Poisson bracket under mappings

Dedicated to Prof. Yoshie Katsurada on her sixtieth birthday

By Takayuki NÔNO and Fumitake MIMURA

1. Introduction

The main purpose of this paper is to investigate the behavior of Poisson bracket under mappings in terms of differential geometry. The formalism of Poisson bracket relates with the problem in quantizing of dynamical systems. Hence this formalism plays an important role in quantum physical theory. The basic concepts in this paper are given in the literatures by R. Hermann [1], [2].

2. The Poisson bracket defined by a closed 2-differential form

Let M be a differentiable manifold, ω a given closed 2-differential form on M, i. e.,

$$\text{(2. 1)} \qquad d\omega = 0\,,$$

where d denotes the exterior derivative.

Let M_p be the tangent space to M at p, then a vector $v \in M_p$ is said to be a characteristic vector of ω if

$$\text{(2. 2)} \qquad v \lrcorner\, \omega = 0 \quad \text{i. e.,} \quad \omega(v, M_p) = 0\,.$$

The set $C_p(\omega)$ of all these characteristic vectors forms a subspace of M_p.

The following theorem is shown by R. Hermann ([1], 122–123).

THEOREM 2. 1. *Let $v_1, \cdots, v_m$ be a basis for M_p such that $v_1, \cdots, v_n$ form a basis for $C_p(\omega)$; $\omega_1, \cdots, \omega_m$ the dual basis of $v_1, \cdots, v_m$ i. e.,*

$$\text{(2. 3)} \qquad \omega_i(v_j) = \delta_{ij} \qquad \textit{for} \quad 1 \leq i, j \leq m\,,$$

then ω is written at p as follows:

$$\text{(2. 4)} \qquad \omega = \sum_{i,j>n} a^{ij}\omega_i \wedge \omega_j\,, \qquad \det_{n<i,j\leq m}(a^{ij}) \neq 0\,.$$

Let $F(M)$ be the ring of C^∞ real valued functions on M, then a function $f \in F(M)$ is said to be an integral of the characteristics of ω if

(2. 5) $$v(f)=0 \quad \text{for all} \quad v\in C_p(\omega), \quad \text{all} \quad p\in M.$$

The set $F(\omega)$ of all these integrals forms a subring of $F(M)$.

THEOREM 2. 2. *A function $f\in F(M)$ is an integral of the characteristics of ω, i.e., $f\in F(\omega)$ if and only if there exists a vector v_f determined in a unique manner up to modulo $C_p(\omega)$ such that*

(2. 6) $$\mathrm{d}f=v_f \rfloor \omega .$$

PROOF. Let us assume (2. 6), then for any vector $v\in C_p(\omega)$ we see

(2. 7) $$v(f)=v\rfloor \mathrm{d}f=v\rfloor v_f \rfloor \omega=-v_f\rfloor(v\rfloor\omega)=0,$$

i. e., $f\in F(\omega)$.

The converse is already stated by R. Hermann without precise proof ([2], 34). This can be shown as follows. For the basis choosen as in theorem 2. 1, set

(2. 8) $$v_f=\sum_{n<k\leq m}\xi^k v_k,$$

then by theorem 2. 1 we have

(2. 9) $$\begin{aligned}(v_f\rfloor\omega)(v_l) &= \sum_{i,j>n} a^{ij}\omega_i\wedge\omega_j(\xi^k v_k, v_l)\\ &= \begin{cases} 0, & \text{if } 1\leq l\leq n,\\ 2\sum_{n<i\leq m} a^{il}\xi^i, & \text{if } n<l\leq m.\end{cases}\end{aligned}$$

On the other hand, since $v_l\in C_p(\omega)$ for $1\leq l\leq n$, we see

(2 .10) $$(\mathrm{d}f)(v_l)=v_l(f)=\begin{cases}0, & \text{if } 1\leq l\leq n,\\ v_l(f), & \text{if } n<l\leq m.\end{cases}$$

Since $\det_{n<i,l\leq m}(a^{il})\neq 0$, the following linear equations

(2. 11) $$2\sum_{n<i\leq m} a^{il}\xi^i=v_l(f), \quad n<l\leq m,$$

can be solved with respect to variables $\xi^i(n<i\leq m)$; hence the existence of v_f is proved. The rest part of this theorem is clear. Hence the theorem is proved.

For two functions f, $g\in F(\omega)$, the Poisson bracket operation $\{f, g\}$ is defined by the following:

(2. 12) $$\{f, g\}(p)=v_f(g), \quad p\in M,$$

and by this operation $F(\omega)$ makes into Lie algebra ([1], 176–177; [2], 34–35).

3. Behavior of Poisson bracket defined by a closed 2-differential form under mappings

The set $C_p(\omega)$ of all characteristics forms a subspace of M_p. The number $\dim M - \dim C_p(\omega)$ is called the rank of ω at $p \in M$; and if its rank is the same at every point of M, ω is said to be of constant rank on M.

THEOREM 3.1. *Let ω be a closed 2-differential form on M of constant rank on M, then for each point $p \in M$ a vector $v \in M_p$ is a characteristic vector of ω, i.e., $v \in C_p(\omega)$, if and only if*

$$v(f) = 0 \qquad \text{for all} \quad f \in F(\omega). \tag{3.1}$$

PROOF. As well known ([1], 124–125), each point $p \in M$ has a neighborhood U with coordinate system $(x_1, \cdots, x_m)$ such that ω takes the following canonical form in U:

$$\omega = \sum_{n<i<j\leq m} dx_i \wedge dx_j. \tag{3.2}$$

Hence at any point $q \in U$; $\left(\frac{\partial}{\partial x_1}\right)_q, \cdots, \left(\frac{\partial}{\partial x_n}\right)_q$ becomes a basis of $C_q(\omega)$. Moreover we know that there exist a neighborhood $V \subset U$ and C^∞ real valued functions f_i such as $f_i = x_i$ on V and $f_i = 0$ on $M \backslash U$. We see these for functions f_i are integrals of characteristics of ω for $n < i \leq m$, i.e., $f_i \in F(\omega)$ $n < i \leq m$. Let assume (3.1) and set

$$v = \sum_{1\leq j\leq m} \xi^j \left(\frac{\partial}{\partial x_j}\right)_p, \tag{3.3}$$

then we have

$$v(f_i) = (df_i)_p(v) = (dx_i)_p \left(\xi^j \left(\frac{\partial}{\partial x_j}\right)_p\right) = \xi^i. \tag{3.4}$$

Since $f_i (n < i \leq m)$ belong to $F(\omega)$, we see $v(f_i) = 0$ for $n < i \leq m$; hence we have $\xi^i = 0$ for $n < i \leq m$ i.e., $v \in C_p(\omega)$.

The converse is clear from the definition of $F(\omega)$. Thus the theorem is proved.

Let M, M' be two manifolds with C^∞ map ϕ: $M \to M'$, ω' a closed 2-differential form on M'. Define a form ω such as

$$\omega = \phi^*(\omega'), \tag{3.5}$$

where ϕ^* denotes the pull-back map induced from ϕ. Of course ω becomes a closed 2-differential form on M. By this definition it can be seen that

(3. 6) $$\phi_*^{-1}\Big(C_{\phi(p)}(\omega')\Big)\subset C_p(\omega)\,.$$

THEOREM 3. 2. *Let ω' be of constant rank on M', then the relation*

(3. 7) $$C_p(\omega)=\phi_*^{-1}\Big(C_{\phi(p(}(\omega')\Big)\quad \textit{for all}\quad p\in M$$

holds, if and only if

$$\phi^* F(\omega')\subset F(\omega)\,.$$

PROOF. Let assume (3. 8). For any $f'\in F(\omega')$ and any $v\in C_p(\omega)$, since $\phi^*f'\in F(\omega)$ we see

(3. 9) $$\Big(\phi_*(v)\Big)(f')=v(\phi^* f')=0\,.$$

Hence by theorem 3. 1 we have

(3. 10) $$\phi_*(v)\in C_{\phi(p)}(\omega')\quad \text{i. e.,}\quad C_p(\omega)\subset\phi_*^{-1}\Big(C_{\phi(p)}(\omega')\Big).$$

Therefore by (3. 6) and (3. 10) we get (3. 7).

The converse of this theorem (the assumption of constant rank is not necessary) is the result of R. Hermann ([2], 37). Thus the theorem is proved.

The following theorem is shown by R. Hermann ([2], 37–40).

THEOREM 3. 3. *Suppose f', $g'\in F(\omega')$, and set*

(3. 11) $$f=\phi^*(f')\,,\qquad g=\phi^*(g')\,.$$

And if a map ϕ: $M\to M'$ satisfies (3. 7) *and the map*

(3. 12) $$M_p\to M'_{\phi(p)}/C_{\phi(p)}(\omega')$$

is onto, then ϕ^*: $F(\omega')\to F(\omega)$ defines a Poisson bracket homomorphism, i. e.,

(3. 13) $$\phi^*\Big(\{f',\,g'\}\Big)=\Big\{\phi^*(f'),\,\phi^*(g')\Big\}.$$

In the following, suppose $M=M'$ and ϕ is a diffeomorphism on M. For the given closed 2-differential form ω on M we set $\omega'=\phi^*(\omega)$.

THEOREM 3. 4. *Suppose ϕ is a diffeomorphism on M, then we have*

(3. 14) $$\phi^*\Big(F(\omega)\Big)=F(\omega')\,.$$

PROOF. Since ϕ is a diffeomorphism we see $\phi_*(M_p)=M_{\phi(p)}$, hence it follows that ([2], 36)

(3. 15) $$\phi_*^{-1}\Big(C_{\phi(p)}(\omega)\Big)=C_p(\omega')\quad \text{for all}\quad p\in M\,,$$

therefore we have ([2], 37)

$$\phi^*\big(F(\omega)\big)\subset F(\omega'). \tag{3.16}$$

By the same way, ϕ^{-1} is a diffeomorphism we get

$$F(\omega')\subset\phi^*\big(F(\omega)\big), \tag{3.17}$$

and hence (3.16) and (3.17) imply (3.14). Thus the theorem is proved.

THEOREM 3.5. *Suppose ϕ is a diffeomorphism on M and ω is of constant rank on M, then the relation*

$$C_p(\omega)=C_p(\omega') \qquad \text{for all } p\in M, \tag{3.18}$$

holds if and only if

$$F(\omega)=\phi^*\big(F(\omega)\big). \tag{3.19}$$

PROOF. Let assume (3.18), then we see

$$F(\omega)=F(\omega'), \tag{3.20}$$

and hence by theorem 3.4 we have (3.19).

Conversely assume (3.19), by theorem 3.4 we have (3.20). On the other hand, for any $f\in F(\omega)$ and any $v'\in C_p(\omega')$, since $f\in F(\omega')$ we see $v'(f)=0$. Hence by theorem 3.1 we have $v'\in C_p(\omega)$ i.e.,

$$C_p(\omega')\subset C_p(\omega). \tag{3.21}$$

By the same method as the above it follows that

$$C_p(\omega)\subset C_p(\omega'), \tag{3.22}$$

and hence (3.21) and (3.22) imply (3.18). Thus the theorem is proved.

THEOREM 3.6. *The relation* (3.18) *holds, if and only if*

$$\bar{\omega}\big(C_p(\omega), M_p\big)=\bar{\omega}\big(C_p(\omega'), M_p\big)=0 \qquad \text{for all } p\in M, \tag{3.23}$$

where

$$\phi^*(\omega)=\omega'=\omega+\bar{\omega}. \tag{3.24}$$

PROOF. Suppose (3.18), then it follows that

$$\omega^*\big(C_p(\omega), M_p\big)=\omega'\big(C_p(\omega), M_p\big)=0, \tag{3.25}$$

and hence we see

$$(\omega-\omega')\big(C_p(\omega), M_p\big)=0, \tag{3.26}$$

therefore we have (3.23).

Conversely suppose (3.23), then we see

$$\omega'\left(C_p(\omega),\, M_p\right)=\omega\left(C_p(\omega),\, M_p\right)+\bar{\omega}\left(C_p(\omega),\, M_p\right)=0\,, \tag{3.27}$$

and hence we have

$$C_p(\omega)\subset C_p(\omega')\,. \tag{3.22}$$

On the other hand, it follows that

$$\omega'\left(C_p(\omega'),\, M_p\right)=\omega\left(C_p(\omega'),\, M_p\right)+\bar{\omega}\left(C_p(\omega), M_p\right), \tag{3.28}$$

hence we have $\omega(C_p(\omega'), M_p)=0$, i.e.,

$$C_p(\omega')\subset C_p(\omega)\,, \tag{3.21}$$

therefore (3.21) and (3.22) imply (3.18). Thus the theorem is proved.

THEOREM 3.7. *Suppose ω is of constant rank on M, then the form ω is invariant under the mapping ϕ^*, i.e.,*

$$\phi^*(\omega)=\omega\,, \tag{3.29}$$

if and anly if the Poisson bracket is invariant under the mapping ϕ^*, i.e.,

$$F(\omega)=\phi^*F(\omega), \tag{3.30}$$

$$\{f,\, g\}^{\omega}=\{f,\, g\}^{\omega'} \qquad \textit{for all} \quad f,\, g\in F(\omega), \tag{3.31}$$

where $\{f, g\}^{\omega}$ means the Poisson bracket defined by the form ω.

PROOF. Suppose (3.30) and (3.31). By (3.30) and the theorem 3.4 we see $F(\omega)=F(\omega')$; hence for any $f, g\in F(\omega)$, the Possion bracket $\{f, g\}^{\omega'}$ can be defined. Let

$$df=v_f^{\omega}\lrcorner\,\omega\,, \qquad df=v_f^{\omega'}\lrcorner\,\omega'\,, \tag{3.32}$$

at $p\in M$, then it follows that

$$\{f,\, g\}^{\omega}(p)=v_f^{\omega}(g)\,, \qquad \{f,\, g\}^{\omega'}(p)=v_f^{\omega'}(g)\,, \tag{3.33}$$

and hence we have

$$(v_f^{\omega'}-v_f^{\omega})(g)=0 \qquad \text{for all } g\in F(\omega)\,. \tag{3.34}$$

Hence by theorem 3.1 we see

$$v_f^{\omega}\equiv v_f^{\omega'} \quad \left(\mathrm{mod}\ C_p(\omega)\right). \tag{3,35}$$

Therefore from theorem 3.5 and theorem 3.6, it follows that

(3.36)
$$v_f^{\omega'} \lrcorner \omega' = \left(v_f^{\omega} + C_p(\omega)\right) \lrcorner (\omega + \bar{\omega})$$
$$= v_f^{\omega} \lrcorner \omega + v_f^{\omega} \lrcorner \bar{\omega},$$

hence we see

(3.37)
$$v_f^{\omega} \lrcorner \bar{\omega} = 0.$$

Now, choose the neighborhood U of p with coordinate system $(x_1, \cdots, x_m)$ as in the proof of theorem 3.1 such that ω takes the canonical form (3.2), then $\left(\frac{\partial}{\partial x_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_n}\right)_p$ form a basis for $C_p(\omega)$ and there exist functions $f_i \in F(\omega) (n < i \leq m)$ such that $(\mathrm{d}f_i)_p = (\mathrm{d}x_i)_p$. Since $f_i \in F(\omega)$, there exist vectors $v_{f_i}^p$ such that

(3.38)
$$(\mathrm{d}f_i)_p = v_{f_i}^{\omega} \lrcorner \omega \qquad n < i \leq m.$$

Hence by (3.37) we see

(3.39)
$$v_{f_i}^{\omega} \lrcorner \bar{\omega} = 0 \qquad n < i \leq m,$$

and since $v_{f_i}^{\omega}$ are linearly independent vectors which are determined in a unique manner up to modulo $C_p(\omega)$, the equation (3.39) implies $\omega = 0$. Hence we have (3.29).

The converse of this theorem is clear, therefore the theorem is proved.

Let G be a group of diffeomorphism of M, then G is said to be a group of symmetries with respect to ω if the form ω is invariant under any element of G. The theorem 3.7 indicates that G becomes a group of symmetries if and only if the Poisson bracket is invariant under any element of G.

4. Behavior of Poisson bracket difined by an arbitrary differential form under mappings

In the preceeding sections we defined the Poisson bracket by using of a closed 2-differential form and discussed the behavior of it under mappings. In this section we shall consider some probrems along the same line for a closed $(r+1)$-differential form ω on M.

Let $F^r(M)$ be a space of r-differential forms on M, $V(M)$ be a space of vector fields on M. Set

(4.1)
$$A^{r-1}(\omega) = \left\{\theta \in F^{r-1}(M) :\ \mathrm{d}\theta \in V(M) \lrcorner \omega\right\},$$

then for two forms θ, $\rho \in A^{r-1}(\omega)$ the Poisson bracket operation $\{\theta, \rho\}$ is defined by the following ([2], 167–171):

$$\{\theta, \rho\} = \left[X(\rho)\right], \tag{4.2}$$

where X is a vector field such as

$$d\theta = X \lrcorner \omega, \tag{4.3}$$

and $[X(\rho)]$ denote the class of $X(\rho)$ modulo $dF^{r-2}(M)$, i.e., an element of $A^{r-1}(\omega)/dF^{r-2}(M)$.

Let $C(\omega)$ denote a space of characteristic vector fields of ω, i.e., the vector fields $X \in V(M)$ such that

$$X \lrcorner \omega = 0. \tag{4.4}$$

Let M, M' be manifolds with C^∞ map ϕ: $M \to M'$, ω' given closed $(r+1)$-differential form on M', and define a closed $(r+1)$-differential form on M by

$$\omega = \phi^*(\omega'), \tag{4.5}$$

then we have

$$\phi_*^{-1}\big(C(\omega')\big) \subset C(\omega). \tag{4.5}$$

The mapping ϕ is said to be of maximal rank if

$$\phi_*\big(V(M)\big) = V(M'). \tag{4.6}$$

THEOREM 4.1. *Suppose ϕ is of maximal rank, then it follows that*

$$\phi_*^{-1}\big(C(\omega')\big) = C(\omega). \tag{4.7}$$

PROOF. Let $X \in C(\omega)$, then we see $X \lrcorner \omega = 0$. Let $X_1', \cdots, X_r'$ be vector fields on M'; then since ϕ is of maximal rank, there exist vector fields $X_1, \cdots, X_r$ on M such that

$$X_1' = \phi_*(X_i) \qquad 1 \leq i \leq r, \tag{4.8}$$

hence we have

$$\begin{aligned}\big(\phi_*(X_i) \lrcorner \omega'\big)(X_1', \cdots, X_r') &= \omega'\big(\phi_*(X), \phi_*(X_1), \cdots, \phi_*(X_r)\big)\\ &= \phi_*(\omega')(X, X_1, \cdots, X_r)\\ &= (X \lrcorner \omega)(X_1, \cdots, X_r)\\ &= 0.\end{aligned} \tag{4.9}$$

Therefore it follows that

$$\phi_*(X) \in C(\omega') \quad \text{i.e.,} \quad C(\omega) \subset \phi_*^{-1}\big(C(\omega')\big). \tag{4.10}$$

Relations (4.5) and (4.10) imply (4.7). Hence the theorem is proved.

THEOREM 4.2. *Suppose ϕ is of maximal rank, then it follows that*

$$\phi^*\left(A^{r-1}(\omega')\right) \subset A^{r-1}(\omega). \tag{4.11}$$

PROOF. Let $\theta' \in A^{r-1}(\omega')$, then since $d\theta' = X' \lrcorner\, \omega'$, we see

$$d\left(\phi^*(\omega')\right) = \phi^*(X' \lrcorner\, \omega'). \tag{4.12}$$

Since ϕ is of maximal rank, there exist $X \in V(M)$ such as $\phi_*(X) = X'$; hence it follows that

$$\begin{aligned} \phi^*(X' \lrcorner\, \omega')(X_1, \cdots, X_r) &= \omega'\left(X', \phi_*(X_1), \cdots, \phi_*(X_r)\right) \\ &= \omega'\left(\phi_*(X), \phi_*(X_1), \cdots, \phi_*(X_r)\right) \\ &= (X \lrcorner\, \omega)(X_1, \cdots, X_r), \end{aligned} \tag{4.13}$$

where $X_1, \cdots, X_r$ are vector fields on M.

Therefore by (4.12) and (4.13) we see

$$d\left(\phi^*(\theta)\right) = X \lrcorner\, \omega \quad \text{i.e.,} \quad \phi^*\left(A^{r-1}(\omega')\right) \subset A^{r-1}(\omega). \tag{4.14}$$

Hence the theorem is proved.

THEOREM 4.3. *Suppose ϕ is of maximal rank, then ϕ^* is a Poisson bracket homomorphism, i.e., for two forms θ', $\rho' \in A^{r-1}(\omega')$,*

$$\phi^*\left(\{\theta', \rho'\}\right) = \left\{\phi^*(\theta'), \phi^*(\rho')\right\}. \tag{4.15}$$

PROOF. Since

$$\phi^*\left(dF^{r-2}(M)\right) \subset dF^{r-2}(M'), \tag{4.16}$$

ϕ^* may be induced to a map of quotient classes. Let

$$d\theta' = X' \lrcorner\, \omega', \quad d\rho' = Y' \lrcorner\, \omega', \tag{4.16}$$

then it follows that

$$\{\theta', \rho'\} = \left[X'(\rho')\right] = [X' \lrcorner\, d\rho'] = [X' \lrcorner\, Y' \lrcorner\, \omega']. \tag{4.17}$$

On the other hand as in the proof of theorem 4.2, we see

$$d\left(\phi^*(\theta')\right) = X \lrcorner\, \omega, \quad d\left(\phi^*(\rho')\right) = Y \lrcorner\, \omega, \tag{4.18}$$

where

$$\phi_*(X) = X', \quad \phi_*(Y) = Y', \tag{4.19}$$

hence we have

$$\{\phi^*(\theta'),\ \phi^*(\rho')\} = \left[X\left(\phi^*(\rho')\right)\right] = \left[X \lrcorner \mathrm{d}\left(\phi^*(\rho')\right)\right] = [X \lrcorner Y \lrcorner \omega]\,. \tag{4.20}$$

Let $X_1, \cdots, X_{r-1}$ be vector fields on M, then it follows that

$$\begin{aligned} \phi^*(X' \lrcorner Y' \lrcorner \omega')(X_1, \cdots, X_{r-1}) &= \omega'\left(Y', X', \phi_*(X_1), \cdots, \phi_*(X_{r-1})\right) \\ &= \omega'\left(\phi_*(Y), \phi_*(X), \phi_*(X_1), \cdots, \phi_*(X_{r-1})\right) \\ &= (X \lrcorner Y \lrcorner \omega)(X_1, \cdots, X_{r-1})\,, \end{aligned} \tag{4.21}$$

hence we have

$$\phi^*(X' \lrcorner Y' \lrcorner \omega') = X \lrcorner Y \lrcorner \omega\,, \tag{4.22}$$

therefore by (4.17), (4.20) and (4.22) we see ϕ^* is a Poisson bracket homomorphism. Thus the theorem is proved.

Suppose ϕ is a diffeomorphism of M, then by theorem 4.2; for a given closed $(r+1)$-differential form ω, it follows that

$$\phi^*\left(A^{r-1}(\omega)\right) = A^{r-1}(\omega')\,, \tag{4.23}$$

where $\omega' = \phi^*(\omega)$; and hence by the above theorem, ϕ becomes a Poisson bracket automorphism.

Department of Mathematics
Fukuoka University of Education
Fukuoka, Japan
and
Department of Mathematics
Kyushu Institute of Technology
Kitakyushu, Japan

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(Received June 12, 1972)