On quadratic first integrals of a particular natural system in classical mechanics

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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§1. Introduction. Recently, the first author and Kimura [1] have studied a theory of quadratic first integrals in natural systems [2], in connection with the symmetry problem of classical mechanics (e.g., see references in [3], [4] and [5]). As a result, a necessary and sufficient condition has been established for the existence of a quadratic first integral, and the maximum number of linearly independent quadratic first integrals has been found on the basis of this condition.

For the future development of the theory, it seems useful to study quadratic first integrals of some simple dynamical systems. Along this line of thought, Kimura [6] has examined the case of a multidimensional central force, particular attention having been paid to the number of linarly independent quadratic first integrals and to their connection with the linear first integrals.

The present paper is devoted to a similar discussion of another of the simplest systems, that is, the system in which the configuration space is an N-dimensional Euclidean space and the equi-potential surfaces are hyperplanes parallel to each other. This case is of much interest from a mathematical point of view, because the maximum number of linearly independent linear first integrals can be attained only in this case and the central potential case, if the configuration space is taken to be Euclidean [7].

In §§ 2 and 3 the general form of quadratic first integrals is obtained in the system under consideration. In § 4, the number of linearly independent quadratic first integrals is found and the relation between the linear and quadratic first integrals is made clear. Further, the Poisson brackets between the first integrals are calculated with a view to their applications in the symmetry problem. The final section is devoted to a discussion of the results obtained.

§2. Basic equations. Let us assume that the configuration space is an N-dimensional Euclidean space referred to Cartesian coordinates x^i and that the potential function U depends on the final coordinate x^N

alone.* This system is one of the simplest cases, and will give interesting results concerning the first integrals.

In this paper, we consider quadratic first integrals of the from

$$Q=\tfrac{1}{2}\eta_{ji}(x)\dot{x}^j\dot{x}^i+\zeta(x)\,,^{**} \tag{2.1}$$

where η_{ji} and ζ, functions of x^i, are a symmetric covariant tensor and a scalar respectively. It is well known that (2.1) is a first integral of the motion if and only if

$$\partial_k\eta_{ji}+\partial_j\eta_{ik}+\partial_i\eta_{kj}=0\,,\qquad \partial_i\zeta=\eta_{iN}U' \qquad (U'=dU/dx^N), \tag{2.2}$$

where ∂_i stands for partial differentiation with respect to x^i [5]. These equations will be the basis for the following discussions.

By transforming the first equation of (2.2) suitably it can be seen that all the third derivatives of η_{ji} vanish [1]. Therefore, we have

PROPOSITION 2.1. *The most general solution of the first equation of* (2.2) *is given by*

$$\eta_{ji}=a_{lkji}x^lx^k+a_{kji}x^k+a_{ji}\,, \tag{2.3}$$

where a_{lkji}, a_{kji} *and* a_{ji} *are constants satisfying*

$$\begin{cases} a_{lkji}=a_{klji}\,, \quad a_{lkji}=a_{lkij}\,, \\ a_{lkji}+a_{ljik}+a_{likj}=0\,, \end{cases} \tag{2.4}$$

$$a_{kji}=a_{kij}\,,\qquad a_{kji}+a_{jik}+a_{ikj}=0\,, \tag{2.5}$$

$$a_{ji}=a_{ij}\,. \tag{2.6}$$

In the case $U'=0$, the second equation of (2.2) reduces to $\zeta=$const. Since this constant has nothing to do with the quadratic part of the first integral (2.1), we put it equal to zero. Thus quadratic first integrals in the case $U'=0$ are given by (2.1) with (2.3) to (2.6) and $\zeta=0$. This is the result already obtained by T. Y. Thomas [8] (see also [9]).

We assume $U'\neq 0$ throughout the following discussions. Under this assumption, the final coordinate x^N has a particular meaning and we divide the domain of indices into two parts, N and $1,2,\cdots,N-1$. Correspondingly, we classify the components of a_{lkji}, a_{kji} and a_{ji} according to the number of times N occurs among the indices, this number being called the *type* of the component. Then the following two propositions are valid concerning the constraints on a_{lkji} and a_{kji}.

* Unless stated otherwise, small Latin indices take the values $1,2,\cdots,N$, Greek ones $1,2,\cdots,N-1$ and the summation convention is used.

** $\dot{x}^i$ denote the generalized velocities.

PROPOSITION 2. 2. (2. 4) *can be rewritten as*

$$a_{NNNN}=0\,, \tag{2.7a}$$

$$a_{\kappa NNN}=a_{N\kappa NN}=a_{NN\kappa N}=a_{NNN\kappa}=0\,, \tag{2.7b}$$

$$\begin{cases} a_{N\lambda N\kappa}=a_{\lambda NN\kappa}=a_{N\lambda\kappa N}=a_{\lambda N\kappa N}\,, \\ a_{\lambda\kappa NN}=a_{NN\lambda\kappa}=-2a_{\lambda N\kappa N}\,, \qquad a_{\kappa N\lambda N}=a_{\lambda N\kappa N}\,, \end{cases} \tag{2.7c}$$

$$\begin{cases} a_{\mu\lambda N\kappa}=a_{\mu\lambda\kappa N}\,, \qquad a_{\mu N\lambda\kappa}=a_{N\mu\lambda\kappa}=-2a_{\mu(\lambda\kappa)N}\,, \\ a_{\lambda\mu\kappa N}=a_{\mu\lambda\kappa N}\,, \qquad a_{\mu\lambda\kappa N}+a_{\lambda\kappa\mu N}+a_{\kappa\mu\lambda N}=0\,, \end{cases} \tag{2.7d}$$

$$\begin{cases} a_{\nu\mu\lambda\kappa}=a_{\nu\mu\kappa\lambda}\,, \qquad a_{\nu\mu\lambda\kappa}=a_{\mu\nu\lambda\kappa}\,, \\ a_{\nu\mu\lambda\kappa}+a_{\nu\lambda\kappa\mu}+a_{\nu\kappa\mu\lambda}=0\,. \end{cases} \tag{2.7e}$$

PROPOSITION 2. 3. (2. 5) *can be rewritten as*

$$a_{NNN}=0\,, \tag{2.8a}$$

$$a_{NN\kappa}=a_{N\kappa N}\,, \qquad a_{\kappa NN}=-2a_{N\kappa N}\,, \tag{2.8b}$$

$$a_{\lambda N\kappa}=a_{\lambda\kappa N}\,, \qquad a_{N\lambda\kappa}=-2a_{(\lambda\kappa)N}\,, \tag{2.8c}$$

$$a_{\mu\lambda\kappa}=a_{\mu\kappa\lambda}\,, \qquad a_{\mu\lambda\kappa}+a_{\lambda\kappa\mu}+a_{\kappa\mu\lambda}=0\,. \tag{2.8d}$$

These expressions may be derived in the same manner as the corresponding formulae for η_{lkji} and η_{kji} in [1], §4.

As a remark, we briefly refer to some features of (2. 7) and (2. 8) for the sake of convenience. The surviving components of a_{lkji} and a_{kji} are of type 2, 1 or 0. The components a_{lkji} of type 2 can be expressed in terms of the $a_{\lambda N\kappa N}$ which are symmetric with respect to κ and λ, and those of type 1 in terms of the $a_{\mu\lambda\kappa N}$ which satisfy the last two equations of (2. 7d). The components a_{kji} of types 2 and 1 are expressible in terms of $a_{N\kappa N}$ and $a_{\lambda\kappa N}$ respectively. Finally, $a_{\nu\mu\lambda\kappa}$ and $a_{\mu\lambda\kappa}$ are subject to the conditions (2. 7e) and (2. 8d) respectively.

§3. General form of quadratic first integrals. In this section, we derive the general form of the quadratic first integrals on the basis of the foregoing results. For this purpose we have only to take account of the second equation in (2. 2) and Propositions 2. 1 and 2. 2.

From the second equation in (2. 2), the integrability condition for ζ becomes

$$(\partial_j\eta_{iN}-\partial_i\eta_{jN})U'+(\delta_{jN}\eta_{iN}-\delta_{iN}\eta_{jN})U''=0\,.$$

For $(j, i)=(\lambda, \kappa)$ and (N, κ), this equation is reduced to

$$\partial_\lambda\eta_{\kappa N}-\partial_\kappa\eta_{\lambda N}=0\,, \tag{3.1}$$

(3.2) $$(\partial_N \eta_{\kappa N} - \partial_\kappa \eta_{NN}) U' + \eta_{\kappa N} U'' = 0$$

respectively, where $U' \neq 0$ is used in deriving (3.1).

We now rewrite (3.1) by making use of the expression (2.3) of η_{ji}. As a result we obtain

$$2(a_{\mu\lambda\kappa N} - a_{\mu\kappa\lambda N}) x^\mu + (a_{\lambda\kappa N} - a_{\kappa\lambda N}) = 0\,,$$

where we have used the identity $a_{N\lambda\kappa N} = a_{N\kappa\lambda N}$, which follows from (2.7c). Since the above equation must hold identically, we have

(3.3) $$a_{\mu\lambda\kappa N} = a_{\mu\kappa\lambda N}\,,$$

(3.4) $$a_{\lambda\kappa N} = a_{\kappa\lambda N}\,.$$

With the help of (3.3) the following proposition may be obtained.

PROPOSITION 3.1. *All the components a_{lkji} of type* 1 *vanish, i.e.,*

(3.5) $$a_{N\mu\lambda\kappa} = a_{\mu N\lambda\kappa} = a_{\mu\lambda N\kappa} = a_{\mu\lambda\kappa N} = 0\,.$$

PROOF. If we use (3.3) and the last two equations of (2.7d), we have $a_{\mu\lambda\kappa N} = 0$. Then the proposition can be proved by means of the first half of (2.7d).

(3.2) may be treated in the same way as (3.1). Namely, we substitute (2.3) in (3.2) and make use of the identities in (2.7b), (2.7c), (2.8b) and (3.5). Then we have

$$(6a_{\lambda N\kappa N} x^\lambda + 3a_{N\kappa N}) U' + (2a_{\lambda N\kappa N} x^\lambda x^N + a_{\lambda\kappa N} x^\lambda + a_{N\kappa N} x^N + a_{\kappa N}) U'' = 0\,,$$

which may be rearranged as

$$\left\{6a_{\lambda N\kappa N} U' + (2a_{\lambda N\kappa N} x^N + a_{\lambda\kappa N}) U''\right\} x^\lambda + 3a_{N\kappa N} U' + (a_{N\kappa N} x^N + a_{\kappa N}) U'' = 0\,.$$

This equation must hold for arbitrary values of x^λ, so accordingly we obtain

(3.6) $$\begin{cases} 6a_{\lambda N\kappa N} U' + (2a_{\lambda N\kappa N} x^N + a_{\lambda\kappa N}) U'' = 0\,, \\ 3a_{N\kappa N} U' + (a_{N\kappa N} x^N + a_{\kappa N}) U'' = 0\,. \end{cases}$$

We first consider the case $U'' \neq 0$. If we eliminate U' and U'' from (3.6), we have

$$a_{N\mu N} a_{\lambda\kappa N} - 2a_{\lambda N\kappa N} a_{\mu N} = 0\,,$$

and accordingly

(3.7) $$a_{\lambda\kappa N} = 2c a_{\lambda N\kappa N}\,, \qquad a_{\kappa N} = c a_{N\kappa N} \qquad (c \text{ is a const.}).$$

By substituting these equations into (3.6) we have

$$(3.8)\qquad 2a_{\lambda N\kappa N}\left\{(x^N+c)U''+3U'\right\}=0\,,\quad a_{N\kappa N}\left\{(x^N+c)U''+3U'\right\}=0\,.$$

In view of these expressions, the case $U''\neq 0$ is further classified into two, according to whether $(x^N+c)\,U''+3U'$ vanishes or not.

CASE I. $U''\neq 0$, $(x^N+c)U''+3U'\neq 0$. In this case we have

THEOREM 3.1. *In case I the quadratic first integral is given by*

$$(3.9)\qquad Q_1=\tfrac{1}{2}(a_{\nu\mu\lambda\kappa}x^\nu x^\mu+a_{\mu\lambda\kappa}x^\mu+a_{\lambda\kappa})\dot{x}^\lambda\dot{x}^\kappa+a_{NN}\left\{\tfrac{1}{2}(\dot{x}^N)^2+U(x^N)\right\},$$

where the constant a_{NN} is arbitrary and $a_{\nu\mu\lambda\kappa}$, $a_{\mu\lambda\kappa}$ and $a_{\lambda\kappa}$ are subject to the conditions (2.7e), (2.8d) *and $a_{\lambda\kappa}=a_{\kappa\lambda}$, respectively.*

PROOF. Since we obtain from (3.7) and (3.8),

$$a_{\lambda N\kappa N}=a_{N\kappa N}=a_{\lambda\kappa N}=a_{\kappa N}=0\,,$$

all the a's of type non-zero vanish except a_{NN}, as is seen from the remark at the end of §2. Then (2.3) is reduced to

$$\eta_{\lambda\kappa}=a_{\nu\mu\lambda\kappa}x^\nu x^\mu+a_{\mu\lambda\kappa}x^\mu+a_{\lambda\kappa}\,,$$
$$\eta_{\kappa N}=0\,,\quad \eta_{NN}=a_{NN}\,,\quad \zeta=a_{NN}U\,,$$

and from (2.1) we readily obtain Q_1 in the form given in (3.9).

CASE II. $U''\neq 0$, $(x^N+c)U''+3U'=0$. In this case, the equation for U can easily be integrated, i.e.,

$$(3.10)\qquad U=a(x^N+c)^{-2}+b\,,\quad a\neq 0\,,$$

a and b being integration constants. Further we obtain

THEOREM 3.2. *In Case II, the quadratic first integral is composed of Q_1 in* (3.9) *and Q_2 defined by*

$$(3.11)\qquad \begin{aligned}Q_2=&-a_{\lambda N\kappa N}x^N(x^N+2c)\dot{x}^\lambda\dot{x}^\kappa+(2a_{\lambda N\kappa N}x^\lambda+a_{N\kappa N})(x^N+c)\dot{x}^\kappa\dot{x}^N\\&-(a_{\lambda N\kappa N}x^\lambda x^\kappa+a_{N\kappa N}x^\kappa)(\dot{x}^N)^2+(a_{\lambda N\kappa N}x^\lambda x^\kappa+a_{N\kappa N}x^\kappa)(x^N+c)U'\,,\end{aligned}$$

where $a_{N\kappa N}$ are arbitrary and $a_{\lambda N\kappa N}$ symmetric with respect to κ and λ.

PROOF. (3.6) is satisfied on account of (3.7), and the constraints on a_{lkji}, a_{kji} and a_{ji} are given by (2.6), (2.7), (2.8), (3.4) and (3.7). Accordingly, η_{ji} and ζ take the form

$$\eta_{\lambda\kappa}=a_{\nu\mu\lambda\kappa}x^\nu x^\mu+a_{\mu\lambda\kappa}x^\mu-2a_{\lambda N\kappa N}x^N(x^N+2c)+a_{\lambda\kappa}\,,$$
$$\eta_{\kappa N}=(2a_{\lambda N\kappa N}x^\lambda+a_{N\kappa N})(x^N+c)\,,$$
$$\eta_{NN}=-2a_{\lambda N\kappa N}x^\lambda x^\kappa-2a_{N\kappa N}x^\kappa+a_{NN}\,,$$
$$\zeta=(a_{\lambda N\kappa N}x^\lambda x^\kappa+a_{N\kappa N}x^\kappa)(x^N+c)U'+a_{NN}U\,,$$

from which we can obtain the required result.

Finally, we consider the case $U''=0$.

CASE III. $U''=0$ $(U'\neq 0)$, i.e., $U=ax^N+b$ $(a\neq 0)$. We have

THEOREM 3.3. *In Case III, the quadratic first integral is composed of Q_1 and Q_3 defined by*

$$(3.12)\qquad Q_3=-a_{\lambda\kappa N}x^N\dot{x}^\lambda\dot{x}^\kappa+(a_{\lambda\kappa N}x^\lambda+a_{\kappa N})\dot{x}^\kappa\dot{x}^N+(\tfrac{1}{2}a_{\lambda\kappa N}x^\lambda x^\kappa+a_{\kappa N}x^\kappa)U',$$

where $a_{\kappa N}$ are arbitrary and $a_{\lambda\kappa N}$ symmetric with respect to κ and λ.

PROOF. From (3.6) we obtain $a_{\lambda N\kappa N}=a_{N\kappa N}=0$, and the a_{lkji} and a_{kji} of type 2 vanish (cf. the remark at the end of §2). Thus η_{ji} and ζ are reduced to

$$\eta_{\lambda\kappa}=a_{\nu\mu\lambda\kappa}x^\nu x^\mu+a_{\mu\lambda\kappa}x^\mu-2a_{\lambda\kappa N}x^N+a_{\lambda\kappa},$$

$$\eta_{\kappa N}=a_{\lambda\kappa N}x^\lambda+a_{\kappa N},\qquad \eta_{NN}=a_{NN},$$

$$\zeta=(\tfrac{1}{2}a_{\lambda\kappa N}x^\lambda x^\kappa+a_{\kappa N}x^\kappa)U'+a_{NN}U.$$

This completes the proof.

REMARK. When the integrability condition of (2.2) is satisfied, ζ can be determined to within an arbitrary constant. For the same reason as in the case $U'=0$ (cf. §2), this constant has been put equal to zero in the above three theorems.

§4. Properties of the first integrals. In this section, we begin with the study of the number of arbitrary constants which are contained in the quadratic first integrals of the last section.

THEOREM 4.1. *The number of linearly independent quadratic first integrals (with constant coefficients) is given by $(N^4-N^2+12)/12$ in Case I and $N(N+1)(N^2-N+6)/12$ in Cases II and III.*

PROOF. Of the coefficients of Q_1 in (3.9), $a_{\nu\mu\lambda\kappa}$, $a_{\mu\lambda\kappa}$ and $a_{\lambda\kappa}$ are subject to the conditions (2.7e), (2.8d) and $a_{\lambda\kappa}=a_{\kappa\lambda}$, respectively. Thus the number of independent coefficients is $N(N-1)^2(N-2)/12$ for $a_{\nu\mu\lambda\kappa}$, $N(N-1)(N-2)/3$ for $a_{\mu\lambda\kappa}$, $N(N-1)/2$ for $a_{\lambda\kappa}$ and 1 for a_{NN}. Summing these, we obtain $(N^4-N^2+12)/12$ as the number of linearly independent quadratic first integrals in Case I. Next, the number of $a_{N\kappa N}$ and of symmetric $a_{\lambda N\kappa N}$ in Q_2 are $N-1$ and $N(N-1)/2$ respectively. Thus the number of independent quadratic first integrals in Case II is

$$(N^4-N^2+12)/12+(N-1)+N(N-1)/2=N(N+1)(N^2-N+6)/12.$$

A similar reasoning is applicable to Q_3, and the result in Case III is found to be the same as in Case II.

REMARK. The above result in Cases II and III coincides with the maximum number of independent quadratic first integrals which are admitted by a classical natural system with N degrees of freedom [1].

We next discuss the relation between the linear and quadratic first integrals. In the system under consideration, a linear first integral is linearly expressible in terms of the linear and angular momenta,

$$p_\kappa = \dot{x}^\kappa\,, \qquad L_{\lambda\kappa} = x^\lambda \dot{x}^\kappa - x^\kappa \dot{x}^\lambda = \delta^{\lambda\kappa}_{\nu\mu} x^\nu \dot{x}^\mu\,, \tag{4.1}$$

where $\delta^{\lambda\kappa}_{\nu\mu} = \delta^\lambda_\nu \delta^\kappa_\mu - \delta^\lambda_\mu \delta^\kappa_\nu$. It is obvious that any quadratic form in p_κ and $L_{\lambda\kappa}$ is a first integral. In this connection we have

THEOREM 4.2. *The first integral Q_1 can be expressed as a linear combination of the total energy and a quadratic form in p_κ and $L_{\lambda\kappa}$. More specifically, Q_1 can be written as*

$$\begin{aligned} Q_1 = a_{NN}E + \tfrac{1}{12} a_{\nu\mu\lambda\kappa}(L_{\nu\lambda}L_{\mu\kappa} + L_{\nu\kappa}L_{\mu\lambda}) \\ + \tfrac{1}{6} a_{\mu\lambda\kappa}(L_{\mu\lambda}p_\kappa + L_{\mu\kappa}p_\lambda) + \tfrac{1}{2}(a_{\lambda\kappa} - \delta_{\lambda\kappa}a_{NN})p_\lambda p_\kappa\,, \end{aligned} \tag{4.2}$$

where E is the total energy,

$$E = \tfrac{1}{2}\sum_{i=1}^{N} (\dot{x}^i)^2 + U\,. \tag{4.3}$$

PROOF, If $a_{\nu\mu\lambda\kappa}$ and $a_{\mu\lambda\kappa}$ satisfy (2.7e) and (2.8d) respectively, the following identities are valid.

$$a_{\nu\mu\lambda\kappa} = \tfrac{1}{6}(\delta^{\delta\beta}_{\nu\lambda}\delta^{\gamma\alpha}_{\mu\kappa} + \delta^{\delta\beta}_{\nu\kappa}\delta^{\gamma\alpha}_{\mu\lambda})a_{\delta\gamma\beta\alpha}\,,$$

$$a_{\mu\lambda\kappa} = \tfrac{1}{3}(\delta^{\gamma\beta}_{\mu\lambda}\delta^\alpha_\kappa + \delta^{\gamma\alpha}_{\mu\kappa}\delta^\beta_\lambda)a_{\gamma\beta\alpha}\,.$$

From this we have

$$a_{\nu\mu\lambda\kappa}x^\nu x^\mu \dot{x}^\lambda \dot{x}^\kappa = \tfrac{1}{6} a_{\nu\mu\lambda\kappa}(L_{\nu\lambda}L_{\mu\kappa} + L_{\nu\kappa}L_{\mu\lambda})\,,$$

$$a_{\mu\lambda\kappa}x^\mu \dot{x}^\lambda \dot{x}^\kappa = \tfrac{1}{3} a_{\mu\lambda\kappa}(L_{\mu\lambda}p_\kappa + L_{\mu\kappa}p_\lambda)\,,$$

which proves the required result (4.2).

THEOREM 4.3. *No first integral of type Q_2 or Q_3 can be expressed in terms of the total energy and linear first integrals.*

PROOF. Let a quadratic first integral be expressed in terms of E, p_κ and $L_{\lambda\kappa}$. Then it must be an integral common to all Cases I, II and III, and accordingly is of type Q_1.

These two theorems show that the integrals Q_2 and Q_3 are more interesting than Q_1, since the former are independent of linear first integrals.

Finally, we consider the Poisson brackets between the first integrals previously discussed, excluding those of type Q_1.

CASE II. It is readily seen that any first integral of type Q_2 can be expressed linearly in terms of the integrals

$$(4.4)\qquad \begin{cases} Q_{2|\kappa} = (x^N+c)\dot{x}^\kappa\dot{x}^N - x^\kappa(\dot{x}^N)^2 + x^\kappa(x^N+c)U', \\ Q_{2|\lambda\kappa} = -x^N(x^N+2c)\dot{x}^\lambda\dot{x}^\kappa + (x^N+c)(x^\lambda\dot{x}^\kappa + x^\kappa\dot{x}^\lambda)\dot{x}^N \\ \qquad -x^\lambda x^\kappa(\dot{x}^N)^2 + x^\lambda x^\kappa(x^N+c)U', \end{cases}$$

which themselves are linearly independent. It is easy to calculate the Poisson brackets concerning p_κ, $L_{\lambda\kappa}$, $Q_{2|\kappa}$ and $Q_{2|\lambda\kappa}$. The results are as follows.

$$(4.5)\qquad \begin{cases} \{p_\lambda, p_\kappa\} = 0, \\ \{p_\mu, L_{\lambda\kappa}\} = \delta_{\mu\kappa}p_\lambda - \delta_{\mu\lambda}p_\kappa, \\ \{L_{\nu\mu}, L_{\lambda\kappa}\} = \delta_{\nu\lambda}L_{\mu\kappa} - \delta_{\nu\kappa}L_{\mu\lambda} - \delta_{\mu\lambda}L_{\nu\kappa} + \delta_{\mu\kappa}L_{\nu\lambda}, \end{cases}$$

$$(4.6\text{a})\qquad \begin{cases} \{Q_{2|\lambda}, p_\kappa\} = -\delta_{\lambda\kappa}(2E - \sum_{\alpha=1}^{N-1} p_\alpha^2), \\ \{Q_{2|\mu\lambda}, p_\kappa\} = \delta_{\mu\kappa}Q_{2|\lambda} + \delta_{\lambda\kappa}Q_{2|\mu}, \\ \{Q_{2|\mu}, L_{\lambda\kappa}\} = \delta_{\nu\kappa}Q_{2|\lambda} - \delta_{\mu\lambda}Q_{2|\kappa}, \\ \{Q_{2|\nu\mu}, L_{\lambda\kappa}\} = \delta_{\nu\kappa}Q_{2|\mu\lambda} + \delta_{\mu\kappa}Q_{2|\nu\lambda} - \delta_{\nu\lambda}Q_{2|\mu\kappa} - \delta_{\mu\lambda}Q_{2|\nu\kappa}, \end{cases}$$

$$(4.6\text{b})\qquad \begin{cases} \{Q_{2|\lambda}, Q_{2|\kappa}\} = 2L_{\lambda\kappa}(2E - \sum_{\alpha=1}^{N-1} p_\alpha^2), \\ \{Q_{2|\mu\lambda}, Q_{2|\kappa}\} = 2(\delta_{\mu\kappa}p_\lambda + \delta_{\lambda\kappa}p_\mu)(c^2E - a - \frac{1}{2}c^2 \sum_{\alpha=1}^{N-1} p_\alpha^2) \\ \qquad + 2p_\mu Q_{2|\lambda\kappa} + 2p_\lambda Q_{2|\mu\kappa} - 2p_\kappa Q_{2|\mu\lambda} - 2c^2 p_\mu p_\lambda p_\kappa, \\ \{Q_{2|\nu\mu}, Q_{2|\lambda\kappa}\} = -2(\delta_{\nu\lambda}L_{\mu\kappa} + \delta_{\nu\kappa}L_{\mu\lambda} + \delta_{\mu\lambda}L_{\nu\kappa} + \delta_{\mu\kappa}L_{\nu\lambda})(c^2E \\ \qquad -a - \frac{1}{2}c^2 \sum_{\alpha=1}^{n-1} p_\alpha^2) - (L_{\nu\lambda}Q_{2|\mu\kappa} + L_{\nu\kappa}Q_{2|\mu\lambda} \\ \qquad + L_{\mu\lambda}Q_{2|\nu\kappa} + L_{\mu\kappa}Q_{2|\nu\lambda}) + c^2(L_{\nu\lambda}p_\mu p_\kappa + L_{\nu\kappa}p_\mu p_\lambda \\ \qquad + L_{\mu\lambda}p_\nu p_\kappa + L_{\mu\kappa}p_\nu p_\lambda). \end{cases}$$

CASE III. A quadratic first integral of type Q_3 can be expressed in terms of the linearly independent integrals

$$(4.7)\qquad \begin{cases} Q_{3|\kappa} = \dot{x}^\kappa\dot{x}^N + x^\kappa U', \\ Q_{3|\lambda\kappa} = -2x^N\dot{x}^\lambda\dot{x}^\kappa + (x^\lambda\dot{x}^\kappa + x^\kappa\dot{x}^\lambda)\dot{x}^N + x^\lambda x^\kappa U'. \end{cases}$$

The Poisson brackets for p_κ, $L_{\lambda\kappa}$, $Q_{3|\kappa}$ and $Q_{3|\lambda\kappa}$ are given by (4.5) and the following.

$$(4.8\text{a})\qquad \begin{cases} \{Q_{3|\lambda}, p_\kappa\} = \delta_{\lambda\kappa}U', \\ \{Q_{3|\mu\lambda}, p_\kappa\} = \delta_{\mu\kappa}Q_{3|\lambda} + \delta_{\lambda\kappa}Q_{3|\mu}, \end{cases}$$

$$\begin{cases} \{Q_{3|\mu},\ L_{\lambda\kappa}\} = \delta_{\mu\kappa}Q_{3|\lambda} - \delta_{\mu\lambda}Q_{3|\kappa}\,, \\ \{Q_{3|\nu\mu},\ L_{\lambda\kappa}\} = \delta_{\nu\kappa}Q_{3|\mu\lambda} + \delta_{\mu\kappa}Q_{3|\nu\lambda} - \delta_{\nu\lambda}Q_{3|\mu\kappa} - \delta_{\mu\lambda}Q_{3|\nu\kappa}\,, \end{cases}$$

$$\text{(4.8b)} \quad \begin{cases} \{Q_{3|\lambda},\ Q_{3|\kappa}\} = 0\,, \\ \{Q_{3|\mu\lambda},\ Q_{3|\kappa}\} = (\delta_{\mu\kappa}p_\lambda + \delta_{\lambda\kappa}p_\mu)(2E - \sum_{\alpha=1}^{N-1} p_\alpha{}^2) - 2p_\mu p_\lambda p_\kappa\,, \\ \{Q_{3|\nu\mu},\ Q_{3|\lambda\kappa}\} = (\delta_{\nu\lambda}L_{\mu\kappa} + \delta_{\nu\kappa}L_{\mu\lambda} + \delta_{\mu\lambda}L_{\nu\kappa} + \delta_{\mu\kappa}L_{\nu\lambda})(2E - \sum_{\alpha=1}^{N-1} p_\alpha{}^2) \\ \qquad\qquad + p_\nu p_\lambda L_{\mu\kappa} + p_\nu p_\kappa L_{\mu\lambda} + p_\mu p_\lambda L_{\nu\kappa} + p_\mu p_\kappa L_{\nu\lambda}\,. \end{cases}$$

We here remark that, in all Cases I, II and III, a new first integral cannot be produced by calculating the Poisson brackets concerning the linear and quadratic first integrals.

§ 5. Further outlook. In the above, the general form of the quadratic first integrals has been obtained for the system in which the configuration space is Euclidean and the potential U depends only on the final Cartesian coordinate x^N. We are much interested in the case where U is given by

$$U = a(x^N + c)^{-2} + b \quad \text{or} \quad ax^N + b\,,$$

$a(\neq 0)$, b and c being arbitrary constants. In both cases, there is a quadratic first integral independent of the total energy and the linear first integrals, and the number of linearly independent quadratic first integrals is equal to $N(N+1)(N^2-N+6)/12$. We have already found that this is the maximum number which is attained in a classical natural system with N degrees of freedom [1]. Furthermore, we shall prove in a forthcoming paper [10] that only the above two systems and the isotropic harmonic oscillator admit the maximum number of quadratic first integrals, if the configuration space is Euclidean.

It is to be noted that, in the system studied in this paper, the Poisson brackets involving the linear and quadratic first integrals cannot be expressed linearly in terms of the integrals themselves. Therefore, these integrals do not form a Lie algebra, in contrast to the case of the hydrogen atom or the harmonic oscillator. A further investigation will be needed concerning the problem of the dynamical symmetry group of the system under consideration.

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Note added in proof. In calculating the Poisson brackets (4. 6) and (4. 8), we omitted the additive constant b of the total energy E.

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