Notes on the balayaged measure on the Kuramochi boundary*)

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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1. Let p^{μ}, p^{ν} be two Green potentials on a hyperbolic Riemann surface R. Let G be an open set in R. It is well-known that if $p^{\mu}=p^{\nu}+h$ on G for some harmonic function h on G, then the restriction of μ on G equals the restriction of ν on G.

In this paper, we shall prove a similar result to the above is also valid for Kuramochi's potentails and an open set in the Kuramochi compactification R_N^* of R (Theorem 1). As applications, we shall prove the followings: (a) The support of the canonical measure associated with $\tilde{g}_b$ for a non-minimal Kuramochi boundary point b is contained in the closure of the set of all non-minimal Kuramochi boundary points (Theorem 2). As for a non-minimal Martin boundary point, T. Ikegami [2] had obtained an analogous result to (a). (b) Let K be a compact set in $R_0^*=R_N^*-K_0$ (K_0 is a closed disk in R) and $\tilde{C}$ be the Kuramochi capacity on R_0^*. If we denote by $\mathrm{Int}(K)$ the set of all interior points of K in R_0^*, then we have $\tilde{C}(K)=\tilde{C}(K-\mathrm{Int}(K))$ (Theorem 3).

2. Let R be a hyperbolic Riemann surface. We shall use the same notation as in [1], for instance, $\tilde{g}_b$, $\tilde{p}^{\mu}$, f^F, R_N^*, $\varDelta_N$ etc. For a subset A of R, we denote by ∂A the relative boundary of A in R and by $\bar{A}$ the closure of A in R_N^*. The Kuramochi boundary $\varDelta_N$ is decomposed into two mutually disjoint parts: the minimal part $\varDelta_1$ and the non-minimal part $\varDelta_0$. By a measure μ on R_0^*, we always mean a positive measure μ on R_N^* such that $\mu(K_0)=0$. For a measure μ on R_0^*, we denote by $S\mu$ the support of μ and by $\mu|E$ the restriction of μ on a Borel set E in R_N^*. If a measure μ on R_0^* satisfies $\mu(\varDelta_0)=0$, then it is called *canonical*. It is known that if μ is a measure on R_0^*, then there exists a unique canonical measure ν such that $\tilde{p}^{\mu}=\tilde{p}^{\nu}$. For a closed set F in R and measure μ on R_0^*, we denote by μ_F the canonical associated measure with $\tilde{p}^{\mu}{}_{\tilde{F}}$. We note that $S\mu_F\subset\bar{F}$.

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A subset A of R is called *polar* if there exists a positive superharmonic function s on R such that $s(a)=+\infty$ at every point a in A. It is known that a polar set is locally of Lebesgue measure zero. We shall say that a property hold $q{\cdot}p{\cdot}$ on a set E if it holds on E except for a polar set.

The following properties are known ([1]).

(a) Let F be a non-polar closed set in R and f be a Dirichlet function[1] on R. If G is a component of $R-F$, then $f^F=f^{\partial G}$ on G.

(b) Let F be a closed set in R. If s is a Dirichlet function on R, $s=0$ on K_0 and s is a non-negative full-superharmonic function[2] on R_0, then

$$s_{\tilde{F}}=s^{K_0\cup F} \qquad \text{on } R_0-F.$$

(c) Let b be any point in $R_0\cup\Delta_1$. If F is a closed set in R such that $\bar{F}$ is a neighborhood of b in R_N^*, then $(\tilde{g}_b)_{\tilde{F}}=\tilde{g}_b$.

(d) Let μ be a measure on R_0^*. If F is a closed set in R, then

$$\left(\int\tilde{g}_b d\mu(b)\right)_{\tilde{F}}=\int(\tilde{g}_b)_{\tilde{F}}d\mu(b).$$

By the aid of (a) and (b), we shall prove

LEMMA. *Let s be a non-negative full-superharmonic function on R_0. If F is a closed subset of R_0, then*

$$s_{\tilde{F}}=s_{\widetilde{\partial F}} \qquad \text{on } R_0-F.$$

PROOF. We can find an open disk D in R such that $K_0\subset D$ and $(D\cup\partial D)\cap F=\emptyset$. For each integer $n>0$, we set $s_n=\min(s_{\widetilde{R_0-D}}, n)$. Since s_n is bounded and the total mass of the associated measure with s_n is finite, it follows from Satz 17.3 in [1] that s_n is a Dirichlet function. Hence it follows from (a) and (b) that

$$(s_n)_{\tilde{F}}=(s_n)_{\widetilde{\partial F}} \qquad \text{on } R_0-F.$$

Since $s_{\widetilde{R_0-D}}=s$ on $R_0-(D\cup\partial D)$, by letting $n\to\infty$, we obtain that $s_{\tilde{F}}=s_{\widetilde{\partial F}}$ on R_0-F.

3. PROPOSITION. *Let F be a closed subset of R_0 and μ be a canonical measure on R_0^*. If we set $\nu=\mu|\overline{R-F}$ and $\lambda=\mu-\nu$, then $\mu_F=\nu_F+\lambda$ and $S\nu_F\subset\bar{F}\cap\overline{R-F}$.*

PROOF. (i) First we shall prove that $S\nu_F\subset\bar{F}\cap\overline{R-F}$. Since $S\nu_F\subset\bar{F}$, it is sufficient to prove that $S\nu_F\subset\overline{R-F}$. Let b be an arbitrary point of $R_0^*-\overline{R-F}$. Then there is an open neighborhood U of b in R_0^* such that

1) This is called eine Dirichletsche Funktion in [1].

2) This is called eine positive vollsuperharmonische Funktion in [1].

$\bar{U}\cap\overline{R-F}=\emptyset$. We set $G=U\cap R$. Since $\bar{G}\cap\overline{R-F}=\emptyset$, $\overline{R_0-G}$ is a neighborhood of each b' in $\overline{R-F}\cap R_0^*$. Hence it follows from (c) and (d) that $\tilde{p}^\nu{}_{\widetilde{R_0-G}}=\tilde{p}^\nu$ on R_0. By the Lemma, we obtain that $\tilde{p}^\nu{}_{\widetilde{R_0-G}}=\tilde{p}^\nu{}_{\widetilde{\partial G}}$ and $\tilde{p}^{\nu F}{}_{\widetilde{R_0-G}}=\tilde{p}^{\nu F}{}_{\widetilde{\partial G}}$ on G. Since $\tilde{p}^{\nu F}=\tilde{p}^\nu$ q.p. on F and $\partial G\subset F$, we have

$$\tilde{p}^{\nu F}{}_{\widetilde{\partial G}}=\tilde{p}^\nu{}_{\widetilde{\partial G}} \qquad \text{on } R_0.$$

Thus we obtain that

$$\tilde{p}^\nu{}_{\widetilde{R_0-G}}=\tilde{p}^\nu{}_{\widetilde{\partial G}}=\tilde{p}^{\nu F}{}_{\widetilde{\partial G}}=\tilde{p}^{\nu F}{}_{\widetilde{R_0-G}} \qquad \text{on } G.$$

Since $\tilde{p}^{\nu F}{}_{\widetilde{\partial G}}=\tilde{p}^\nu=\tilde{p}^{\nu F}$ q. p. on G, we see that

$$\tilde{p}^{\nu F}{}_{\widetilde{R_0-G}}=\tilde{p}^{\nu F}{}_{\widetilde{\partial G}}=\tilde{p}^{\nu F} \qquad \text{q. p. on } G.$$

This shows that

$$\tilde{p}^{\nu F}{}_{\widetilde{R_0-G}}=\tilde{p}^{\nu F} \qquad \text{q. p. on } R_0.$$

Hence we have

$$\tilde{p}^{\nu F}{}_{\widetilde{R_0-G}}=\tilde{p}^{\nu F} \qquad \text{on } R$$

and $(\nu_F)_{R_0-G}=\nu_F$. Thus $\nu_F(U)=0$. Since b is arbitrary, we see that $S\nu_F\subset\overline{R-F}$.

(ii) Secondly we shall prove that $\lambda_F=\lambda$. Let b be an arbitrary point in $R_0^*-\overline{R-F}$. Then there exists an open neighborhood U of b in R_N^* such that $\bar{U}\cap\overline{R-F}=\emptyset$. Since $U\cap R\subset F$, $\bar{F}$ is a neighborhood of b. Hence it follows from (c) and (d) that $\tilde{p}^\lambda{}_{\tilde{F}}=\tilde{p}^\lambda$ on R_0. This shows that $\lambda_F=\lambda$. Therefore we complete the proof.

Corollary 1. $\mu_F|(R_0^*-\overline{R-F})=\mu|(R_0^*-\overline{R-F})$.

Corollary 2 ([3]). *If $S\mu\cap\bar{F}=\emptyset$, then $S\mu_F$ is contained in $\bar{F}\cap\overline{R-F}$.*

Theorem 1. *Let μ, ν be canonical measures on R_0^* and s be a non-negative full-superharmonic function on R_0. Let G be an open subset of R_N^* such that $K_0\cap\bar{G}=\emptyset$. If $\tilde{p}^\mu=\tilde{p}^\nu+s$ on $G\cap R_0$, then $\mu|G\geqq\nu|G$.*

Proof. We can find an open disk D in R such that $K_0\subset D$ and $\bar{D}\cap\bar{G}=\emptyset$. Then $s_{\widetilde{R_0-D}}$ is equal to a potential $\tilde{p}^\lambda$. Hence $\tilde{p}^\mu=\tilde{p}^\nu+\tilde{p}^\lambda$ on $G\cap R_0$. Let b be an arbitrary point in G. Then there is an open neighborhood U of b in R_N^* such that $\bar{U}\subset G$. If we set $F=\bar{U}\cap R_0$ $(\subset G\cap R_0)$, then $\tilde{p}^\mu=\tilde{p}^\nu+\tilde{p}^\lambda$ on F. Hence it follows from Corollary 1 to Proposition that $\mu|U=(\nu+\lambda)|U$. Since b is arbitrary, we obtain that $\mu|G=(\nu+\lambda)G\geqq\nu|G$.

Corollary. *If $s_{\widetilde{R_0-G}}=s$ on R_0 and $\tilde{p}^\mu=\tilde{p}^\nu+s$ on $G\cap R_0$, then $\mu|G=\nu|G$. In particular, if $\tilde{p}^\mu=\tilde{p}^\nu$ on $G\cap R_0$, then $\mu|G=\nu|G$.*

As an application of the above corollary, we shall prove

THEOREM 2. *Let b_0 be an arbitrary point in Δ_0. If μ is the canonical measure associated with $\tilde{g}_{b_0}$, then $S\mu$ is contained in $\bar{\Delta}_0$.*

PROOF. Suppose $S\mu$ is not contained in $\bar{\Delta}_0$. Then there exists an open set U in R_N^* such that $\bar{F}\cap\bar{\Delta}_0=\emptyset$ and $\mu(U\cap\Delta_N)>0$. We can find a closed subset F of R_0 such that $\bar{F}\cap\bar{\Delta}_0=\emptyset$ and $\bar{F}$ is a neighborhood of $\bar{U}$. Since b_0 is contained in $\overline{R_0-F}$, it follows from the Lemma in [3] that there exists a measure ν on R_0^* such that $S\nu\subset\bar{F}\cap\overline{R-F}$ and $(\tilde{g}_{b_0})_{\bar{F}}\leqq\tilde{p}^{\nu}\leqq\tilde{g}_{b_0}$ on R_0. Since $\bar{F}\cap\bar{\Delta}_0=\emptyset$, $S\nu\cap\bar{\Delta}_0=\phi$. Hence ν is canonical. Since $\tilde{p}^{\nu}=\tilde{g}_{b_0}$ q.p. on F and $U\cap R_0\subset F$, we see that $\tilde{p}^{\nu}=\tilde{g}_{b_0}=\tilde{p}^{\mu}$ on $U\cap R_0$. It follows from the Corollary to Theorem 1 that $\nu|U=\mu|U$. Since $S\nu\subset\bar{F}\cap\overline{R-F}$, $S\nu\cap\bar{U}=\emptyset$. Hence $\nu(U\cap\Delta_N)=0$. This contradicts the assumption on μ. Therefore we complete the proof.

4. For a compact set K in R_0^*, the (Kuramochi) capacity $\tilde{C}(K)$ is defined by sup $\{\mu(K);\ \mu$ is canonical and $p^{\mu}\leqq 1\}$. It is known ([1]) that there exists a unique canonical measure χ^K on K such that $\tilde{p}^{\chi^K}\leqq 1$, $\tilde{p}^{\chi^K}=1$ on K except for an F_σ-set with capacity zero and $\tilde{C}(K)=\chi^K(K)$.

THEOREM 3. *If K is a compact set in R_0^*, then $\tilde{C}(K)=\tilde{C}(K-Int(K))$.*

PROOF. Since $\tilde{C}(K-\mathrm{Int}(K))\leqq\tilde{C}(K)$, it is sufficient to prove the converse inequality. Since $\tilde{p}^{\chi^K}=1$ on K except for an F_σ-set with capacity zero, we see that $\tilde{p}^{\chi^K}=1$ on $\mathrm{Int}(K)\cap R_0$. Hence, by setting $\mu=\chi^K$, $\nu=0$ and $s=1$ in the Corollary to Theorem 1, we have that $\chi^K(\mathrm{Int}(K))=0$. Thus we obtain that

$$\tilde{C}\left(K-\mathrm{Int}(K)\right)=\sup\left\{\mu\left(K-\mathrm{Int}(K)\right);\ \mu \text{ is canonical and } \tilde{p}^{\mu}\leqq 1\right\}$$
$$\geqq\chi^K\left(K-\mathrm{Int}(K)\right)=\chi^K(K)=\tilde{C}(K).$$

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