On periodic orbits of stable flows

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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Introduction. Let M be a compact C^∞ Riemannian manifold of dimension $n \geq 2$ without boundary and X a C^1 vector field on M. Let $\{f^t\}$ be the one-parameter group of C^1-diffeomorphisms f^t of M generated by X. $\{f^t\}$ is called a *differentiable flow* (or dynamical system) on M. More generally, a one-parameter group of homeomorphisms $\{g^t\}$ is called a *continuous flow* on M if the map $g: M \times \mathbf{R} \to M$ defined by $g(x, t) = g^t(x)$ ($x \in M$, $t \in \mathbf{R}$) is continuous.

A point $x \in M$ is called a *periodic point* of $\{f^t\}$ if there is a $t_0 > 0$ such that $f^{t_0}(x) = x$ holds. We denote by Per $(\{f^t\})$ the set of all periodic points of $\{f^t\}$. The orbit $\{f^t(x) | t \in \mathbf{R}\}$ is called a *periodic orbit* if $x \in \mathrm{Per}(\{f^t\})$.

A point $x \in M$ is called a *non-wandering point* of $\{f^t\}$ if for any neighborhood U of x and any $k > 0$ we can find a $t_0 \geq k$ such that $f^{t_0}(U) \cap U \neq \phi$ holds. We denote by $\Omega(\{f^t\})$ the set of all non-wandering points of $\{f^t\}$. Clearly $\Omega(\{f^t\})$ is closed in M and we have

$$\mathrm{Per}\left(\{f^t\}\right) \subset \Omega\left(\{f^t\}\right).$$

Let Map(M) be the set of all continuous maps f of M into M. For $f, g \in \mathrm{Map}(M)$ we define the metric $d(f, g)$ by

$$d(f, g) = \sup_{x \in M} d\left(f(x), g(x)\right),$$

where d denotes the metric on M induced by the Riemannian metric on M. For any continuous function μ on M we define the norm $\|\mu\|$ by

$$\|\mu\| = \mathop{\mathrm{Max}}_{x \in M} \left|\mu(x)\right|.$$

DEFINITION 1. $\{f^t\}$ is called to be *topologically stable*, if there exists a positive number ε_0 having the following property: For any positive $\varepsilon < \varepsilon_0$, there exists a positive $\delta = \delta(\varepsilon)$ such that for any continuous flow $\{g^t\}$ with $d(f^t, g^t) < \delta$ for $t \in \left[\frac{1}{4}, 1\right]$, there exist a continuous function p on $M \times \mathbf{R}$ and a surjective map $u \in \mathrm{Map}(M)$ such that

$$u\left(g^t(x)\right) = f^{p(x,t)}\left(u(x)\right)$$

holds for every $x \in M$ and $t \in \mathrm{R}$ and that the following conditions are satisfied:

(i) $d(u, 1_M) + \|(1/t)p_t - 1\| < \varepsilon$

for $t \in \left[\frac{1}{4}, 1\right]$, where $p_t(x) = p(x, t)$ ($x \in M$, $t \in \mathrm{R}$),

(ii) $p(x, t+1) = p(g^t(x), 1) + p(x, t) \qquad (x \in M,\ t \in \mathrm{R})$,

(iii) $\|p_t\| \leq 2 \qquad$ for $\quad t \in [0, 1]$.

In this note we shall prove the following

THEOREM 1. *If* $\{f^t\}$ *is topologically stable, then* $Per(\{f^t\})$ *is dense in* $\Omega(\{f^t\})$.

We have proved in [2] the following

THEOREM A. *Any Anosov flow* $\{f^t\}$ *is topologically stable.*

For Anosov flows, see [1], [3], [5].

In fact, we have proved the uniqueness of u and p in Definition 1 under certain conditions on u, which we shall not use in what follows.

Combining Theorem 1 and Theorem A we obtain the following

COROLLARY. *If* $\{f^t\}$ *is an Anosov flow, then* $Per(\{f^t\})$ *is dense in* $\Omega(\{f^t\})$.

Anosov [1] proved the above corollary by making use of stable manifold theory.

It is conjectured that $\Omega(\{f^t\}) = M$ holds for Anosov flow $\{f^t\}$ (cf. [5]).

The idea of the proof of Theorem 1 was inspired by that of Theorem 4 [6].

§1. Preliminary Lemmas. We shall first prove the following lemma which is intuitively clear.

LEMMA 1. *Let* δ_i, ε_i $(i=1,2)$ *and* a, b *be real numbers with* $\delta_1 < \delta_2$, $\varepsilon_1 < a < \varepsilon_2$, *and* $\varepsilon_1 < b < \varepsilon_2$. *Then for each* $y \in \mathrm{R}$ *we can find a continuous curve* $c_y(t)$ $(t \in \mathrm{R})$ *in* R^2 *satisfying the following conditions* (a)~(e)

(a) $c_y(t) = (t, g_y(t))$, $(y, t) \in \mathrm{R}^2$

where g_y *is a differentiable function on* R *and*

$$g_y(t) = \begin{cases} y & \text{for } t \leq \delta_1 \text{ or } y \notin [\varepsilon_1, \varepsilon_2], \\ g_y(\delta_2) & \text{for } t \geq \delta_2 \text{ and } y \in [\varepsilon_1, \varepsilon_2]. \end{cases}$$

(b) $\varepsilon_1 \leq g_y(t) \leq \varepsilon_2 \qquad$ *for* $\quad t \in \mathrm{R}$ *and* $y \in [\varepsilon_1, \varepsilon_2]$.

(c) $g_a(\delta_2) = b$.

(d) *For each* $x \in \mathrm{R}^2$, *we can find one and only one* $(t, y) \in \mathrm{R}^2$ *such that* $x = c_y(t)$.

(e) *For each* $s\in\mathbf{R}$ *we can define the map* ψ^s: $\mathbf{R}^2\to\mathbf{R}^2$ *by* $\psi^s(c_y(t))=c_y(t+s)$. $\{\psi^s\}$ *is a continuous flow on* $\mathbf{R}^2$.

PROOF. Let $g\in C^\infty(\mathbf{R})$ be a differentiable function on $\mathbf{R}$ such that

$$g(t)=\begin{cases} 0 & t\leq\delta_1, \\ 1 & t\geq\delta_2, \end{cases}$$

and that $g(t)<g(s)$ for $\delta_1\leq t<s\leq\delta_2$.

Next, we can find a differentiable monotone increasing function h on $\mathbf{R}$ satisfying the following conditions (i)~(iv).

(i) $h([\varepsilon_1, \varepsilon_2])=[\varepsilon_1, \varepsilon_2]$.

(ii) $h(a)=b$.

(iii) $h|_S=1_S$,

where $S=\mathbf{R}-[\varepsilon_1, \varepsilon_2]$.

(iv) $h(t)\leq t$ (resp. $h(t)\geq t$) for $t\in[\varepsilon_1, \varepsilon_2]$ if $a\geq b$ (resp. $a\leq b$).

Put $g_y(t)=y+(h(y)-y)g(t)$ for $(t, y)\in\mathbf{R}^2$. It is easily seen that $g_y(t)$ is an increasing function of y for fixed $t\in\mathbf{R}$, from which we can verify (d). It is also readily seen that (a)~(c) hold.

By the property (d) the map ψ^s is well defined. The map φ, φ_s of $\mathbf{R}^2$ onto $\mathbf{R}^2$ defined by $\varphi(t, y)=(t, g_y(t))$ and $\varphi_s(t, y)=(t+s, g_y(t+s))$ for $(t, y)\in\mathbf{R}^2$ are both homeomorphisms of $\mathbf{R}^2$ onto $\mathbf{R}^2$. Since $\psi^s=\varphi_s\circ\varphi^{-1}$, we see that $\{\psi^s\}$ is a continuous flow on $\mathbf{R}^2$. Q. E. D.

We can now prove the following lemma which is a generalization of Lemma 1.

LEMMA 2. (Detour Lemma). *Let* ε_i, δ_i $(i=1, 2)$ *be real numbers with* $\delta_1<\delta_2$ *and* $\varepsilon_1<\varepsilon_2$. *Let* $Q=(\varepsilon_1, \varepsilon_2)^{n-1}$ *be the cube in* $\mathbf{R}^{n-1}$, *where* $(\varepsilon_1, \varepsilon_2)=\{t\in\mathbf{R}|\varepsilon_1<t<\varepsilon_2\}$. *Let* $A, B\in Q$.

Then, for each $y\in\mathbf{R}^{n-1}$ *we can find a continuous curve* $C_y(t)$ $(t\in\mathbf{R})$ *in* $\mathbf{R}^n$ *satisfying the following conditions* (a)~(e).

(a) $C_y(t)=(t, G_y(t))$ $t\in\mathbf{R}$,

where G_y: $\mathbf{R}\to\mathbf{R}^{n-1}$ *is differentiable and*

$$G_y(t)=\begin{cases} y & t\leq\delta_1 \text{ or } y\notin\bar{Q} \\ G_y(\delta_2) & t\geq\delta_2 \text{ and } y\in\bar{Q}. \end{cases}$$

(b) $G_y(t)\in\bar{Q}$ $t\in\mathbf{R}$, $y\in\bar{Q}$.

(c) $G_A(\delta_2)=B$.

(d) *For each* $x\in\mathbf{R}^n$ *we can find one and only one* $(t, y)\in\mathbf{R}\times\mathbf{R}^{n-1}$ *such that* $x=C_y(t)$.

(e) *For each* $s\in\mathbf{R}$ *we can define the map* Ψ^s: $\mathbf{R}^n\to\mathbf{R}^n$ *by* $\Psi^s(C_y(t))=C_y(t+s)$. $\{\Psi^s\}$ *is a continuous flow on* $\mathbf{R}^n$.

PROOF. We denote $g_y(t)$ in Lemma 1 by $g_y(t)=g_{a,b,y}(t)$, since g_y depends on a, b. Put $A=(a_2,\cdots,a_n)$, $B=(b_2,\cdots,b_n)$ and $y=(y_2,\cdots,y_n)$ with $a_i, b_i, y_i\in\mathbf{R}$ $(i=2,\cdots,n)$.

We define $G_y(t)$ in the following manner:

$$G_y(t)=\left(g_y^2(t),\cdots,g_y^n(t)\right),$$

where $g_y^i(t)=g_{a_i,b_i,y_i}(t)$ for $i=2,\cdots,n$.

We can see as in Lemma 1 that the family of curves $C_y(t)=(t, G_y(t))$ satisfies the conditions (a)～(e). Q. E. D.

REMARK. We see that the flows $\{\phi^t\}$ and $\{\Psi^t\}$ above are differentiable flows on $\mathbf{R}^2$ and $\mathbf{R}^n$ respectively.

LEMMA 3. *Take* $\varepsilon_0\leq\frac{1}{2}$ *in Definition* 1. *Then the function* $p(x,t)$ *in Def.* 1 *takes positive values for* $t\geq 4$ *and* $x\in M$.

PROOF. Using the property (ii) in Def. 1. we can prove by induction on k that for $t\in[k,k+1]$ we have

$$p(x,t)=p(x,t-k)+\sum_{i=0}^{k-1}p\left(g^{t-i}(x),1\right)$$

for $x\in M$. Since $1-\varepsilon<p(x,1)<1+\varepsilon$ and $|p(x,t)|\leq 2$ for every $(x,t)\in M\times[0,1]$ we get

$$p(x,t)>k(1-\varepsilon)-2,$$

from which the lemma follows. Q. E. D.

§2. Proof of Theorem 1.

Take a point $x_0\in\Omega(\{f^t\})-\mathrm{Per}(\{f^t\})$ and fix a positive $\varepsilon<\varepsilon_0$ and $\delta_0=\delta(\varepsilon)$, where ε_0 is as in Definition 1. We can assume $\delta_0\leq\varepsilon$. Since $x_0\notin\mathrm{Per}(\{f^t\})$ we have $X(x_0)\neq 0$.

Assume first that X is of class C^2. Then, we can find a coordinate neighborhood U of x_0 with C^2-coordinate system $\{x^1,\cdots,x^n\}$ such that $x^i(x_0)=0$, $|x^i|<\delta_1\leq\frac{1}{2}$ for $i=1,2,\cdots,n$ and that

$$X|_U=\partial/\partial x^1|_U \tag{1}$$

(cf. [4] p. 115). For $\delta_1>\delta>0$, we put $U_\delta=\{x\in U \mid |x^i(x)|\leq\delta\ (i=1,\cdots,n)\}$. We take a positive $\delta_2<\delta_1$ such that

(2) $$\operatorname{diam}(U_{\delta_2}) < \delta_0 .$$

We assert that there is a positive $\delta < \frac{\delta_2}{3}$ such that

(3) $$f^t(U_\delta) \cap U_\delta = \phi \quad \text{for} \quad \frac{2\delta_2}{3} \leq t \leq 6 .$$

If not, there would be sequences $\{t_\nu\}$ and $\{p_\nu\} \subset U_{\delta_2}$ such that $p_\nu \to x_0$ $(\nu \to \infty)$, $2\delta_2/3 \leq t_\nu \leq 6$ and that $f^{t_\nu}(p_\nu) \to x_0$ $(\nu \to \infty)$. Then, we can assume that $t_\nu \to t_0$ $(\nu \to \infty)$ with some $t_0 \in [2\delta_2/3, 6]$. Hence $f^{t_0}(x_0) = x_0$, whence $x_0 \in \operatorname{Per}(\{f\})$. Thus our assertion is verified.

We can also assume that δ satisfies the following condition:

(4) $$d_0(x, y) < \delta ;\ x, y \in U \quad \text{imply}$$
$$d\big(f^t(x), f^t(y)\big) < \delta_0$$

for $0 \leq t \leq 1$, where $d_0(x, y) = \operatorname{Max}_i |x^i(x) - x^i(y)|$.

If not, there would be sequences $\{t_\nu\}$ and x_ν, $y_\nu \in U$ $(\nu = 1, 2, \cdots)$ such that

$$d_0(x_\nu, y_\nu) \to 0 , \qquad d\big(f^{t_\nu}(x_\nu), f^{t_\nu}(y_\nu)\big) \geq \delta_0 .$$

We can assume that $t_\nu \to t_0$, $x_\nu \to x^0$, $y_\nu \to y^0$ $(\nu \to \infty)$ with $t_0 \in [0, 1]$, $x^0, y^0 \in M$. Then we have $x^0 = y^0$ and $d(f^{t_0}(x^0), f^{t_0}(y^0)) \geq \delta_0$, which is a contradiction.

Now, since $x_0 \in \Omega(\{f^t\})$, there is a $t_1 \geq 6$ such that $f^{t_1}(U_\delta^0) \cap U_\delta^0 \neq \phi$, where U_δ^0 denotes the interior of U_δ. Hence there are two points $v, w \in U_\delta^0$ such that $f^{t_1}(v) = w$ holds. Put $t_0^* = \inf\{t \geq 6 \mid f^t(v) = w\}$. Then we have $f^{t_0^*}(v) = w$. Consider two points $A', B' \in U_\delta$ defined by

(5) $$A' = f^{-x^1(w) - \delta}(w) , \qquad B' = f^{\delta - x^1(v)}(v) .$$

Clearly we have

(6) $$x^1(A') = -\delta , \qquad x^1(B') = \delta .$$

Hence in the coordinate system $\{x^1, \cdots, x^n\}$, we have $A' = (-\delta, A), B' = (\delta, B)$ with $A, B \in (-\delta, \delta)^{n-1}$.

By making use of the Detour Lemma for $\varepsilon_1 = \delta_1 = -\delta$, $\varepsilon_2 = \delta_2 = \delta$, and $Q = (-\delta, \delta)^{n-1}$, we can construct a continuous flow $\{g^t\}$ on M by patching up the restriction of the flows $\{f^t|_{M - U_\delta}\}$ and $\{\Psi^t|_{U_\delta}\}$. The flow $\{g^t\}$ has the following properties:

(α) $g^{2\delta}(A') = B'$.

(β) $x \in M$, $f^t(x) \notin U_\delta$ for $t \in [0, 1]$ imply $g^t(x) = f^t(x)$ for $t \in [0, 1]$.

(γ) If $f^{t_0}(x) \in U_\delta$ and $f^t(x) \notin U_\delta$ for $0 < t < t_0$, then $g^t(x) = f^t(x)$ for

$0 \leq t \leq t_0$ and $g^t(x) = \Psi^{t-t_0}(f^{t_0}(x))$ for $t_0 + 2\delta \geq t \geq t_0$.

Next, we assert that

$$(7) \qquad d(f^t, g^t) < \delta_0 \qquad \text{for} \quad t \in [0, 1].$$

Take a point $x \in M$ and fix it. If $f^t(x) \notin U_\delta$ for all $t \in [0, 1]$, then we have $f^t(x) = g^t(x)$ and so $d(f^t(x), g^t(x)) = 0$.

Assume that there is a $t_1 \in [0, 1]$ such that $f^{t_1}(x) \in U_\delta$. Put $t_1 = \inf\{t \in [0, 1] | f^t(x) \in U_\delta\}$ and $t_2 = t_1 + 2\delta$.

In case $t_2 \in [0, 1]$, we have $f^t(x) = g^t(x)$ for $0 \leq t \leq t_1$, $f^t(x) \in U_\delta$ for $t_1 \leq t \leq t_2$ and $f^t(x) \notin U_\delta$ for $t_2 < t \leq 1$ by virtue of (3) and (1). Hence we get $f^t(x)$, $g^t(x) \in U_\delta$ for $t_1 \leq t \leq t_2$, which implies $d(f^t(x), g^t(x)) < \delta_0$ by (2). For $t_2 < t \leq 1$ we have $g^t(x) = f^{t-t_2}(x')$, where $x' = C_{f^{t_1}(x)}(2\delta)$ by (γ). Put $x'' = f^{t_2}(x)$. Then $x', x'' \in U_\delta$ and $d_0(x', x'') \leq \delta$. Hence we get $d(f^t(x), g^t(x)) = d(f^{t-t_0}(x''), f^{t-t_2}(x')) < \delta_0$ by (4).

In case $t_2 > 1$, we have $f^t(x) \in U_\delta$ for $t_1 \leq t \leq 1$. We get $d(f^t(x), g^t(x)) = 0$ for $0 \leq t \leq t_1$ and $f^t(x)$, $g^t(x) \in U_\delta$ for $t_1 \leq t \leq 1$, which implies $d(f^t(x), g^t(x)) < \delta_0$ by (2).

Thus (7) is proved.

By our assumption and Def. 1 there exist a map $u: M \to M$ and a function p on $M \times \mathrm{R}$ satisfying the following condition:

$$(8) \qquad u\left(g^t(x)\right) = f^{p(x,t)}\left(u(x)\right)$$

for $(x, t) \in M \times \mathrm{R}$ and $d(u, 1_M) < \varepsilon$.

Put $t_0' = t_0^* + x^1(v) - x^1(w) - 2\delta$. Then, since $f^{t_0^*}(v) = w$ we have

$$(9) \qquad f^{t_0'}(B') = A'$$

by virtue of (5).

Clearly we have $t_0^* \geq t_0' \geq 4$ since $t_0^* \geq 6$. Using (3) and (1) we see that

$$f^t(B') \notin U_\delta \qquad \text{for} \quad 0 < t < t_0'.$$

By (β) and (9) we have $g^{t_0'}(B') = f^{t_0'}(B') = A'$. By (α) we get $B' = g^{2\delta}(A') = g^{t_0'+2\delta}(B') = g^{t_0''}(B')$, where we put $t_0'' = t_0' + 2\delta$. By (8) we obtain

$$u(B') = u\left(g^{t_0''}(B')\right) = f^{p(B', t_0'')}\left(u(B')\right).$$

Now, by virtue of Lemma 3 we have $p(B', t_0'') > 0$, since $t_0'' \geq t_0' \geq 4$. Hence the point $u(B')$ is a periodic point of $\{f^t\}$. Since $d(u, 1_M) < \varepsilon$, we have $d(x_0, u(B')) \leq d(x_0, B') + d(B', u(B')) < 2\varepsilon$. Thus, we have proved that there is a periodic point of $\{f^t\}$ in the 2ε-neighborhood of x_0. Since $\varepsilon > 0$ can be taken arbitrarily small, we have proved $\mathrm{Per}(\{f^t\})$ is dense in $\Omega(\{f^t\})$.

In the case X is not of class C^2, we construct a "flow box" around x_0, namely, a homeomorphism Φ of an open cube $V=(-\delta, \delta)^n$ in R^n onto a neighborhood U of x_0 satisfying the following conditions

(i) $\Phi(0)=x_0$,

(ii) $(\Phi(x_1+t, x_2, \cdots, x_n))=f^t(\Phi(x_1, \cdots, x_n))$

for $(x_1, \cdots, x_n)$, $(x_1+t, x_2, \cdots, x_n)\in V$. Using the Detour lemma for V and transporting it into U by Φ we can construct a continuous flow $\{g^t\}$ satisfying (7). Therefore, we can prove Theorem 1 in the same way as in the case when X is of class C^2. Q. E. D.

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