On some commutor theorems of rings

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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Introduction

In their papers [7] and [4], Hattori and Endo-Watanabe proved that in a central separable C-algebra, there exists a one to one correspondence of the class of semisimple C-subalgebras to itself (See Theorem 3.5 [7] and Theorem 4.2 [4]). The author tried to extend this theorem to the case of separable extension, and obtained a partial extension of their theorem (See §2). Let $\Lambda|\Gamma$ be a ring extension with ${}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda<\oplus_\Lambda(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$ (we call this extension H-separable extension), and denote $\mathfrak{B}_l=\{B|B\supset\Gamma, {}_BB_\Gamma<\oplus {}_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits$\}$, $\mathfrak{D}_l=\{D|C\subset D, {}_DD<\oplus_D\Delta$ and $D\otimes_C\Delta\to\Delta$ splits$\}$, $\mathfrak{B}=\{B|$separable extension of Γ, ${}_BB_B<\oplus_B\Lambda_B\}$ and $\mathfrak{D}=\{D|$separable C-subalgebras of $\Delta\}$. In §0 we state some important properties of H-separable extension which have been obtained already for convenience to readers. In §1 we shall prove that there exist one to one correspondences between $\mathfrak{B}_l$ and $\mathfrak{D}_l$ and between $\mathfrak{B}$ and $\mathfrak{D}$. The latter correspondence has been proved by the same author under the additional condition that Λ is left or right Γ-*f.g.* projective. In §2 we shall prove that B in $\mathfrak{B}_l$ is left (resp. right) semisimple over Γ, $D=V_\Lambda(B)$ is right (resp. left) semisimple over C under the condition that Λ is left Γ-*f.g.* projective and a C-generator. In §3 we shall give an example of separable extension which is not a Frobenius extension.

0. Preliminaries

All rings in this paper shall be assumed to have unities and all subrings have the same identities as the over rings. First, we shall recall the definitions. Let Λ be a ring and Γ a subring of Λ, C the center of Λ, $\Delta=V_\Lambda(\Gamma)=\{x\in\Lambda|xr=rx$ for every $r\in\Gamma\}$.

DEFINITION. Λ is a separable extension of Γ if the map π: $\Lambda\otimes_\Gamma\Lambda\to\Lambda$ defined by $\pi(x\otimes y)=xy$ splits as $\Lambda-\Lambda$-map.

DEFINITION. Λ is an H-separable extension of Γ if $\Lambda\otimes_\Gamma\Lambda$ is $\Lambda-\Lambda$-isomorphic to a $\Lambda-\Lambda$-direct summand of a finite direct sum of copies of Λ.

DEFINITION. Λ is a left semisimple extension of Γ if every left Λ-module is (Λ, Γ)-projective, or equivalently, if every left Λ-module is (Λ, Γ)-

injective. Λ is a semisimple extension of Γ if Λ is both left and right semisimple extension of Γ.

DEFINITION. Let R be a commutative ring and Λ an R-algebra. Λ is a left semisimple R-algebra if every finitely generated left Λ-module is (Λ, Γ)-projective.

It has been proved that in the case where R is Noetherian and Λ is finitely generated over R, Λ is a left semisimple R-algebra if and only if Λ is a right semisimple R-algebra.

Now we shall pick up some main properties of H-separable extension which have been obtained in [9], [10], [14], [15], and [18].

(0.1) *If Λ is an H-separable extension of Γ, Λ is a separable extension of Γ* (Theorem 2.2 [9]).

(0.2) *The following three conditions are equivalent;*

(1) *Λ is an H-separable extension of Γ*

(2) *Δ is C-f.g. projective and the map η of $\Lambda\otimes_{\Gamma}\Lambda$ to Hom $({}_C\Delta, {}_C\Lambda)$ defined by $\eta(x\otimes y)(d)=xdy$ for $x, y\in\Lambda$ and $d\in\Delta$ is a Λ-Λ-isomorphism*

(3) *For every Λ-Λ-module M, the map g: of $\Delta\otimes_C M^{\Lambda}$ to M^{Γ} such that $g(d\otimes m)=dm$ for $d\in\Delta$ and $m\in M$ is an isomorphism. Here $M^{\Omega}=\{m\in M|$ $mx=xm$ for every $x\in\Omega\}$ for a subring Ω of Λ.* (Theorem 1.1 and Theorem 1.3 [14]).

(0.3) *If Λ is H-separable over Γ, then the maps η_r of $\Lambda\otimes_C\Delta^0$ to End (Λ_{Γ}) defined by $\eta_r(x\otimes d^0)(y)=xdy$, η_l of $\Delta\otimes_C\Lambda^0$ to End $({}_{\Gamma}\Lambda)$ defined by $\eta_l(d\otimes x^0)(y)=dyx$ and η_t of $\Delta\otimes_C\Delta^0$ to End $({}_{\Gamma}\Lambda_{\Gamma})$ defined by $\eta_t(d\otimes e^0)(y)=dye$ for $x, y\in\Lambda$ and $d, e\in\Delta$ are ring isomorphisms* (Prop. 3.1 and 4.7 [10]).

(0.4) *If Λ is H-separable over Γ, and if ${}_{\Gamma}\Gamma<\oplus_{\Gamma}\Lambda$ or $\Gamma_{\Gamma}<\oplus\Lambda_{\Gamma}$, then $V_{\Lambda}(V_{\Lambda}(\Gamma))=\Gamma$* (Prop. 1.2 [14]).

(0.5) *If Λ is H-separable over Γ, and if B is a subring of Λ such that $B\supset\Gamma$ and ${}_BB_{\Gamma}<\oplus_B\Lambda_{\Gamma}$, then $V_{\Lambda}(V_{\Lambda}(B))=B$ and the map η_B of $B\otimes_{\Gamma}\Lambda$ to Hom $({}_D\Delta, {}_D\Lambda)$ defined by $\eta_B(b\otimes x)(d)=bdx$ for $b\in B$, $x\in\Lambda$ and $d\in D$, where $D=V_{\Lambda}(B)$, is a $B-\Lambda$-isomorphism* (Prop. 1.3 [14]).

(0.6) *If Λ is H-separable over Γ, and if ${}_{\Gamma}\Gamma_{\Gamma}<\oplus{}_{\Gamma}\Lambda_{\Gamma}$ (resp. ${}_{\Gamma}\Lambda_{\Gamma}<\oplus{}_{\Gamma}(\Gamma\oplus\cdots\oplus\Gamma)_{\Gamma}$), Δ is a separable (resp. a central separable) C-algebra* (Prop. 4.7 [10]).

(0.7) *Let Λ, B and Γ be rings such that $\Gamma\subset B\subset\Lambda$. Then if B is a separable (resp. an H-separable) extension of Γ, we have ${}_DD_D<\oplus{}_D\Delta_D$ (resp. ${}_D\Delta_D<\oplus_D(D\oplus\cdots\oplus D)_D$)* (Prop. 1.1 [15]).

Let B be a subring of Λ with $\Gamma\subset B$. Then we shall simply say that $B\otimes_{\Gamma}\Lambda\to\Lambda$ splits if the map π_B: $B\otimes_{\Gamma}\Lambda\to\Lambda$ such that $\pi_B(b\otimes x)=bx$ $(b\in B, x\in\Lambda)$ splits as $B-\Lambda$-map.

(0.8) *If Λ is H-separable over Γ, and if B is a subring of Λ such that $\Gamma\subset B$ and $B\otimes_\Gamma\Lambda\to\Lambda$ or $\Lambda\otimes_\Gamma B\to\Lambda$ splits, then Λ is an H-separable extension of B* (Prop. 2.2 [15]).

1. Commutor theory on separable subextensions

Let $\mathfrak{B}_l$ (resp. $\mathfrak{B}_r$) be the set of subrings B of Λ such that $\Gamma\subset B$, ${}_BB_\Gamma<\oplus {}_B\Lambda_\Gamma$ (resp. ${}_\Gamma B_B<\oplus{}_\Gamma\Lambda_B$) and $B\otimes_\Gamma\Lambda\to\Lambda$ (resp. $\Lambda\otimes_\Gamma B\to\Lambda$) splits, and $\mathfrak{D}_l$ (resp. $\mathfrak{D}_r$) the set of C-subalgebras D of Δ such that ${}_DD<\oplus{}_D\Delta$ (resp. $D_D<\oplus\Delta_D$) and $D\otimes_C\Delta\to\Delta$ (resp. $\Delta\otimes_C D\to\Delta$) splits. Furtheremore, let $\mathfrak{B}$ be the set of subrings B of Λ such that B is a separable extension of Γ and ${}_BB_B<\oplus{}_B\Lambda_B$, and $\mathfrak{D}$ the set of separable C-subalgebras of Δ.

(1.1) *Let R be a commutative ring and Ω an R-algebra which is an R-f.g. projective module. Then any separable R-subalgebra Λ of Ω is a $\Lambda-\Lambda$-direct summand of Ω.*

Proof. Let C be the center of Λ, $V_\Omega(C)=B$ and $V_\Omega(\Lambda)=\Gamma$, Then by (0.7) and Theorem 2.3 [1], ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma B_\Gamma$ and ${}_BB_B<\oplus{}_B\Omega_B$. Hence, B and Γ are R-*f.g.* projective, consequently, C-*f.g.* projective. Then ${}_CC<\oplus{}_C\Gamma$. But $B\cong\Lambda\otimes_C\Gamma\oplus>\Lambda$ as $\Lambda-\Lambda$-module by Theorem 3.1 [1]. Therefore, ${}_\Lambda\Lambda_\Lambda<\oplus{}_\Lambda\Omega_\Lambda$.

(1.2) *Let Λ be an H-separable extension of Γ. Then, for any $B\in\mathfrak{B}_l$, $V_\Lambda(B)\in\mathfrak{D}_l$, and for any $D\in\mathfrak{D}_l$, $V_\Lambda(D)\in\mathfrak{B}_l$* (Prop. 2 [16]).

Now we can obtain more general results than Theorem 2.1 [15] and Theorem 1 [18].

(1.3) Theorem. *Let Λ be an H-separable extension of Γ, and consider the correspondence V: $A\rightsquigarrow V_\Lambda(A)$ for subring A of Λ. Then we have;*

(1) *V yields a one to one correspondence between $\mathfrak{B}_l$ and $\mathfrak{D}_l$ (resp. $\mathfrak{B}_r$ and $\mathfrak{D}_r$) such that $V^2=$identity.*

(2) *V yields a one to one correspondence between $\mathfrak{B}$ and $\mathfrak{D}$ such that $V^2=$identity.*

Proof. (1). By (0.5) we have $V_\Lambda(V_\Lambda(B))=B$ for $B\in\mathfrak{B}_l$. Thus by (1.2) we need only to prove $V_\Lambda(V_\Lambda(D))=D$ for every $D\in\mathfrak{D}_l$. Let $B=V_\Lambda(D)$ and $D'=V_\Lambda(B)$. Then we have

$$D'\otimes_D\Delta\cong\mathrm{Hom}({}_B\Lambda_\Lambda,{}_B\Lambda_\Lambda)\otimes_D\Delta\cong\mathrm{Hom}\left({}_B\mathrm{Hom}({}_D\Delta,{}_D\Lambda)_\Lambda,{}_B\Lambda_\Lambda\right)$$

$$\cong\mathrm{Hom}({}_BB\otimes_\Gamma\Lambda_\Lambda,{}_B\Lambda_\Lambda)\quad\left(\text{By }(0.5)\right)$$

$$\cong\mathrm{Hom}\left({}_\Gamma\Lambda_\Lambda,{}_\Gamma\mathrm{Hom}({}_BB,{}_B\Lambda)_\Lambda\right)\cong\mathrm{Hom}({}_\Gamma\Lambda_\Lambda,{}_\Gamma\Lambda_\Lambda)\cong\Delta$$

The composition of the above map is given by $d'\otimes d\to d'd$ for $d'\in D'$ and

$d\in\Delta$. Since $B\otimes_\Gamma\Lambda\to\Lambda$ splits by (1.2), ${}_{D'}D'<\oplus_{D'}\Delta$ by (1.2). Hence $D'\otimes {}_DD'\subset D'\otimes_D\Delta\cong\Delta$, and we have $D'\otimes_DD'\cong D'$. Then since ${}_DD<\oplus_DD'$, we have $D=V_{D'}(V_{D'}(D))=V_{D'}$ (the center of D')$=D'$ by (0.4)

(2). For any $B\in\mathfrak{B}$ ($\subset\mathfrak{B}_l$), Λ is an H-separable extension of B with ${}_BB_B<\oplus_B\Lambda_B$ by (0.8), hence $V_\Lambda(B)\in\mathfrak{D}$ by (0.6) and $V_\Lambda(V_\Lambda(B))=B$ by (0.4). Since $\mathfrak{D}\subset\mathfrak{D}_l$ by (1.1), for any $D\in\mathfrak{D}$, $V_\Lambda(V_\Lambda(D))=D$ by (1), and $V_\Lambda(D)=B\in\mathfrak{B}_l$ by (1.2). Since ${}_BB_B<\oplus_B\Lambda_B$ by (0.7), we need only to prove the next lemma.

(1.4) *Let Λ, B and Γ be rings such that $\Gamma\subset B\subset\Lambda$. Then if $B\otimes_\Gamma\Lambda\to\Lambda$ splits and ${}_BB_B<\oplus_B\Lambda_B$, B is a separable extension of Γ.*

PROOF. Let $\Lambda=B\oplus M$ as $B-B$-module. Since $B\otimes_\Gamma\Lambda\to\Lambda$ splits, there exists $\sum b_i\otimes x_i\in(B\otimes_\Gamma\Lambda)^B$ such that $\sum b_ix_i=1$. Let $x_i=a_i+m_i$ with $a_i\in B$, $m_i\in M$. Then $M\ni\sum b_im_i=1-\sum b_ia_i\in B$, and $\sum b_ia_i=1$. On the other hand, since $\sum b_i\otimes x_i\in(B\otimes_\Gamma\Lambda)^B$, $\sum b_i\otimes a_i\in(B\otimes_\Gamma B)^B$. Thus B is separable over Γ.

We end this section by giving a remark on a separable extension of simple ring. Here, simple ring means simple artinean ring. The next proposition almost depends on the classical fundamental theorem on simple rings.

(1.5) THEOREM. *Let Γ be a simple ring and Λ an H-separable extension of Γ. Then, we have;*

(1) *Λ is also a simple ring, and $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$.*

(2) *Δ is a simple C-algebra.*

(3) $[\Lambda:\Gamma]_l=[\Lambda:\Gamma]_r=[\Delta:C]$.

Thus the correspondence V provides a one to one correspondence of the class of simple subrings which contain Γ to the class of simple C-subalgebras of Δ.

PROOF. (1). Since Γ is simple, ${}_\Gamma\Gamma<\oplus_\Gamma\Lambda$. Thus $C\subset V_\Lambda(V_\Lambda(\Gamma))=\Gamma$. Hence C contains no idempotent. On the other hand, Λ is a semisimple ring, since Λ is a semisimple extension of a semisimple ring Γ (See Cor. 1.7 and Prop. 2.6 [8]). Hence Λ is a simple ring.

(2). C is a field by (1), and Δ is a C-algera with $[\Delta:C]<\infty$. Hence $\Lambda\otimes_\Gamma\Lambda\cong\mathrm{Hom}({}_C\Delta,{}_C\Lambda)\cong(\Lambda\oplus\cdots\oplus\Lambda)$ as $\Lambda-\Lambda$-module. Let $\Lambda=\sum_{\alpha\in A}\oplus\mathfrak{l}_\alpha$ with each $\mathfrak{l}_\alpha$ a simple Γ-submodule of Λ. Then $\mathfrak{l}_\alpha\cong\mathfrak{l}_\beta$, since Γ is a simple ring, and we see $\Lambda\otimes_\Gamma\mathfrak{l}_\alpha\cong\Lambda\otimes_\Gamma\mathfrak{l}_\beta\neq0$. Thus $\sum\oplus\Lambda\otimes_\Gamma\mathfrak{l}_\alpha\cong\Lambda\oplus\cdots\oplus\Lambda$ as left Λ-module. This implies that $|A|<\infty$. Hence Λ is left Γ-finitely generated and similarly Λ is right Γ-finitely generated. Since $\Delta\otimes_C\Lambda^0\cong\mathrm{Hom}({}_\Gamma\Lambda,{}_\Gamma\Lambda)$ and Λ is a generator of ${}_\Gamma\mathfrak{M}$, $\Delta\otimes_C\Lambda_0$ is a simple ring by Morita Theorem. Then, Δ is a simple C-algebra.

(3). Since $[\Delta:C]<\infty$ and $V_\Lambda(\Delta)=\Gamma$, we can apply the 'fundamental theorem on simple rings' (See §16 [2], Theorem 16.1).

2. Semisimple subextensions in separable extension

In this section, we shall study the centralizers of semisimple extensions in H-separable extension. Let Λ be an H-separable extension of Γ with ${}_{\Gamma}\Gamma_{\Gamma}<\oplus_{\Gamma}\Lambda_{\Gamma}$, and D be any right semisimple C-subalgebra of Δ. Then $D\in\mathfrak{D}_{r}$, and $V_{\Lambda}(D)\in\mathfrak{B}_{r}$. Now we shall consider the semisimplicity of $B\in\mathfrak{B}_{r}$ over Γ.

(2.1) PROPOSITION. *Let Λ be an H-separable extension of Γ, and suppose that Λ is right Γ-f.g. projective. Then for any B in $\mathfrak{B}_{r}$, B is a left (resp. right) semisimple extension of Γ if and only if $\Lambda\otimes_{C}D^{0}$ is a left (resp. right) semisimple extension of Λ, where $D=V_{\Lambda}(B)$.*

PROOF. Since $B\in\mathfrak{B}_{r}$, we have $D\in\mathfrak{D}_{r}$, $V_{\Lambda}(D)=B$ and $\mathrm{End}(\Lambda_{B})=\Lambda\otimes_{C}D^{0}$. Denote that $P=\Lambda\otimes_{C}D^{0}$ and $Q=\Lambda\otimes_{C}\Delta^{0}$. First suppose that P is left semisimple over Λ, and lef M be any left P-module. Consider the next left P-map

$$\pi_{(\Lambda,M)}:\ \Lambda\otimes_{\Gamma}\mathrm{Hom}\,({}_{P}\Lambda,\ {}_{P}M)\to M$$

defined by $\pi_{(\Lambda,M)}(x\otimes f)=xf$ for $x\in\Lambda$, $f\in\mathrm{Hom}\,({}_{P}\Lambda,\ {}_{P}M)$. Since Λ is right Γ-f.g. projective and $Q\cong\mathrm{End}\,(\Lambda_{\Gamma})$ by (0.3), we have

$$\Lambda\otimes_{\Gamma}\mathrm{Hom}\,({}_{P}\Lambda,\ {}_{P}M)\cong\mathrm{Hom}\big({}_{P}\mathrm{Hom}\,(\Lambda_{\Gamma},\ \Lambda_{\Gamma}),\ {}_{P}M\big)\cong\mathrm{Hom}\,({}_{P}Q,\ {}_{P}M)$$

Hence $\pi_{(\Lambda,M)}$ is factored through the map ϕ of $\mathrm{Hom}\,({}_{P}Q,\ {}_{P}M)$ to M such that $\phi(\alpha)=\alpha(1)$ for $\alpha\in\mathrm{Hom}\,({}_{P}Q,\ {}_{P}M)$. Since P is a $P-\Lambda$-direct summand of Q, ϕ splits as left Λ-map. Since P is a left semisimple extension of Λ, ϕ splits as P-map. Thus for any left P-module M, $\pi_{(\Lambda,M)}$ splits as left P-map and Λ is left P-*f.g.* projective. Therefore, $B=\mathrm{End}\,({}_{P}\Lambda)$ is a left semisimple extension of Γ by Theorem 2 [17]. Next, suppose that P is right semisimple extension of Λ, and let N be any right P-module, and consider the next right P-map

$$\iota_{(\Lambda,N)}:\ N\longrightarrow\mathrm{Hom}\,(\Lambda_{\Gamma},\ N\otimes_{P}\Lambda_{\Gamma})$$

defined by $\iota_{(\Lambda,N)}(n)(x)=n\otimes x$ for $n\in N$, $x\in\Lambda$. Since Λ is right Γ-*f.g.* projective, we have a right P-isomorphism σ: $N\otimes_{P}\mathrm{Hom}\,(\Lambda_{\Gamma},\Lambda_{\Gamma})\to\mathrm{Hom}\,(\Lambda_{\Gamma},N\otimes_{P}\Lambda_{\Gamma})$ such that $\sigma(n\otimes g)(x)=n\otimes g(x)$ for $x\in\Lambda$, $n\in N$ and $g\in\mathrm{End}\,(\Lambda_{\Gamma})$. (Let $\{f_{i},x_{i}\}$ be a dual basis of Λ_{Γ}. Then $h\to\sum h(x_{i})\circ f_{i}$ for $h\in\mathrm{Hom}(\Lambda_{\Gamma},\ N\otimes_{P}\Lambda_{\Gamma})$ is the inverse map of σ). Thus $\iota_{(\Lambda,N)}$ is equivalent to ι_{N}: $N\to N\otimes_{P}\mathrm{End}(\Lambda_{\Gamma})\cong N\otimes_{P}Q$ such that $\iota_{N}(n)=n\otimes 1$ for $n\in N$. Since ${}_{P}P_{\Lambda}<\oplus_{P}Q_{\Lambda}$, ι_{N} (hence $\iota_{(\Lambda,N)}$) splits as right Λ-map. Then ι_{N} (hence $\iota_{(\Lambda,N)}$) splits as P-map, since P is a right semisimple extension of Λ. Hence by Theorem 2 [17], B is a right semisimple extension of Γ.

Thus we have proved the 'if' parts. Now we shall prove the 'only if' parts. Since $B \in \mathfrak{B}_r$ and Λ is right Γ-*f.g.* projective by assumption, Λ is right B-*f.g.* projective. Suppose that B is a right semisimple extension of Γ, and let M be any right B-module. Consider the next map

$$\pi'_{(\Lambda,M)}: \operatorname{Hom}(\Lambda_B, M_B) \otimes_\Gamma \Lambda_B \longrightarrow M_B$$

defined by $\pi'_{(\Lambda,M)}(f \otimes x) = f(x)$ for $x \in \Lambda$, $f \in \operatorname{Hom}(\Lambda_B, M_B)$. This map is equivalent to the map ψ of $\operatorname{Hom}(\Lambda_B, M_B)$ to M such that $\psi(f) = f(1)$. Since ${}_\Gamma B_B < \oplus {}_\Gamma \Lambda_B$, ψ and $\pi'_{(\Lambda,M)}$ splits as Γ-maps. Then since B is right semisimple over Γ by assumption, $\pi'_{(\Lambda,M)}$ splits as B-map for any right B-module M. Hence $P = \operatorname{End}(\Lambda_B)$ is a right semisimple extension of Γ by Theorem 2 [17]. Similarly we can prove that if B is a left semisimple extension of Γ, P is a left semisimple extension of Λ, since the map

$$\iota'_{(\Lambda,N)}: N \longrightarrow \operatorname{Hom}({}_\Lambda\Lambda, {}_\Lambda\Lambda \otimes_B N)$$

defined by $\iota'_{(\Lambda,N)}(n)(x) = x \otimes n$ for $x \in \Lambda$, $n \in N$, splits as B-map under the given conditions.

By this proposition and Proposition 2 [12], we obtain a partial extension of Hattori's theorem.

(2. 2) THEOREM. *Let Λ be an H-separable extension of Γ such that Λ is right Γ-f.g. projective and also a C-generator. Then if B in $\mathfrak{B}_r$ is a left (resp. right) semisimple extension of Γ, D is a right (resp. left) semisimple extension of C.*

PROOF. Since $\Lambda \otimes_C D^0$ is left (resp. right) semisimple over Λ, and ${}_C C < \oplus {}_C \Lambda$, D^0 is left (resp. right) semisimple over C by Prop. 2 [12]. Thus D is right (resp. left) semi-simple over C.

3. A separable extension which is not a quasi-Frobenius extension

In [6], Endo-Watanabe proved that every separable R-algebra which is a finitely generated projective R-module is a symmetric, hence a Frobenius R-algebra. But in the case of ring extension of non commutative ring, we can show this is not always true. More generally, we can give an example of a separable extension which is not a left quasi-Frobenius extension. A ring Λ is a left quasi-Frobenius extension of Γ if $\Lambda \supset \Gamma$, Λ is left Γ-*f.g.* projective and ${}_\Lambda(\sum \oplus \operatorname{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma))_\Gamma \oplus > {}_\Lambda\Lambda_\Gamma$. Right quasi-Frobenius extension is similarly defined, and both left and right quasi-Frobenius extension is called quasi-Frobenius extension (See [11]). The next lemma has been proved in §3 [15].

(3. 1) *Let Λ be a ring and Γ a subring of Λ. Then the following*

conditions are equivalent;

(1) *Λ is a left quasi-Frobenius extension of Γ.*

(2) *Λ is a left Γ-f.g. projective, and there exist $\sum x_i^{(k)}\otimes y_i^{(k)}\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ and $\alpha_k\in Hom(_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$, $k=1,\cdots,n$, such that $\sum_{k,i} x_i^{(k)}\alpha_k(y_i^{(k)})=1$.* (*See Lemma* 3.1 [15]).

(3.2) THEOREM. *Let Λ be a ring and Γ a subring of Λ such that the map π: $\Lambda\otimes_\Gamma\Lambda\to\Lambda$ defined by $\pi(x\otimes y)=xy$ (x, $y\in\Lambda$) is an isomorphism. Then if Λ is a left* (*or right*) *quasi-Frobenius extension of Γ, $\Lambda=\Gamma$.*

PROOF. By (3.1), there exist $\sum x_i^{(k)}\otimes y_i^{(k)}\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ and $\alpha_k\in\mathrm{Hom}(_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ with $\sum_{i,k} x_i^{(k)}\alpha_k(y_i^{(k)})=1$. Since the map π is isomorphic, we have

$$V_\Lambda(\Gamma)\cong\mathrm{Hom}\,(_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda, {}_\Lambda\Lambda_\Lambda)\cong\mathrm{Hom}\,(_\Lambda\Lambda_\Lambda, {}_\Lambda\Lambda_\Lambda)\cong C\,.$$

This implies $V_\Lambda(\Gamma)=C$. Then π induces also

$$C=V_\Lambda(\Gamma)\cong\mathrm{Hom}\,(_\Gamma\Lambda_\Lambda, {}_\Gamma\Lambda_\Lambda)\cong\mathrm{Hom}\,(_\Gamma\Lambda\otimes_\Gamma\Lambda_\Lambda, {}_\Gamma\Lambda_\Lambda)$$
$$\cong\mathrm{Hom}\,(_\Gamma\Lambda_\Gamma, {}_\Gamma\mathrm{Hom}\,(\Lambda_\Lambda, \Lambda_\Lambda)_\Gamma)=\mathrm{Hom}\,(_\Gamma\Lambda_\Gamma, {}_\Gamma\Lambda_\Gamma)\supset\mathrm{Hom}\,(_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)\,.$$

Hence each α_k is given by the multiplication of some $d_k\in C$ with $d_k(\Lambda)\subset\Gamma$. Let $c_k=\sum x_i^{(k)}y_i^{(k)}\in C$. Then $1=\sum_{i,k} x_i^{(k)}d_i(y_i^{(k)})=\sum d_ic_i$. Therefore, $\Lambda=\sum d_k(c_k\Lambda)\subset\sum d_k(\Lambda)\subset\Gamma$. Thus $\Lambda=\Gamma$.

Example. A ring extension Λ/Γ which satisfies the condition that Λ is left Γ-*f.g.* projective and $\Lambda\otimes_\Gamma\Lambda\cong\Lambda$ really exists. This condition is equivalent to the condition that Λ is a finite left localization of Γ in the sence of L. Silver [13]. The existance of such a ring extension is shown in §2 and §3 (See e.g. Prop. 3.10) [13]. Clearly Λ is a separable extension of Γ but not a left (nor right) quasi-Frobenius extension of Γ in this case.

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