Complete surfaces in 3-dimensional space forms

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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For surfaces in a Euclidean 4-space E^4, the author [4] proved the following

THEOREM. *A complete, connected, oriented and pseudo-umbilical surfaces immersed in* E^4 *with non-vanishing constant mean curvature* H *and the Gaussian curvature* K *which does not change its sign is necessarily either a Clifford flat torus in* E^4 *or a sphere with radius* $1/H$ *in a hyperplane* E^3.

In this case, by Lemma 2.2 in [4] we see that surfaces are minimal in a hypersphere S^3 in E^4. In this paper, the author will study surfaces with constant mean curvature H in a 3-dimensional Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$. Our main result is the following

THEOREM. *Let* M *be a complete, connected and oriented 2-dimensional Riemannian manifold isometrically immersed in a 3-dimensional oriented Riemannian manifold* $\bar{M}$ *of constant curvature* $\bar{c}$. *If* $H^2+\bar{c}$ *is positive constant and the Gaussian curvature* K *does not change its sign, then we have*

(I) *M is umbilic free and* $K=0$ *on* M,

or

(II) *M is tottally umbilic and* $K=H^2+\bar{c}$ *on* M.

By this theorem, we can verify the following results:

COROLLARY 1. *Let* M *be a complete, oriented and connected 2-dimensional Riemannian manifold isometrically immersed in a unit 3-sphere* S^3 *in* E^4. *If the mean curvature* H *is constant and the Gaussian curvature* K *does not change its sign, then* M *is a sphere or a Clifford flat torus.*

COROLLARY 2. (*T. Klotz and R. Osserman* [3]) *Let* M *be a complete, oriented and connected 2-dimensional Riemannian manifold isometrically immersed in a Euclidean 3-space* E^3. *If* H *is non-zero constant and* K *does not change its sign, then* M *is a sphere or a right circular cylinder.*

COROLLARY 3. *Let* M *be a complete, oriented and connected 2-dimensional Riemannian manifold isometrically immersed in a hyperbolic 3-space* H^3 *of constant curvature* -1. *If* H^2-1 *is positive constant and* K *does not change its sign, then* M *is a sphere or a right circular cylinder.*

Let $\bar{M}$ be a 3-dimensional Riemannian manifold of constant curvature

$\bar{c}$ and M be a 2-dimensional Riemannian manifold isometrically immersed in $\bar{M}$ with the immersion $x: M \to \bar{M}$. Let $F(\bar{M})$ and $F(M)$ be the bundles of all orthonormal frames over $\bar{M}$ and M respectively. Let B be the set of all elements $b=(p, e_1, e_2, e_3) \in F(\bar{M})$ such that $(p, e_1, e_2) \in F(M)$, identifying $p \in M$ with $x(p)$ and e_i with $dx(e_i)$, $i=1, 2$. Then B is cosidered as a smooth submanifold of $F(\bar{M})$. We have, as is well known, a system of differential 1-forms ω_1, ω_2, $\omega_{12}=-\omega_{21}$, $\omega_{13}=-\omega_{31}$, $\omega_{23}=-\omega_{32}$ on B associated with the immersion x such that

$$
(0.1) \qquad \begin{cases} d\omega_i = \omega_{ij} \wedge \omega_j, & (i,j=1,2,\ i \neq j) \\ d\omega_{12} = -\omega_{13} \wedge \omega_{23} - \bar{c}\omega_1 \wedge \omega_2, & \\ d\omega_{i3} = \omega_{ij} \wedge \omega_{j3}, & (i,j=1,2,\ i \neq j) \end{cases}
$$

and

$$
(0.2) \qquad \omega_{i3} = \sum_j A_{ij}\omega_j, \qquad A_{ij} = A_{ji}, \qquad (i,j=1,2)
$$

We call $H=1/2\sum_i A_{ii}$ *the mean curvature.* M is said to be *umbilic* at p if $A_{11}=A_{22}=H$ and $A_{12}=0$ at p. We say M to be *totally umbilic* if M is umbilic at each point of M. We may consider M as a Riemann surface, because M is a 2-dimensional oriented Riemannian manifold. We say M to be *parabolic* if there are non-constant negative subharmonic functions on M. We shall prove the theorem for the case (1) $K \leqq 0$ and the case (2) $K \geqq 0$.

§ 1. The proof of the theorem.

We first prove the following

PROPOSITION 1. *Let M be a complete, oriented and connected 2-dimensional Riemannian manifold immersed in a 3-dimensional oriented Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$. If $H^2+\bar{c}$ is positive constant and K is not greater than zero, then M is umbilic free and $K=0$ on M.*

PROOF. The Gaussian curvature K is given by the equation $d\omega_{12} = -K\omega_1 \wedge \omega_2$. On the other hand, by (0.1) and (0.2) we have $d\omega_{12} = -(\bar{c}+detA)\omega_1 \wedge \omega_2$, where A is the matrix (A_{ij}). Writting $\omega_{13}=(H+h_1)\omega_1+h_2\omega_2$ and $\omega_{23}=h_2\omega_1+(H-h_1)\omega_2$, we have

$$
(1.1) \qquad K = \bar{c} + H^2 - (h_1^2 + h_2^2),
$$

which, together with $K \leqq 0$ and $H^2+\bar{c}>0$, implies that M is umbilic free. Hence, we can choose locally frames $b \in B$ such that A is given by

$$
(1.2) \qquad A = \begin{pmatrix} H+h & 0 \\ 0 & H-h \end{pmatrix},
$$

where the function h is differentiable and defined globally on M, because $\det A = H^2 - h^2$ is a global differentiable function on M. Since M is umbilic free, we may suppose $h>0$ on M. Using the structure equations (0.1) for ω_{i3}, we have

$$2h d\omega_1 + dh \wedge \omega_1 = 0\,,$$
$$2h d\omega_2 + dh \wedge \omega_2 = 0\,,$$

which show that we have a neighborhood U of a point $p \in M$ in which there exist the following isothermal coordinates (u, v):

$$ds^2 = \lambda\{du^2 + dv^2\}\,, \quad \omega_1 = \sqrt{\lambda}\, du\,, \quad \omega_2 = \sqrt{\lambda}\, dv\,, \quad h\lambda = 1\,, \tag{1.3}$$

where $\lambda = \lambda(u, v)$ is a positive function on U. Now, we get the following

LEMMA 1. *The universal covering surface $\widetilde{M}$ of M is conformally equivalent to the entire plane, so that M is parabolic.*

PROOF OF LEMMA. Since $H^2 + \bar{c}$ is positive constant, the conformal metric $\sqrt{H^2+\bar{c}}\, ds^2$ is complete on M. However, since $\sqrt{H^2+\bar{c}} \leqq h$, the conformal metric hds^2 is also complete on M. Furthermore, the metric hds^2 is flat from (1.3). Hence the covering surface $\widetilde{M}$ with the lifted metric from hds^2 on M is isometric to the entire plane. Thus $\widetilde{M}$ is conformally equivalent to the entire plane, so that $\widetilde{M}$ is parabolic. Hence M is also parabolic.

As is well known, the Gaussian curvature K is given by

$$K = -(1/2\lambda)\Delta \log \lambda\,, \quad \Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2\,,$$

with respect to the isothermal coordinates (u, v). Since $K \leqq 0$ and $h\lambda = 1$, we have

$$\Delta \log h = -\Delta \log \lambda \leqq 0\,,$$

which implies that the function $\log h$ is a superharmonic function on M. Since $0 < H^2 + \bar{c} \leqq h^2$, the superharmonic function $\log h$ on M is bounded from below by $(1/2)\log(H^2+\bar{c})$, so that $\log h$ must be constant, because M is parabolic by Lemma 1. Therefore, K is identically zero on M. Thus we have proved Proposition 1.

We next prove the following

PROPOSITION 2. *Let M be a complete, oriented and connected 2-dimensional Riemannian manifold immersed in a 3-dimensional oriented Riemannian manifold $\bar{M}$ of consttant curvature $\bar{c}$. If $H^2+\bar{c}$ is positive constant and K is not less than zero, then we have*

(i) *M is umbilic free and $K=0$ on M,*

or

(ii) *M is totally umbilic and $K=H^2+\bar{c}$ on M.*

PROOF. We first prove.

LEMMA 2. *K is a superharmonic function on M.*

PROOF of LEMMA. Let M_0 be the set of all points at which M is umbilic, i.e., $A_{11}=A_{22}=H$ and $A_{12}=0$. Since M_0 is closed in M, $M_1=M-M_0$ is open in M. Then, analogously in the proof of Proposition 1, we can choose a neighborhood U of a point $p\in M_1$ in M_1 where there exist isothermal coordinates (u, v) such that

$$(1.4)\qquad \begin{cases} ds^2=\lambda\{du^2+dv^2\}, \quad \omega_1=\sqrt{\lambda}\,du\,, \quad \omega_2=\sqrt{\lambda}\,dv\,, \\ A=\begin{pmatrix} H+h & 0 \\ 0 & H-h \end{pmatrix}, \qquad h>0\,, \quad h\lambda=1\,, \end{cases}$$

where h is a differentiable function on U: Since K is given by

$$K=-(1/2\lambda)\Delta\log\lambda=(h/2)\Delta\log h\leqq 0 \quad \text{and} \quad h>0\,,$$

we have $\Delta h\geqq 0$, so that we get

$$\Delta K=-\Delta h^2=-2\left\{(\partial h/\partial u)^2+(\partial h/\partial v)^2\right\}-2h\Delta h\leqq 0\,.$$

Thus we have $\Delta K\leqq 0$ on M_1. We next prove that $\Delta K\leqq 0$ at any point of M_0. Take a point p_0 of M_0 and consider the isothermal coordinates (u, v) and frames on a neighborhood V of p_0 such that

$$ds^2=\lambda\{du^2+dv^2\}, \quad \omega_1=\sqrt{\lambda}\,du\,, \quad \omega_2=\sqrt{\lambda}\,dv\,.$$

In this case, the second foundamental form A may be represented by

$$A=\begin{pmatrix} H+h_1 & h_2 \\ h_2 & H-h_1 \end{pmatrix},$$

where h_1 and h_2 are functions on V. Then we have

$$K=\bar{c}+H^2-(h_2^2+h_2^2) \qquad \text{on} \quad V\,.$$

Hence, with respect to the isothermal coordinates (u, v), we get on V

$$(1.5)\qquad \begin{aligned} \Delta K=&-2\left\{(\partial h_1/\partial u)^2+(\partial h_1/\partial v)^2+(\partial h_2/\partial u)^2+(\partial h_2/\partial v)^2\right\} \\ &-2h_1\Delta h_1-2h_2\Delta h_2\,. \end{aligned}$$

Since h_1 and h_2 attain zero at p_0, we have

$$\Delta K\leqq 0 \quad \text{at} \quad p_0\,.$$

Thus we have $\Delta K\leqq 0$ at a point of M_0. We have proved Lemma.

Now, if M is compact, the superharmonic function K on M attains its minimum at some point on M, so that K must be constant on M. On the other hand, if M is non compact, M is parabolic by Theorem 15 in Huber [2], because $K \geqq 0$. Since K is non-negative superharmonic function on M, K must be constant on M. Thus K is constant on M. Since $K=\bar{c}+H^2-(h_1^2+h_2^2)=\text{constant}$ and $H^2+\bar{c}=\text{constant}>0$, we can consider the following two cases:

Case (a): M_0 is not empty.

Case (b): M_0 is empty.

We first consider the case (a). If M_0 is not empty, $H^2+\bar{c}-K$ attains zero at points of M_0, so that $H^2+\bar{c}-K$ must be identically zero on M. Hence, $K=H^2+\bar{c}=\text{constant.}>0$ holds identically on M.

We next consider the case (b). If M_0 is empty, in the same manner as the proof of Proposition 1, we can choose a neighborhood U of a point $p \in M$ satisfying (1.2) and (1.3). Then, since $K=\bar{c}+H^2-h^2$ is constant, h^2 is also constant, which implies $K=0$, because $K=(h/2)\Delta \log h$. Thus, we have proved Proposition 2.

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