On certain integral formulas for hypersurfaces in a constant curvature space

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 0. Introduction.

Let V^m be a closed orientable hypersurface twice differentiably imbedded in an $(m+1)$-dimensional Euclidean space $E^{m+1}(m+1\geqq 3)$ and $k_1, \cdots, k_m$ be the m principal curvatures at a point P of V^m. The ν-th mean curvature H_ν of V^m at P is defined by

$$\binom{m}{\nu}H_\nu = \sum k_1 \cdots k_\nu \qquad (\nu=1, 2, \cdots, m),$$

where the right hand member denotes the ν-th elementary symmetric function of $k_1, \cdots, k_m$. It is convenient to define $H_0=1$. C. C. Hsiung [1][1)] proved

$$\int_{V^m}(H_{\nu+1}p+H_\nu)dA = 0 \qquad (\nu=0, 1, \cdots, m-1), \tag{0. 1}$$

where p denotes the oriented distance from a fixed point O in E^{m+1} to the tangent space of V^m at P and dA is the area element of V^m. Let $\bar{V}^m$ be a closed orientable hypersurface parallel to the given V^m. Then, the integral formulas (0. 1) have been derived by comparison between associated quantities of V^m and $\bar{V}^m$.

Let R^{m+1} be an $(m+1)$-dimensional Riemann space of class $C^r(r\geqq 3)$, which admits an infinitesimal conformal transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta\tau . \tag{0. 2}$$

We assume that a closed orientable hypersurface V^m does not pass through any singular point of a tangent vector field of the paths with respect to the infinitesimal transformation (0. 2). Since the transformation is conformal, there exists a scalar field Φ and the vector ξ^i satisfies the relation

$$\xi_{i;j}+\xi_{j;i} = 2\Phi g_{ij}, \tag{0. 3}$$

where $\xi_i=g_{ij}\xi^j$ and the symbol "$;$" means covariant differentiation with respect to Riemann connection determined by the metric tensor g_{ij} of R^{m+1}

1) Numbers in brackets refer to the references at the end of the paper.

(K. Yano [2]). As the generalizations of (0.1) for a Riemann space, Y. Katsurada [3] derived

$$\int_{V^m}(H_1 p+\Phi)dA=0 \tag{0.4}$$

for V^m in R^{m+1} and when R^{m+1} is a constant curvature space, proved

$$\int_{V^m}(H_{\nu+1}p+H_\nu\Phi)dA=0 \qquad (\nu=1,2,\cdots,m-1), \tag{0.5}$$

where $p=n^i\xi_i$ and n^i is the unit normal vector of V^m. The integral formulas (0.4) and (0.5) have been derived by applying Stokes' theorem to the relations obtained by exterior differentiation of certain differential forms on V^m.

Certain generalizations of (0.4) and (0.5) for a closed orientable submanifold of codimension greater than 1 have been given by Y. Katsurada and H. Kôjyô [4].

These integral formulas have been applied by many authors to the study of closed orientable submanifolds with constant ν-th mean curvature in a Euclidean space and a Riemann space.

Recently, K. Amur [5] derived (0.1) in a different way and also proved for V^m in E^{m+1}

$$\int_{V^m}(\nabla H_\nu\cdot X)dA+m\int_{V^m}(H_1H_\nu-H_{\nu+1})p\,dA=0\,. \qquad (\nu=0,1,\cdots,m-1) \tag{0.6}$$

where the integrand of the first term in the left hand member denotes inner product of *grad* H_ν and the position vector X of V^m.

Some generalizations of (0.6) for a closed orientable submanifold of codimension greater than 1 in E^{m+1} have been derived by K. Yano and B. Y. Chen [6].

The main purpose of the present paper is to give an integral formula which is similar to (0.6) and valid for a closed orientable hypersurface V^m in a constant curvature space R^{m+1}. In accordance with the idea given by Y. Katsurada [3], we also assume that R^{m+1} admits a conformal Killing vector field ξ^i and use it in place of the position vector X in (0.6). The method of calculations is learned much from the paper of K. Amur [5]. §1 is devoted to give some notations and fundamental relations which will be used in the following section. Some integral formulas will be given in §2.

The present author wishes to express his sincere appreciation to Professor Y. Katsurada for her kind guidance.

§1. Preliminaries.

Let R^{m+1} $(m+1\geqq 3)$ be an $(m+1)$-dimensional Riemann space and x^i

and g_{ij} be the local coordinate and the positive definite metric tensor of R^{m+1}. We consider that a closed orientable hypersurface V^m in R^{m+1} is expressed locally by parametric equations

$$x^i = x^i(u^\alpha). \qquad (i=1,2,\cdots,m+1:\ \alpha=1,2,\cdots,m)^{2)}$$

If we put

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \qquad (\alpha=1,2,\cdots,m),$$

the m vectors $B_\alpha=(B^1_\alpha,\cdots,B^{m+1}_\alpha)$ are linearly independent and span the tangent space of V^m. The induced metric tensor $g_{\alpha\beta}$ of V^m is given by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta$$

and $g^{\alpha\beta}$ is defined by $g^{\alpha\beta}g_{\beta\gamma}=\delta^\alpha_\gamma$, where δ^α_γ denotes the Kronecker delta.

Denoting by $N=(n^1,\cdots,n^{m+1})$ a contravariant vector such that

$$(1.1) \qquad g_{ij}B^i_\alpha n^j = 0, \qquad g_{ij}n^i n^j = 0,$$

$$(1.2) \qquad \det.(B_1,\cdots,B_m,N)>0,$$

then we can see that N is determined uniquely at each point of V^m and is the unit normal vector of V^m.

If we denote by the symbol ";" the covariant differentiation due to van der Waerden-Bortolotti, we have the following Gauss and Weingarten formula:

$$(1.3) \qquad B^i_{\alpha;\beta} = b_{\alpha\beta}n^i,$$

$$(1.4) \qquad n^i{}_{;\alpha} = -b^\gamma_\alpha B^i_\gamma,$$

where $b_{\alpha\beta}$ is the second fundamental tensor of V^m and $b^\gamma_\alpha=b_{\alpha\beta}g^{\gamma\beta}$.

Let $k_1,\cdots,k_m$ be the roots of the characteristic equation

$$\det.(b_{\alpha\beta}-kg_{\alpha\beta})=0,$$

then the ν-th mean curvature H_ν of V^m is defined to be the ν-th elementary symmetric function of $k_1,\cdots,k_m$ divided by the number of terms, i.e.

$$\binom{m}{\nu}H_\nu = \sum k_1\cdots k_\nu \qquad (\nu=1,2,\cdots,m).$$

As usual we put $H_0=1$.

We denote by $\varepsilon_{i_1\cdots i_{m+1}}$ and $\varepsilon^{i_1\cdots i_{m+1}}$ the ε-tensor in R^{m+1}, that is

2) Throughout the present paper the Latin indices run from 1 to $m+1$ and the Greek indices run from 1 to m.

$$\varepsilon_{i_1\cdots i_{m+1}} = \sqrt{G}\; e_{i_1\cdots i_{m+1}},$$
$$\varepsilon^{i_1\cdots i_{m+1}} = (\sqrt{G})^{-1} e^{i_1\cdots i_{m+1}},$$

where $G = \det.(g_{ij})$ and the quantites $e_{i_1\cdots i_{m+1}} = e^{i_1\cdots i_{m+1}}$ are defined to be zero, when two or more of the indices are the same, and to be 1 or -1 when the indices are obtainable from the natural sequence $1, 2, \cdots, m+1$ by an even or odd permutation.

Let

$$V_{(\lambda)} = (v^1_{(\lambda)}, \cdots, v^{m+1}_{(\lambda)}), \qquad (\lambda = 1, 2, \cdots, p)$$
$$W_{(\mu)} = (w^1_{(\mu)\alpha} du^\alpha, \cdots, w^{m+1}_{(\mu)\alpha} du^\alpha), \qquad (\mu = p+1, p+2, \cdots, m)$$

be contravariant vectors and vector valued differential forms in R^{m+1}, then we define a combined product $[\ ]_i$ and its exterior differential $\delta[\ ]_i$ by

$$[V_{(1)}, \cdots, V_{(p)}, W_{(p+1)}, \cdots, W_{(m)}]_i$$
$$= \varepsilon_{i_1\cdots i_m i} v^{i_1}_{(1)} \cdots v^{i_p}_{(p)} w^{i_{p+1}}_{(p+1)\alpha_{p+1}} \cdots w^{i_m}_{(m)\alpha_m} du^{\alpha_{p+1}} \wedge \cdots \wedge du^{\alpha_m},$$
$$\delta[V_{(1)}, \cdots, V_{(p)}, W_{(p+1)}, \cdots, W_{(m)}]_i$$
$$= (\varepsilon_{i_1\cdots i_m i} v^{i_1}_{(1)} \cdots v^{i_p}_{(p)} w^{i_{p+1}}_{(p+1)\alpha_{p+1}} \cdots w^{i_m}_{(m)\alpha_m})_{;\alpha} du^\alpha \wedge du^{\alpha_{p+1}} \wedge \cdots \wedge du^{\alpha_m},$$

where $\wedge$ means exterior product.

By means of (1.1) and (1.2), we have

$$\varepsilon_{i_1\cdots i_m i} B^{i_1}_1 \cdots B^{i_m}_m n^i = \sqrt{g}, \tag{1.5}$$

where $g = \det.(g_{\alpha\beta})$. Making use of (1.5), we can see that

$$[B_1, \cdots, B_m]_i = \sqrt{g}\; n_i, \tag{1.6}$$

$$[N, B_1, \cdots, B_{\alpha-1}, B_{\alpha+1}, \cdots, B_m]_i = (-1)^\alpha \sqrt{g}\; B^\alpha_i, \tag{1.7}$$

where $B^\alpha_i = g_{ij} g^{\alpha\beta} B^j_\beta$.

If we put

$$U_\alpha = (-1)^\alpha \sqrt{g}\; du^1 \wedge \cdots \wedge du^{\alpha-1} \wedge du^{\alpha+1} \wedge \cdots \wedge du^m,$$

we can verify that U_α are transformed under parameter transformation $\bar{u}^\lambda = \bar{u}^\lambda(u^\alpha)$ such that

$$U_\alpha = \frac{\partial \bar{u}^\lambda}{\partial u^\alpha} \bar{U}_\lambda,$$

where $\bar{U}_\lambda = (-1)^\lambda \sqrt{\bar{g}}\; d\bar{u}^1 \wedge \cdots \wedge d\bar{u}^{\lambda-1} \wedge d\bar{u}^{\lambda+1} \wedge \cdots \wedge d\bar{u}^m$. Therefore U_α is a covariant vector.

Denoting by dx the vector valued differential form

$$dx = (dx^1, \cdots, dx^{m+1}),$$

where $dx^i=B^i_\alpha du^\alpha$, then by means of (1.7) we get

$$[N, dx, \cdots, dx]_i=(m-1)!\,B^\alpha_i U_\alpha\,. \tag{1.8}$$

From (1.8) it follows that

$$dx^i[N, dx, \cdots, dx]_i=-m!\,dA\,, \tag{1.9}$$

where we put $dA=\sqrt{g}\ du^1\wedge\cdots\wedge du^m$ and dA is the area element of V^m.

From (1.4) we have $\delta n^i=-b^\gamma_\alpha B^i_\gamma du^\alpha$. Therefore, we obtain

$$\delta n^i[N, dx, \cdots, dx]_i=m!\,H_1 dA\,, \tag{1.10}$$

$$[\overbrace{\delta N, \cdots, \delta N}^{\nu},\ \overbrace{dx, \cdots, dx}^{m-\nu}]_i=(-1)^\nu m!\,H_\nu n_i dA\,. \tag{1.11}$$

If f is a scalar field on V^m, by means of (1.8) we have

$$df\wedge[N, dx, \cdots, dx]_i=-(m-1)!\,\frac{\partial f}{\partial u^\alpha}B^\alpha_i dA\,. \tag{1.12}$$

§ 2. Integral formulas.

THEOREM 2.1. *Let R^{m+1} be an $(m+1)$-dimensional Riemann space which admits a conformal Killing vector field ξ^i and V^m a closed orientable hypersurface in R^{m+1}. Then*

$$\int_{V^m}\frac{\partial H_\nu}{\partial u^\alpha}\xi^\alpha dA+m\int_{V^m}(H_\nu\Phi+H_1H_\nu p)dA=0\,, \qquad (\nu=0, 1, \cdots, m) \tag{2.1}$$

where $\xi^\alpha=B^\alpha_i\xi^i$.

PROOF. We have

$$\delta\big(H_\nu[N, dx, \cdots, dx]_i\big)=dH_\nu\wedge[N, dx, \cdots, dx]_i+H_\nu\delta[N, dx, \cdots, dx]_i\,.$$

Making use of (1.11) and (1.12) we get

$$\delta\big(H_\nu[N, dx, \cdots, dx]_i\big)=-(m-1)!\,\frac{\partial H_\nu}{\partial u^\alpha}B^\alpha_i dA-m!\,H_1H_\nu n_i dA\,.$$

Therefore we have

$$\xi^i\delta\big(H_\nu[N, dx, \cdots, dx]_i\big)=-(m-1)!\,\frac{\partial H_\nu}{\partial u_\alpha}\xi^\alpha dA-m!\,H_1H_\nu p dA\,. \tag{2.2}$$

If we put

$$S=H_\nu\xi^i[N, dx, \cdots, dx]_i\,,$$

then we have

(2. 3) $$\xi^i\delta\left(H_\nu[N, dx, \cdots, dx]_i\right) = dS - H_\nu\delta\xi^i \wedge [N, dx, \cdots, dx]_i .$$

In consequence of (0. 3) and (1. 7) it follows that

(2. 4) $$\delta\xi^i \wedge [N, dx, \cdots, dx]_i = -m\,!\,\Phi dA .$$

By means of (2. 2), (2. 3) and (2. 4) we get

$$\frac{\partial H_\nu}{\partial u^\alpha}\xi^\alpha dA + m(H_\nu\Phi + H_1 H_\nu p)dA + \frac{dS}{(m-1)!} = 0 .$$

Since V^m is a closed orientable hypersurface, applying Stokes' theorem to the last relation, we obtain (2. 1).

THEOREM 2. 2. *Let R^{m+1} be an $(m+1)$-dimensional constant curvature space which admits a conformal Killing vector field ξ^i and V^m a closed orientable hypersurface in R^{m+1}. Then*

(2. 5) $$\int_{V^m}\frac{\partial H_\nu}{\partial u^\alpha}\xi^\alpha dA + m\int_{V^m}(H_1 H_\nu - H_{\nu+1})p\,dA = 0 . \qquad (\nu = 0, 1, \cdots, m-1)$$

PROOF. We put

$$(\Delta_\nu)_i = [N, \overbrace{\delta N, \cdots, \delta N}^{\nu}, \overbrace{dx, \cdots, dx}^{m-\nu-1}]_i .$$

Since R^{m+1} is a constant curvature space, we have $\delta\delta N = 0$. Therefore, by means of (1. 11) we get

(2. 6) $$\xi^i\delta(\Delta_\nu)_i = (-1)^{\nu+1} m\,!\, H_{\nu+1} p\,dA .$$

On the other hand, as in [5] we obtain

(2. 7) $$(\Delta_\nu)_i = (-1)^\nu \frac{m}{m-\nu} H_\nu [N, dx, \cdots, dx]_i + (-1)^\nu \nu\,!(m-\nu-1)!\sum_{p=1}^{\nu}(-1)^p \binom{m}{\nu-p} H_{\nu-p} K_{(p)i} ,$$

where we put

$$K_{(p)i} = B_i^1 (k_1)^p U_1 + B_i^2 (k_2)^p U_2 + \cdots + B_i^m (k_m)^p U_m .$$

Making use of (1. 11) and (1. 12), we get from (2. 7)

(2. 8) $$\xi^i\delta(\Delta_\nu)_i = (-1)^{\nu+1}\frac{m}{m-\nu}(m-1)!\left(\frac{\partial H_\nu}{\partial u^\alpha}\xi^\alpha + mH_1 H_\nu p\right)dA + (-1)^\nu \nu\,!(m-\nu-1)!\sum_{p=1}^{\nu}(-1)^p \binom{m}{\nu-p}\xi^i\delta(H_{\nu-p} K_{(p)i}) .$$

If we put

$$T=\xi^{i}H_{\nu-p}K_{(p)i},$$

it follows that

$$\xi^{i}\delta(H_{\nu-p}K_{(p)i})=dT-H_{\nu-p}\delta\xi^{i}\wedge K_{(p)i}. \tag{2.9}$$

By virtue of (0.3), we can see that

$$\delta\xi^{i}\wedge K_{(p)i}=-\Phi\sum_{\lambda=1}^{m}(k_{\lambda})^{p}dA. \tag{2.10}$$

In consequence of (2.9) and (2.10), (2.8) can be rewritten as follows:

$$\begin{aligned}\xi^{i}\delta(\Delta_{\nu})_{i}&=(-1)^{\nu+1}\frac{m}{m-\nu}(m-1)!\left(\frac{\partial H_{\nu}}{\partial u^{\alpha}}\xi^{\alpha}+mH_{1}H_{\nu}p\right)dA\\&\quad+(-1)^{\nu}\nu!(m-\nu-1)!\sum_{p=1}^{\nu}(-1)^{p}\binom{m}{\nu-p}\left(dT+\Phi H_{\nu-p}\sum_{\lambda=1}^{m}(k_{\lambda})^{p}dA\right).\end{aligned} \tag{2.11}$$

According to the identity of Newton for the elementary symmetric functions, we have

$$\sum_{p=1}^{\nu}(-1)^{p}\binom{m}{\nu-p}H_{\nu-p}\sum_{\lambda=1}^{m}(k_{\lambda})^{p}=-\nu\binom{m}{\nu}H_{\nu}. \qquad \text{(See [5])} \tag{2.12}$$

Making use of (2.6), (2.11) and (2.12), we obtain

$$\begin{aligned}&\left(\frac{\partial H_{\nu}}{\partial u^{\alpha}}\xi^{\alpha}+mH_{1}H_{\nu}p-(m-\nu)H_{\nu+\nu}p+\nu\Phi H_{\nu}\right)dA\\&\qquad-\binom{m}{\nu}^{-1}\sum_{p=1}^{\nu}(-1)^{p}\binom{m}{\nu-p}dT=0.\end{aligned}$$

Since V^{m} is a closed orientable hypersurface, applying Stokes' theorem to the last relation we obtain

$$\int_{V^{m}}\frac{\partial H_{\nu}}{\partial u^{\alpha}}\xi^{\alpha}dA+m\int_{V^{m}}(H_{1}H_{\nu}-H_{\nu+1})p\,dA+\nu\int_{V^{m}}(H_{\nu+1}p+H_{\nu}\Phi)dA=0. \tag{2.13}$$

Eliminating the term $\int_{V^{m}}\frac{\partial H_{\nu}}{\partial u^{\alpha}}\xi^{\alpha}dA$ from (2.1) and (2.13), we obtain

$$\int_{V^{m}}(H_{\nu+1}p+H_{\nu}\Phi)dA=0. \tag{2.14}$$

(2.14) is the integral formulas (0.5) obtained by Y. Katsurada [3]. In consequence of (2.13) and (2.14), we obtain (2.5).

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