On f-three-structures

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

By Shigeru ISHIHARA and Mariko KONISHI

Recently, differentiable manifolds with three structures of the same kind have been studied, when the structures have properties somewhat related to a quaternion structure. For examples, almost contact three-structures and normal contact three-structures are studied by Kuo [4], Kuo and Tachibana [5], Tachibana and Yu [7], Tanno [8] and the present authors [2], and f-three-structures with complementary frames by Kashiwada [3]. In any of these cases, when there are given two structures of the same kind with suitable properties, the third structure can be constructed of the given two and be of the same kind. In the present paper, we shall study triples of f-structures which have not necessarily complementary frames.

§ 1. f-structures.

Let M be a differentiable manifold of dimension n and assume that there is given in M a tensor field f of type (1, 1) satisfying

$$f^3+f=0 \tag{1.1}$$

and being of rank r everywhere in M.*) Then we call f an *f-structure of rank r*, where r is necessarily even (Yano [9]). If we put

$$F=f^2+I, \quad L=-f^2, \tag{1.2}$$

I being the unit tensor field of type (1, 1) in M, then we have

$$\begin{aligned} &F^2=F, \quad L^2=L, \quad F+L=I, \\ &fF=Ff=0, \quad fL=Lf=f. \end{aligned} \tag{1.3}$$

Thus the F and L can be considered as projection operators acting on each tangent space of M and defining complementory distributions in M. We denote the distribution defined by F and that defined by L respectively also by F and L. When f is of rank r, these distributions F and L are of dimensions $n-r$ and r, respectively.

*) Manifolds and geometric objects we discuss are assumed to be differentiable and of class C^∞.

Let $\{U; x^h\}$ be a coordinate neighborhood of M with local coordinates (x^h).*) If we take a local base $\{f_y\}$ for the distribution F in $\{U; x^h\}$, f_y being for each fixed index y a local vector field in $\{U; x^h\}$, then there are in $\{U; x^h\}$ $n-r$ local covector fields f^x such that $F=\sum_x f^x \otimes f_x$ and $f^x(f_y) = \delta_y^x$.**) Then we have by (1.3)

$$ff_y = 0, \quad f^x \circ f = 0, \tag{1.4}$$

where the 1-form $f_x \circ f$ is defined by $(f^x \circ f)(X) = f_x(fX)$ for any vector field X in M. Thus we have from (1.2), (1.3) and (1.4)

$$\begin{aligned} &f^2 = -I + \textstyle\sum_x f^x \otimes f_x, \quad ff_y = 0, \\ &f^x \circ f = 0, \quad f^x(f_y) = \delta_y^x. \end{aligned} \tag{1.5}$$

The coframe $\{f^x\}$ is said to be *dual to the frame* $\{f_y\}$ *with respect to the f-structure f.*

For the given f-structure f of rank r in M, we denote by $V_F(M)$ the set of all tangent vectors belonging to the distribution F. Then $V_F(M)$ is a subbundle of the tangent bundle $T(M)$ over M, where we denote by $p_F: V_F(M) \to M$ the bundle projection. If we take an arbitrary element $v \in p_F^{-1}(U)$ $(\subset V_F(M))$ and assume that $v = \sum_x v^x f_x$, then we see easily that the correspondence $v \to (x^h, v^x)$ defines a system of local coordinates (x^h, v^x) in $p_F^{-1}(v, x^h)$, (x^h) being local coordinates of the point $P = p_F(v)$ in $\{U; x^h\}$. Let there be given a linear connection ω in the vector bundle $V_F(M)$ and denote by Γ_{jy}^x components of ω with respect to the local base $\{f_y\}$ of $V_F(M)$ in $\{U; x^h\}$. Then, given a cross-section W in $V_F(M)$ and a vector field X in M, the covariant derivative $\nabla_X W$ of W with respect to ω has local expression

$$\nabla_X W = \sum_x X^j \left(\frac{\partial W^x}{\partial x^j} + \sum_y \Gamma_{jy}^x W^y \right) f_x$$

in $\{U; x^h\}$, where $W = \sum_x W^x f_x$ and $X = X^j \partial/\partial x^j$ in $\{U; x^h\}$. We note that

$$\begin{pmatrix} \delta_i^h, & 0 \\ -\Gamma_i^x & \delta_y^x \end{pmatrix}^{-1} = \begin{pmatrix} \delta_i^h & 0 \\ \Gamma_i^x & \delta_y^x \end{pmatrix}$$

holds. Thus, denoting by $f_i^h, f^h{}_y$ and f_i^x respectively components of f, f_y and f^x in $\{U; x^h\}$, we consider a local tensor field f_U of type (1, 1) in $p_F^{-1}(U)$ $(\subset V_F(M))$ with components***)

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*) The indices h, i, j, k, l run over the range $\{1, 2, \cdots, n\}$ and the summation convension will be used with respect to this system of indices.

**) The indices x, y, z, u, v run over the range $\{n+1, n+2, \cdots, 2n-r\}$.

***) The indices λ, μ, ν run over the range $\{1, 2, \cdots, 2n-r\}$.

$$(1.6)\qquad \begin{aligned}(\tilde{f}_{\mu}{}^{\lambda}) &= \begin{pmatrix} \delta_k^h & 0 \\ -\Gamma_k^x & \delta_u^x \end{pmatrix} \begin{pmatrix} f_j{}^k & -f^k{}_v \\ f_j{}^u & 0 \end{pmatrix} \begin{pmatrix} \delta_i^j & 0 \\ \Gamma_i^v & \delta_y^v \end{pmatrix} \\ &= \begin{pmatrix} f_i^h - \Gamma_i^y f^h{}_y & -f^h{}_y \\ f_i^x - f_i^k \Gamma_k^x + \Gamma_i^z f^k{}_z \Gamma_k^x & f^k{}_y \Gamma_k^x \end{pmatrix}\end{aligned}$$

with respect to local coordinates $(\xi^{\lambda})=(x^h, v^x)$ in $p_F{}^{-1}(U)$, Γ_i^x being defined by $\Gamma_i^x=\sum_y \Gamma_{iy}^x v^y$, where $\xi^1=x^1, \cdots, \xi^n=x^n$, $\xi^{n+1}=v^{n+1}, \cdots, \xi^{2n-r}=v^{2n-r}$. Then it is easily verified that the tensor field f_U defined locally in each $p_F{}^{-1}(U)$ determines a global tensor field $\tilde{f}$ of type (1, 1) in $V_F(M)$. Using (1.5) and (1.6), we can easily prove that $\tilde{f}^2=-I$, i.e., that $\tilde{f}$ is an almost complex structure in $V_F(M)$ (Ishihara [1]). When the almost complex structure $\tilde{f}$ is integrable, we say that the given f-structure f is *normal with respect to the linear connection* ω *in the vector bundle* $V_F(M)$. The f-structure f is normal with respect to ω if and only if

$$N_{ji}{}^h + \sum_x \left[\left(\frac{\partial}{\partial x^j} f_i^x + \sum_y \Gamma_{jy}^x f_i^y\right) - \left(\frac{\partial}{\partial x^i} f_j^x + \sum_y \Gamma_{iy}^x f_j^y\right)\right] f^h{}_x = 0,$$

where $N_{ji}{}^h$ are components of the Nijenhuis tensor of f (Ishihara [1]).

In a manifold M with f-structure f, there is a Riemannian metric γ satisfying

$$(1.7)\qquad \begin{aligned} &\gamma(fX, fY)+\gamma(FX, Y)=\gamma(X, Y), \\ &\gamma(fX, Y)+\gamma(X, fY)=0, \quad \gamma(FX, Y)-\gamma(X, FY)=0, \\ &\gamma(LX, Y)-\gamma(X, LY)=0, \quad \gamma(fX, FY)=0 \end{aligned}$$

for any vector fields X and Y in M. We call such a Riemannian metric γ a *Riemannian metric associated with the f-structure f* (Yano [9], Yano and Ishihara [11]).

§ 2. *f*-three-structures.

Let there be given two f-structures f and g of the same rank r in a manifold M of dimension n and put

$$F=f^2+I, \qquad G=g^2+I.$$

Suppose that f and g satisfy the conditions

$$(2.1)\qquad GF=FG=0,$$

$$(2.2)\qquad fGf=gEg,$$

$$(2.3)\qquad gf+fg=Gfg+gfG.$$

The conditions (2.1) and (2.2) imply

$$(2.4)\qquad gfG=Fgf,\quad Gfg=fgF.$$

Thus (2.3) can be also written as

$$(2.5)\qquad gf-Gfg=gf-fgF=-fg+gfG=-fg+Fgf.$$

If we put

$$(2.6)\qquad h=gf-Gfg,$$

then we have

$$h^2+I=(-fg+gfG)(gf-Gfg)+I=-fGf$$

by virtue of (1.2), (1.3), (2.1) and (2.2) (use (2.10)′). Moreover, we find

$$(2.7)\qquad h^3+h=(-gf+fgF)fGf=0,$$

which shows that h is another f-structure in M. On putting

$$H=h^2+I,$$

we can easily verify the following equations:

$$(2.8)\qquad GF=FG=0,\qquad HG=GH=0,\qquad FH=HF=0;$$

$$(2.9)\qquad F=-gHg=-hGh,\quad G=-hFh=-fHf,\quad H=-fGf=-gGg;$$

$$(2.10)\qquad f=hg-Hgh,\qquad g=fh-Fhf,\qquad h=gf-Gfg$$

and

$$(2.10)'\qquad \begin{aligned} f&=hg-ghG & g&=fh-hfH & h&=gf-fgF\\ &=-gh+hgH & &=-hf+fhF & &=-fg+gfG\\ &=-gh+Ghg, & &=-hf+Hfh, & &=-fg+Fgf. \end{aligned}$$

We see easily from (2.9) that F, G and H are of the same rank $n-r$. Thus the new f-structure h is of rank r. A triple (f, g, h) of three f-structures f, g and h of the same rank r is called an *f-three structure of rank r*, when (f, g, h) satisfies the conditions (2.8), (2.9) and (2.10). Thus, summing up, we have

PROPOSITION 1. *If there are given in M two f-structures f and g of the same rrnk r satisfying* (2.1), (2.2) *and* (2.3), *then there exists in M an f-three-structure (f, g, h) of rank r, where h is defined by* (2.6).

Let there be given in M an f-three-structure (f, g, h) of rank r. If we take, in each coordinate neighborhood $\{U; x^h\}$ of M, a local base $\{f_y\}$ for the distribution F determined by the projection operator $F=f^2+I$ and the local coframe $\{f^x\}$ being dual to $\{f_y\}$ with respect to f, then we have $F=\sum_x f^x\otimes f_x$ and hence by (2.8)

(2.11) $$Gf_y = 0, \quad Hf_y = 0; \quad f^x \circ G = 0, \quad f^x \circ H = 0,$$

where $G = g^2 + I$ and $H = h^2 + I$. Next, putting

(2.12) $$h_y = gf_y, \qquad g_y = fh_y;$$

(2.13) $$h^x = -f^x \circ g, \qquad g^x = -h^x \circ f,$$

we find by (2.11), (2.12) and (2.13)

(2.14) $$f_y = hg_y, \qquad f^x = -g^x \circ h$$

and by (2.9)

(2.15) $$F = \sum_x f^x \otimes f_x, \quad G = \sum_x g^x \otimes g_x, \quad H = \sum_x h^x \otimes h_x,$$

which shows that $\{g_y\}$ and $\{h_y\}$ are in $\{U; x^h\}$ local bases for the distributions determined respectively by the projection operators G and H, and, that $\{g^x\}$ is dual to $\{g_y\}$ with respect to g and $\{h^x\}$ is dual to $\{h_y\}$ with respect to h. If we take account of (2.10)~(2.14), we have

(2.16) $$\begin{aligned} &f_y = hg_y = -gh_y, && g_y = fh_y = -hf_y, && h_y = gf_y = -fg_y; \\ &f^x = h^x \circ g = -g^x \circ h, && g^x = f^x \circ h = -h^x \circ f, && h^x = g^x \circ f = -f^x \circ g. \end{aligned}$$

We now denote by $p_F: V_F(M) \to M$, $p_G: V_G(M) \to M$ and $p_H: V_H(M) \to M$ the vector bundles defined in §1 with respect to f, g and h, respectively. In each coordinate neighborhood $\{U; x^h\}$ of M, we take local bases $\{f_y\}$ of $V_F(M)$, $\{g_y\}$ of $V_G(M)$ and $\{h_y\}$ of $V_H(M)$, satisfying (2.16). Then we can define bundle isomorphisms $f^*: V_H(M) \to V_G(M)$, $g^*: V_F(M) \to V_H(M)$ and $h^*: V_G(M) \to V_F(M)$ respectively by $f^*(w) = fw$, $g^*(u) = gu$, $h^*(v) = hv$ for any $u \in V_F(M)$, $v \in V_G(M)$ and $w \in V_H(M)$. Moreover we have $g^* \circ h^* = -f^*$, $h^* \circ f^* = -g^*$, $f^* \circ g^* = -h^*$.

Let there be given a linear connection ω_F in $V_F(M)$ and denote by ω_G the connection induced in $V_G(M)$ by h^{*-1} from ω_F and by ω_H the connection induced in $V_H(M)$ by g^* from ω_F. Thus, if ω_F has components Γ_{jy}^x with respect to $\{f_y\}$, then ω_G and ω_H have the same components Γ_{jy}^x with respect to $\{g_y\}$ and $\{h_y\}$, respectively. We denote by $\tilde{f}$ the almost complex structure in $V_F(M)$ constructed from f and ω_F by (1.6), by $\tilde{g}'$ the one in $V_G(M)$ constructed from g and ω_G and by $\tilde{h}'$ the one in $V_H(M)$ constructed from h and ω_H. If we denote by $\tilde{g}$ and $\tilde{h}$ respectively the almost complex structures $(dh^*) \circ \tilde{g}' \circ (dh^*)^{-1}$ and $(dg^*)^{-1} \circ \tilde{h}' \circ (dg^*)$ in $V_F(M)$, then we have three almost complex structures $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ in $V_F(M)$. With respect to local coordinates $(\xi^\lambda) = (x^h, v^h)$, they have local components

$$
(2.17)\qquad
\begin{aligned}
(f_\mu{}^\lambda) &= \begin{pmatrix} \delta_k^h & 0 \\ -\Gamma_k{}^x & \delta_u^x \end{pmatrix} \begin{pmatrix} f_j{}^k & -f^k{}_v \\ f_j{}^u & 0 \end{pmatrix} \begin{pmatrix} \delta_i^j & 0 \\ \Gamma_i{}^v & \delta_y^v \end{pmatrix}, \\
(\tilde{g}_\mu{}^\lambda) &= \begin{pmatrix} \delta_k^h & 0 \\ -\Gamma_k{}^x & \delta_u^x \end{pmatrix} \begin{pmatrix} g_j{}^k & -g^k{}_v \\ g_j{}^u & 0 \end{pmatrix} \begin{pmatrix} \delta_i^j & 0 \\ \Gamma_i{}^v & \delta_y^v \end{pmatrix}, \\
(\tilde{h}_\mu{}^\lambda) &= \begin{pmatrix} \delta_k^h & 0 \\ -\Gamma_k{}^x & \delta_u^x \end{pmatrix} \begin{pmatrix} h_j{}^k & -h^k{}_v \\ h_j{}^u & 0 \end{pmatrix} \begin{pmatrix} \delta_i^j & 0 \\ \Gamma_i{}^v & \delta_y^v \end{pmatrix},
\end{aligned}
$$

where $f_j{}^h$, $f^h{}_y$, $f_j{}^x$; $g_j{}^h$, $g^h{}_y$, $g_j{}^x$; $h_j{}^h$, $h^h{}_y$, $h_j{}^x$ are local components of f, f_y, f_x; g, g_y, g^x; h, h_y, h^x in $\{U; x^h\}$, respectively. On the other hand, using (2.15) and (2.16), we find

$$Gfg=(\sum_x g^x \otimes g_x)fg=\sum_x\big((g^x\circ f)\circ g\big)\otimes g_x=(\sum_x (h^x\circ g)\otimes g_x=\sum_x f^x\otimes g_x$$

and other similar equations. Thus, taking account of (2.10), (2.16) and (2.17), we have $\tilde{g}\tilde{f}=-\tilde{f}\tilde{g}=\tilde{h}$. By similar devices, we have

$$\tilde{f}=\tilde{h}\tilde{g}=-\tilde{g}\tilde{h}\,,\quad \tilde{g}=\tilde{f}\tilde{h}=-\tilde{h}\tilde{f}\,,\quad \tilde{h}=\tilde{g}\tilde{f}=-\tilde{f}\tilde{g}\,,$$

which means that the triple $(\tilde{f}, \tilde{g}, \tilde{h})$ of almost complex structures is an almost quaternion structure. Summing up, we have

PROPOSITION 2. *Let a manifold M admit an f-three-structure (f, g, h) of rank r. Then there exists, in the vector bundle $V_F(M)$ determined by projection operator $F=f^2+I$, an almost quaternion ftructure $(\tilde{f}, \tilde{g}, \tilde{h})$ with components* (2.17), *if a linear connection ω_F is given in $V_F(M)$.*

Let there be given in M an f-three-structure (f, g, h) and a linear connection ω_F in the vector bundle $V_F(M)$. Suppose that the two f-structures f and g are normal with respect to ω_F and ω_G, where ω_G is the connection induced in $V_G(M)$ by h^{*-1} from ω_F. Then, by definition, the almost complex structures $\tilde{f}$ and $\tilde{g}'$ are integrable in $V_F(M)$ and in $V_G(M)$, respectively. That is, the two almost complex structures $\tilde{f}$ and $\tilde{g}=(dh^*)\circ\tilde{g}'\circ(dh^*)^{-1}$ are both integrable in $V_F(M)$. Hence, by virtue of a well-known theorem (cf. Obata [6], Yano and Ako [10]), the third almost complex structure $\tilde{h}$ is also integrable in $V_F(M)$. Consequently, the almost complex structure $\tilde{h}'=(dg^*)\circ\tilde{h}\circ(dg^*)^{-1}$ is integrable in $V_H(M)$, which shows that the third f-structure h is normal with respect to the induced connection ω_H. Thus we have

PROPOSITION 3. *Let a manifold M admit an f-three-structure of rank r and suppose that there is given a linear connection ω_F in $V_F(M)$. If the f-structures f and g are normal respectively with respect to ω_F and ω_G, where ω_G is the linear connection induced in $V_G(M)$ by h^{*-1} from ω_F, then the third f-structure h defined by* (2.6) *is also normal with respect to the* *linear connection ω_H induced in $V_H(M)$ by g^* from ω_F.*

Let (f, g, h) be an f-three-structure in M and a a Riemannian metric associated with the f-structure f in M. On putting $M=g^2$ and $N=h^2$, we define in M Riemannian metrics b and c succesively by

$$b(X, Y)=a(MX, MY)+a(GX, GY),$$
$$c(X, Y)=b(NX, NY)+b(HX, HY),$$

X and Y being arbitrary vector fields in M. If we put

$$\gamma(X, Y)=c(X, Y)+c(fX, fY)+c(gX, gY)+c(hX, hY)$$
$$+c(FX, FY)+c(GX, GY)+c(HX, HY),$$

X and Y being arbitrary vector fields in M, then we have a Riemannian metric γ in M. We can verify that this Riemannian metric γ is associated with all of the f-structures f, g and h. Such a Riemannian metric γ is said to be *associated with the f-three-structure* (f, g, h).

§ 3. Reduction of the structure group of the tangent bundle.

Let there be given an f-three-structure (f, g, h) of rank r in a manifold of dimension n and take a Riemannian metric γ associated with (f, g, h). Then we have

$$\begin{aligned} &\gamma(fX, fY)+\gamma(FX, Y)=\gamma(X, Y), \quad \gamma(fX, FY)=0;\\ &\gamma(gX, gY)+\gamma(GX, Y)=\gamma(X, Y), \quad \gamma(gX, GY)=0;\\ &\gamma(hX, hY)+\gamma(HX, Y)=\gamma(X, Y), \quad \gamma(hX, HY)=0 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\gamma(fX, Y)+\gamma(X, fY)=0, \quad \gamma(FX, Y)=\gamma(X, FY);\\ &\gamma(gX, Y)+\gamma(X, gY)=0, \quad \gamma(GX, Y)=\gamma(X, GY);\\ &\gamma(hX, Y)+\gamma(X, hY)=0, \quad \gamma(HX, Y)=\gamma(X, HY) \end{aligned} \tag{3.2}$$

for any vector fields X and Y in M. Using (2.10) and (3.1), we see that, for each point P of M, three subspaces $F(T_P(M))$, $G(T_P(M))$ *and* $H(T_P(M))$ are orthogonal to each other, where $T_P(M)$ is the tangent space of M at P. Thus we can decompose the tangent space $T_P(M)$ in such away that

$$T_P(M)=W_P\oplus F\big(T_P(M)\big)\oplus G\big(T_P(M)\big)\oplus H\big(T_P(M)\big) \quad \text{(direct sum)},$$

where W_P is the orthogonal complement of $F(T_P(M))\oplus G(T_P(M))\oplus H(T_P(M))$ in $T_P(M)$ with respect to the Riemannian metric γ. If we take account of (2.14), we have

$$(hg)X=-(gh)X=fX\,,\quad (fh)X=-(hf)X=gX\,,\quad (gf)X=-(fg)X=hX$$

for any vector X belonging to W_P. Therefore, there are orthonormal vectors $X_1, \cdots, X_s$ in W_P such that $\{X_1, \cdots, X_s; fX_1, \cdots, fX_s; gX_1, \cdots, gX_s; hX_1, \cdots, hX_s\}$ is an orthonormal base in W_P, where $4s=3r-2n$. If we take an othonormal base $\{f_y\}$ in $F(T_P(M))$, then, using (2.16) and (3.1), we see that $\{g_y\}$ and $\{h_y\}$ are orthonormal bases respectively in $G(T_P(M))$ and in $H(T_P)(M)$, where g_y and h_y are defined by (2.16). Thus we have in $T_P(M)$ an orthonormal base $\{X_1, \cdots, X_s; fX_1, \cdots, fX_s; gX_1, \cdots, gX_s; hX_1, \cdots, hX_s; f_1, \cdots, f_{n-r}; g_1, \cdots, g_{n-r}; h_1, \cdots, h_{n-r}\}$, which is called a *frame adapted to the f-three-structure* (f, g, h) *and* γ. Therefore, if we take a point P of M two frames adapted to (f, g, h) and γ, the transformation of these two frames is expressed by an orthogonal matrix T of the form

$$T=\left(\begin{array}{c|ccc} A & & O & \\ \hline & B & O & O \\ O & O & B & O \\ & O & O & B \end{array}\right), \tag{3.3}$$

where $A \in S_P(s)$ $(4s=3r-2n)$ and $B \in O(n-r)$. Thus we have

PROPOSITION 4. *A necessary and sufficient condition for a manifold M of dimension n to admit an f-three-structure of rank r is that the structure group of the tangent bundle $T(M)$ over M is reducible to the group $S_P(s)\times O(n-r)$ consisting of all matrices of the form* (3.3), *where* $4s=3r-2n$.

Department of Mathematics
Tokyo Institute of Technology

Bibliography

[1] ISHIHARA, S.: Normal structure f satisfying $f^3+f=0$, Kōdai Math. Sem. Rep., 18 (1966); 36–47.

[2] ISHIHARA, S. and M. KONISI: Fibred Riemannian space with Sasakian 3-structure, Differential Geometry, in honor of Kentaro Yano, 1972, Kinokuniya, Tokyo, 179–194.

[3] KASHIWADA, T.: On three framed f-structures with some relations, to appear in Nat. Sei. Rep., Ochanomizu Univ., 22 (1972), 91–99.

[4] KUO, Y. Y.: On almost contact 3-structure, Tōhoku Math. Jour., 22 (1970), 325–332.

[5] KUO, Y.Y. and S. TACHIBANA: On the distribution appeared in contact 3-structure, Kaita Jour. Math., 2 (1970), 17–24.

[6] OBATA, M.: Affine connection on manifold with almost complex, quaternion, hermitian structure, Jour. Math. Soc. Japan, 26 (1956), 43–77.

[7] TACHIBANA, S. and W. N. Yu,: On a Riemannian space admitting more than one Sasakian structure, Tōhoku Math. Jour., 22 (1970), 536–540.

[8] TANNO, S.: Killing vectors on contact Riemannian manifolds and fibering related to the Hopf fibrations, Tōhoku Math. Jour. 23 (1971), 313–334.

[9] YANO, K.: On a structure defined by a tensor field f of type $(1,1)$ satisfying $f^3+f=0$, Tensor, 14 (1963), 99–109.

[10] YANO, K. and M. AKO: Integrability conditions for almost quaternion structure, Hokkaido Math. Jour. 1 (1972), 63–86.

[11] YANO, K. and S. ISHIHARA: The f-structure induced on submanifolds of complex and almost complex spaces, Kōdai Math. Sem. Rep. 18 (1966), 120–160.

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