B}$ by an isomorphism that is the identity on $\mathbf{A}_0$.

Proof. Given such an F we construct G as follows. On objects and horizontal arrows G is equal to F. Given a vertical arrow $v : A \longrightarrow\!\!\bullet\!\!\longrightarrow \bar{A}$ in $\mathbb{A}$, we have the isomorphisms $\delta_0 v$ and $\delta_1 v$ so there exists a vertical arrow Gv and an invertible cell tv

$$\begin{array}{ccc} GA = FA & \xrightarrow{\delta_0 v} & \partial_0 Fv \\ {\scriptstyle Gv}\downarrow\bullet & tv & \bullet\downarrow{\scriptstyle Fv} \\ G\bar{A} = F\bar{A} & \xrightarrow[\delta_1 v]{} & \partial_1 Fv \end{array}$$

This defines G on vertical arrows, once we've made a choice of these for each v. On cells

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle v}\downarrow\bullet & \alpha & \bullet\downarrow{\scriptstyle v'} \\ \bar{A} & \xrightarrow[\bar{f}]{} & ovA' \end{array}$$

$G_1 : \mathbf{A}_1 \longrightarrow \mathbf{B}_1$ is defined by transporting $F_1 : \mathbf{A}_1 \longrightarrow \mathbf{B}_1$ along the isomorphisms tv, i.e.

$$G\alpha = t(v')^{-1} F(\alpha) t(v)$$

which is horizontally functorial. Naturality of δ_0 gives

$$\begin{array}{rcl} \partial_0 G\alpha & = & \partial_0 t(v')^{-1} \partial_0 F(\alpha) \partial_0 tv \\ & = & \delta_0(v')^{-1} \partial_0 F(\alpha) \delta_0 v \\ & = & \delta_0(v')^{-1} \delta_0(v') F(\partial_0 \alpha) = F(f). \end{array}$$

That is,

$$\begin{array}{ccc} \mathbf{A}_1 & \xrightarrow{\partial_0} & \mathbf{A}_0 \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G_0} \\ \mathbf{B}_1 & \xrightarrow[\partial_0]{} & \mathbf{B}_0 \end{array}$$

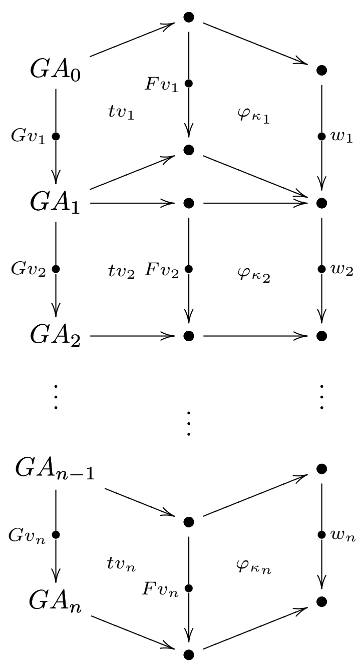
commutes, as does the similar one with ∂_1. It follows that G can be extended to all

$$G_n : \mathbf{A}_n \longrightarrow \mathbf{B}_n$$

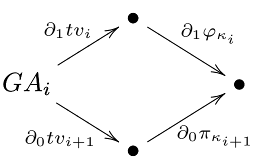
This G_n is isomorphic to F_n. Take

$$A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \dots \xrightarrow{v_n} A_n$$

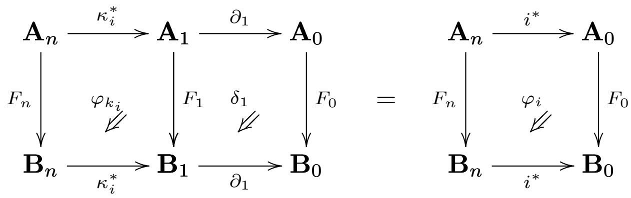
in $\mathbf{A}_n$. For each $i > 0$ consider the increments $\kappa_i : [1] \longrightarrow [n]$ as well as the functions $i : [0] \longrightarrow [n]$ which have the value i.

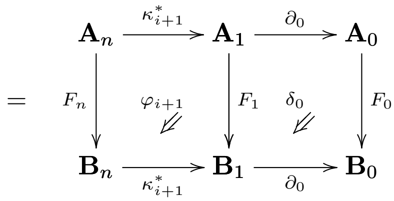


For this to be a morphism in $\mathbf{B}_n$ we need that the lozenges



commute. By definition, $\partial_1 tv_i = \delta_1 v_i$ and $\partial_0 tv_{i+1} = \delta_0 v_{i+1}$. The coherence conditions on φ give





If we apply this to $\langle v_i \rangle$ we get the required commutativity. It follows that $\langle \varphi_{\kappa_i} \cdot tv_i \rangle$ is a morphism $G_n\langle v_i \rangle \longrightarrow F_n\langle v_i \rangle$, and as each component is an isomorphism we get $G_n \cong F_n$.

By Proposition 2.3 G is a wobbly functor isomorphic to F, and since G preserves domains and codomains (strictly) it is a double functor, colax, strict or lax according to what F is.

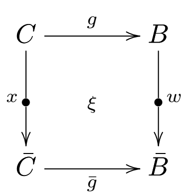
Q.E.D.

Corollary 4.6. If a strong double functor $U : \mathbb{B} \longrightarrow \mathbb{A}$ is a weak equivalence and $\mathbb{B}$ is horizontally invariant, then U is a strong equivalence.

There is a converse to Theorem 4.5.

Proposition 4.7. $\mathbb{B}$ is horizontally invariant if every wobbly functor $F : \mathbb{A} \rightsquigarrow \mathbb{B}$ is isomorphic to a double functor $G : \mathbb{A} \longrightarrow \mathbb{B}$ by an isomorphism that is the identity on $\mathbf{A}_0$.

Proof. Let $w : B \longrightarrow\!\!\!\bullet\; \bar{B}$ be a vertical arrow in $\mathbb{B}$ and $g : C \longrightarrow B$, $\bar{g} : \bar{C} \longrightarrow \bar{B}$ be horizontal isomorphisms. Let $\mathbb{A}$ be the double category $\mathbf{2}_v$ which has two objects, 0 and 1, and one non identity vertical arrow $v : 0 \longrightarrow\!\!\!\bullet\; 1$, and nothing else except the requisite identities. Define $F : \mathbf{2}_v \rightsquigarrow \mathbb{B}$ by $F(0) = C, F(1) = \bar{C}, F(v) = w$. This is a wobbly functor and the $G : \mathbf{2}_v \longrightarrow \mathbb{B}$ gives an $x : C \longrightarrow\!\!\!\bullet\; \bar{C}$ and the isomorphism $G \cong F$ gives the invertible cell



Q.E.D.

REMARK: The condition that the isomorphism $G \cong F$ be the identity on $\mathbf{A}_0$ is needed as can be seen by taking $\mathbb{B}$ to be the double category represented by

$$
\begin{array}{ccc}
0' & \longrightarrow & 0 \\
 & & \big\downarrow\!\bullet \\
1' & \longrightarrow & 1
\end{array}
$$

where $0' \longrightarrow 0$ and $1' \longrightarrow 1$ are isomorphisms.

We now return to the question of adjoints which was the starting point of our discussion.

Theorem 4.8. Let $U : \mathbb{B} \longrightarrow \mathbb{A}$ be a lax functor with $\mathbb{B}$ horizontally invariant. Assume that U_0 and U_1 have left adjoints F_0 and F_1 and that the mates

$$
\begin{array}{ccc}
\mathbf{A}_1 & \xrightarrow{\partial_i} & \mathbf{A}_0 \\
{\scriptstyle F_1}\big\downarrow & \overset{\delta_i}{\swarrow} & \big\downarrow{\scriptstyle F_0} \\
\mathbf{B}_1 & \xrightarrow[\partial_i]{} & \mathbf{B}_0
\end{array}
$$

of the identities

$$
\begin{array}{ccc}
\mathbf{B}_1 & \xrightarrow{\partial_i} & \mathbf{B}_0 \\
{\scriptstyle U_1}\big\downarrow & /\!/ & \big\downarrow{\scriptstyle U_0} \\
\mathbf{A}_1 & \xrightarrow[\partial_i]{} & \mathbf{A}_0
\end{array}
$$

are isomorphisms. Then U has a colax left adjoint $G : \mathbb{A} \longrightarrow \mathbb{B}$ with $G_0 = F_0$ and $G_1 \cong F_1$.

Proof. We take $G_0 = F_0$ and choose G_1 as in the proof of Theorem 4.5. Then we have the isomorphism $t : G_1 \longrightarrow F_1$ such that $\partial_i t = \delta_i$. It follows that G_1 is also left adjoint to U_1 and that the mates of the identities $\partial_i U_1 = U_0 \partial_i$ are themselves identities. In particular we have commutativity

$$
\begin{array}{ccc}
\mathbf{A}_1 & \xrightarrow{\partial_i} & \mathbf{A}_0 \\
{\scriptstyle G_1}\big\downarrow & /\!/ & \big\downarrow{\scriptstyle G_0} \\
\mathbf{B}_1 & \xrightarrow[\partial_i]{} & \mathbf{B}_0
\end{array}
$$

Therefore G_0 and G_1 extend to $G_n : \mathbf{A}_n \longrightarrow \mathbf{B}_n$, which we claim is left adjoint to $U_n : \mathbf{B}_n \longrightarrow \mathbf{A}_n$. Given objects $\langle v_n \rangle$ in $\mathbf{A}_n$ and $\langle w_n \rangle$ in $\mathbf{B}_n$, we have natural bijections

$$\begin{array}{c} G_n\langle v_n\rangle \xrightarrow{\ \beta\ } \langle w_n\rangle \\ \hline \langle G_1 v_n\rangle \xrightarrow{\ \beta\ } \langle w_n\rangle \\ \hline \langle G_1 v_n \xrightarrow{\ \beta_n\ } w_n\rangle \\ \hline \langle v_n \xrightarrow{\ \alpha_n\ } U_1 w_n\rangle \\ \hline \langle v_n\rangle \xrightarrow{\ \alpha\ } \langle U_1 w_n\rangle \\ \hline \langle v_n\rangle \xrightarrow{\ \alpha\ } U_n\langle w_n\rangle \end{array}$$

The middle bijection is between strings of cells whose domains and codomains match up, so we need to check that if the α_i correspond to β_i by the adjointness, then

$$\partial_1\alpha_i = \partial_0\alpha_{i+1} \Leftrightarrow \partial_1\beta_i = \partial_1\beta_{i+1}$$

And this holds precisely because the identities $G_0\partial_i = \partial_i G_1$ and $\partial_i U_1 = U_0\partial_i$ are mates.

Finally, for any translation $f : [m] \longrightarrow [n]$, the colaxity morphisms

$$\begin{array}{ccc} \mathbf{A}_n & \xrightarrow{\ f^*\ } & \mathbf{A}_m \\ {\scriptstyle G_n}\downarrow & \stackrel{\gamma_f}{\swarrow} & \downarrow{\scriptstyle G_m} \\ \mathbf{B}_n & \xrightarrow[\ f^*\ ]{} & \mathbf{B}_m \end{array}$$

are identities, mates of the corresponding identities for U. Thus by Theorem 2.8 U has G as left adjoint, and G is a colax functor $\mathbb{A} \longrightarrow \mathbb{B}$.

Q.E.D.

References

[1] M. Grandis and R. Paré, *Limits in double categories*, Cahiers Topologie Géom. Différentielle Catég. 40, no. 3 (1999), p. 162-220.

[2] M. Grandis and R. Paré, *Adjoint for double categories*, Cahiers Topologie Géom. Différentielle Catég. 45, no. 3 (2004), p. 193-240.

[3] S. Lack, *Notes on Paré's Theory of double categories*, Private communication, Feb. 2014.