On a K-space with certain conditions

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 0. Introduction.

Recently, K. Takamatsu and Y. Watanabe [2][1] proved that a conformally flat K-space is locally symmetric.

The purpose of the present paper is to investigate the analogous problems in a K-space with $C^h{}_{ijk;h}=0$. In §1, we shall give some relations in a K-space to use latter. §2 is devoted to give some results in a K-space with $C^h{}_{ijk;h}=0$.

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§ 1. Preliminaries.

Let M^n be an n-dimensional $(n=2m>2)$ almost Hermitian manifold with Hermitian structure $(F_j{}^i, g_{ij})$, i.e. with an almost complex structure tensor $F_j{}^i$ and a positive definite Riemannian metric g_{ij} satisfying

$$(1.1) \qquad F_j{}^i F_i{}^k = -\delta_j^k$$

$$(1.2) \qquad g_{ab} F_i{}^a F_j{}^b = g_{ij},$$

where δ_j^k is the Kronecker's delta.

If an almost Hermitian structure satisfies

$$(1.3) \qquad F_{ij;k} + F_{ik;j} = 0 \qquad (F_{ij} = g_{hj} F_i{}^h) \ [3],$$

where the symbol " ; " denotes the operator differentiation with respect to the Riemann connection determined by g_{ij}, then the manifold is called a K-space.

From (1.1), (1.2) and (1.3), it follows that

$$(1.4) \qquad F_{ij} = -F_{ji}, \quad F_i{}^j{}_{;j} = 0.$$

Let $R^i{}_{jkl}$, $R_{jk}=R^i{}_{jki}$ and $R=g^{ij}R_{ij}$ be the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. Applying the Ricci's

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1) Numbers in brackets refer to the references at the end of the paper.

identity to F_{hi}, we get

$$F_{hi;j;k}-F_{hi;k;j}=-F_{li}R^{l}{}_{hjk}-F_{hl}R^{l}{}_{ijk}.$$

Multiplying this equation by g^{hk} and summing for h and k, by virtue of (1. 3) and the Bianchi's identity, we have

$$F_{ji;k}{}^{;k}=-\frac{1}{2}F^{hl}R_{hlji}-R_{j}{}^{l}F_{li}\qquad (F^{hl}=g^{hk}F_{k}{}^{l}).$$

If we notice that some tensors in the above equation are anti-symmetric with respect to i and j, we find that

(1. 5) $$R_{i}{}^{l}F_{lj}+R_{j}{}^{l}F_{li}=0$$

(1. 6) $$R_{ab}F_{i}{}^{a}F_{j}{}^{b}=R_{ij}\quad [3].$$

It is well known that in a Riemannian manifold, we have

(1. 7) $$R^{i}{}_{jkl;i}=R_{jk;l}-R_{jl;k}$$

and on multiplying (1. 7) by g^{jk} and summing for j and k, we get

(1. 8) $$R^{i}{}_{j;i}=\frac{1}{2}R_{;j}.$$

Differentiating (1. 5) covariantly, by virtue of (1. 4) and (1. 8), we find that

(1. 9) $$R_{ih;j}F^{ij}=-\frac{1}{2}R_{;i}F_{h}{}^{i}.$$

§ 2. A K-space with $C^{h}{}_{ijk;h}=0$.

Let $C^{h}{}_{ijk}$ be the Weyl's conformal curvature tensor:

(2. 1) $$C^{h}{}_{ijk}=R^{h}{}_{ijk}-\frac{1}{n-2}(\delta_{k}^{h}R_{ij}-\delta_{j}^{h}R_{ik}+g_{ij}R^{h}{}_{k}-g_{ik}R^{h}{}_{j})+\frac{R}{(n-1)(n-2)}(\delta_{k}^{h}g_{ij}-\delta_{j}^{h}g_{ik}).$$

If we assume that M^{n} be a K-space with $C^{h}{}_{ijk;h}=0$, then we have the following theorem:

THEOREM 2. 1. *Let M^{n} be a K-space with $C^{h}{}_{ijk;h}=0$. Then the scalar curvature R of M^{n} is constant.*

PROOF. Multiplying (2. 1) by $F_{l}{}^{j}$ and summing for j, and differentiating covariantly, we have

$$C^h{}_{ijk;m}F_l{}^j + C^h{}_{ijk}F_l{}^j{}_{;m} = R^h{}_{ijk;m}F_l{}^j + R^h{}_{ijk}F_l{}^j{}_{;m}$$
$$-\frac{1}{n-2}\Big[\delta_k^h R_{ij;m}F_l{}^j + \delta_k^h R_{ij}F_l{}^j{}_{;m} - R_{ik;m}F_l{}^h - R_{ik}F_l{}^h{}_{;m} + R^h{}_{k;m}F_{li}$$
$$+ R^h{}_k F_{li;m} - g_{ik}R^h{}_{j;m}F_l{}^j - g_{ik}R^h{}_j F_l{}^j{}_{;m}\Big]$$
$$+\frac{1}{(n-1)(n-2)}\Big[R_{;m}(\delta_k^h F_{li} - g_{ik}F_l{}^h) + R(\delta_k^h F_{li;m} - g_{ik}F_l{}^h{}_{;m})\Big].$$

Interchanging indices m and l in the above equation, by virtue of (1.3) we get

$$C^h{}_{ijk;m}F_l{}^j + C^h{}_{ijk;l}F_m{}^j = R^h{}_{ijk;m}F_l{}^j + R^h{}_{ijk;l}F_m{}^j$$
$$-\frac{1}{n-2}\Big[\delta_k^h R_{ij;m}F_l{}^j + \delta_k^h R_{ij;l}F_m{}^j - R_{ik;m}F_l{}^h - R_{ik;l}F_m{}^h + R^h{}_{k;m}F_{li}$$
$$+ R^h{}_{k;l}F_{mi} - g_{ik}R^h{}_{j;m}F_l{}^j - g_{ik}R^h{}_{j;l}F_m{}^j\Big]$$
$$+\frac{1}{(n-1)(n-2)}\Big[R_{;m}(\delta_k^h F_{li} - g_{ik}F_l{}^h) + R_{;l}(\delta_k^h F_{mi} - g_{ik}F_m{}^h)\Big].$$

Multiplying this equation by $g^{kl}g^{mi}$ and summing for all indices, by making use of $C^h{}_{ijk;h}=0$ and (1.4), we obtain

$$(2.2)\qquad R_h{}^i{}_{jk;i}F^{kj} + R_{hij}{}^k{}_{;k}F^{ij} - \frac{1}{n-2}\Big[3R^i{}_{j;i}F_h{}^j + R_{ij;h}F^{ij} + 3R_{hk;i}F^{ki}\Big]$$
$$+\frac{3}{(n-1)(n-2)}R_{;i}F_h{}^i = 0\,.$$

Since $R_{ij;h}F^{ij}=0$, taking account of (1.7), (1.8) and (1.9), (2.2) can be written as

$$\frac{3n(n-3)}{2(n-1)(n-2)}R_{;i}F_h{}^i = 0\,,$$

from which, it follows that $R = constant$.

THEOREM 2.2. *Let M^n be a K-space with $C^h{}_{ijk;h}=0$. Then $R_{ij;k}$ is symmetric in all indices.*

PROOF. From our assumption and Theorem 2.1, it follows that

$$R_{ij;k} - R_{jk;i} = 0\,.$$

By virtue of Theorem 2.1, 2.2 and (1.7), we have the following

THEOREM 2.3. *Let M^n be a K-space with $C^h{}_{ijk;h}=0$. Then we have*

$$R^i{}_{jkl;i} = 0\,.$$

THEOREM 2.4. *Let M^n be a K-space with $C^h{}_{ijk;h}=0$. Then we have the following relation*:

$$3R_{ij;k}=R_{ab;k}F_i^aF_j^b+R_{ab;i}F_j^aF_k^b+R_{ab;j}F_k^aF_i^b\,. \tag{2.3}$$

PROOF. Differentiation (1.6) covariantly, by virtue of (1.5) we have

$$R_{ij;k}=R_{ab;k}F_i^aF_j^b-R_{jb}F_{i\ ;k}^{a}F_a^b-R_{ai}F_b^aF_{j\ ;k}^{b}\,.$$

Making the cyclic sum with respect to indices i, j and k of the last equation, we have

$$R_{ij;k}+R_{jk;i}+R_{ki;j}=R_{ab;k}F_i^aF_j^b+R_{ab;i}F_j^aF_k^b+R_{ab;j}F_k^aF_i^b\,,$$

from which, by making use of Theorem 2.2, we get (2.3).

THEOREM 2.5. *A K-space with $C^h{}_{ijk;h}=0$ is a Ricci-symmetric space if*

$$\det.\,|\delta^{(a}_{(i}\delta^b_j\delta^{c)}_{k)}-F^{(a}_{(i}F_j{}^b\delta^{c)}_{k)}|\neq 0\,, \tag{2.4}$$

where the symbol ( ) *denotes the symmetric part with respect to indices i, j and k.*

PROOF. (2.3) can be written as

$$R_{ab;c}(\delta^{(a}_{(i}\delta^b_j\delta^{c)}_{k)}-F^{(a}_{(i}F_j{}^b\delta^{c)}_{k)})=0\,.$$

If det. $|\delta^{(a}_{(i}\delta^b_j\delta^{c)}_{k)}-F^{(a}_{(i}F_j{}^b\delta^{c)}_{k)}|\neq 0$, then we have $R_{ab;c}=0$.

The Riemannian manifold is called conformally symmetric [1] if it satisfies $C^h{}_{ijk;l}=0$. A conformally symmetric K-spaec is the special case of a K-space with $C^h{}_{ijk;h}=0$. Therefore, by virtue of Theorem 2.1 and Theorem 2.5, we have the following

COROLLARY 2.6. *A conformally symmetric K-space is locally symmetric if it satisfies* (2.4).

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References

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