# On coclosed mappings

Dedicated to Professor Y. Katsurada on her sixtieth birthday

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Let $N$ be an $n$-dimensional compact orientable Riemannian manifold and $M$ be an $m$-dimensional compact Riemannian manifold. We use local coordinates $u^\kappa$ and $x^h$ to represent points of $N$ and $M$ respectively where indices such as $\kappa, \lambda, \mu, \cdots$ run the range $\{1, \cdots, n\}$ and the indices such as $h, i, j, \cdots$ run the range $\{1, \cdots, m\}$. The fundamental tensors of $N$ and $M$ are respectively denoted by $a_{\mu\lambda}$ and $g_{ji}$ and the summation convention is used.

Let us assume $n \geqq m$. The purpose of the present paper is to study mappings $\mu$: $N \rightarrow M$ satisfying a certain condition by which we shall call them coclosed mappings.

## §1. Coclosed mappings

If $\mu$: $N \rightarrow M$ is a differentiable mapping, a field of connecting tensor

$$B_\lambda^h = \partial_\lambda x^h, \qquad \partial_\lambda = \partial/\partial u^\lambda \tag{1.1}$$

is defined. From this connecting tensor we get another connecting tensor

$$B_{\lambda_1 \cdots \lambda_m}^{i_1 \cdots i_m} = B_{\lambda_1}^{i_1} \cdots B_{\lambda_m}^{i_m} \tag{1.2}$$

and from this connecting tensor we can define on $N$ an $m$-form

$$\phi[\mu]_{\lambda_1 \cdots \lambda_m} = B_{[\lambda_1 \cdots \lambda_m]}^{1 \cdots m} \sqrt{g} \tag{1.3}$$

where $g = \det(g_{ji})$.[1]

DEFINITION. If the form $\phi[\mu]_{\lambda_1 \cdots \lambda_m}$ is a coclosed form, the mapping is called a coclosed mapping. If $\phi[\mu]_{\lambda_1 \cdots \lambda_m}$ vanishes everywhere, $\mu$ is called a trivial coclosed mapping. If, for a non-trivial coclosed mapping, there exist points of $N$ at which $\phi[\mu]_{\lambda_1 \cdots \lambda_m}$ vanishes, such points are called singular points of the mapping.

Then we have the following proposition.

PROPOSITION 1.1. *Let $\mu$ be a non-trivial coclosed mapping and $\mu_0$ be a trivial coclosed mapping. Then $\mu$ is not homotopic to $\mu_0$.*

PROOF. Let $\varphi_{i_1 \cdots i_m}$ be an $m$-form on $M$ such that $\varphi_{1 \cdots m} = \sqrt{g}$. Let $a^{\mu\lambda}$ and $\phi[\mu]^{\lambda_1 \cdots \lambda_m}$ be defined by

1) [ ] means the bracket symbol of Bach and represents the alternating part.

$$a_{\mu\lambda}a^{\mu\kappa}=\delta_{\lambda}^{\kappa}, \quad \psi[\mu]^{\lambda_1\cdots\lambda_m}=\psi[\mu]^{\mu_1\cdots\mu_m}a^{\mu_1\lambda_1}\cdots a^{\mu_m\lambda_m}.$$

Then

$$(\psi[\mu],\ \tilde{\mu}^*\varphi)=\int_N \psi[\mu]^{\lambda_1\cdots\lambda_m}\tilde{B}_{\lambda_1\cdots\lambda_m}^{i_1\cdots i_m}\varphi_{i_1\cdots i_m}\eta(a),$$

where $\tilde{B}_{\lambda}^{h}$ is the connecting tensor of a mapping $\tilde{\mu}$ and $\eta(a)$ means the volume element of $N$, is a homotopy invariant of the mapping $\tilde{\mu}$ since $\psi[\mu]_{\lambda_1\cdots\lambda_m}$ is a coclosed form on $N$ and $\varphi_{i_1\cdots i_m}$ is a closed form on $M$ [2]. As $\psi[\mu]_{\lambda_1\cdots\lambda_m}$ is defined by (1.3), we see

$$(\psi[\mu],\ \mu^*\varphi)=m\,!\int_N \psi[\mu]^{\lambda_1\cdots\lambda_m}\psi[\mu]_{\lambda_1\cdots\lambda_m}\eta(a)$$

and this integral is non-negative. $(\psi[\mu],\mu^*\varphi)$ vanishes only when $\psi[\mu]_{\lambda_1\cdots\lambda_m}=0$ at every point of $N$, but, if this is true, $\mu$ is not a non-trivial coclosed mapping. Hence $(\psi[\mu],\ \mu^*\varphi)>0$. On the other hand we have $(\psi[\mu],\ \mu_0^*\varphi)=0$ since we have $\tilde{B}_{[\lambda_1\cdots\lambda_m]}^{i_1\cdots i_m}=0$ if $\tilde{\mu}=\mu_0$. Hence $\mu$ is not homotopic to $\mu_0$.

We give here some examples of coclosed mappings. For short we write $\psi_{\lambda_1\cdots\lambda_m}$ for $\psi[\mu]_{\lambda_1\cdots\lambda_m}$ if there is no possibility of confusion.

EXAMPLE 1. If $N=M$ and $\mu$ is an identity mapping, $\mu$ is a coclosed mapping, because $\psi_{\lambda_1\cdots\lambda_m}$ is a coclosed form by virtue of

$$\psi_{1\cdots m}=\frac{1}{m\,!}\sqrt{g}=\frac{1}{m\,!}\sqrt{a}\,.$$

EXAMPLE 2. Let $N=M=S^1$. If $\alpha$ and $\beta$ are arguments respectively on $N$ and $M$, a mapping $\mu$ is represented by a function $\beta=\beta(\alpha)$ such that $\beta(\alpha+2\pi)=\beta(\alpha)+2k\pi$ where $k$ is an integer. Taking the natural metric on $S^1$ we have $\sqrt{g}=\sqrt{a}=1$. $\mu$ is a coclosed mapping if and only if $d^2\beta/d^2\alpha=0$. Hence we have

$$\beta=k\alpha+\theta \qquad (k\neq 0,\ \theta \text{ is a constant})$$

for a coclosed mapping.

EXAMPLE 3. Let $N$ be a compact orientable Riemannian manifold of dimension $n\geqq 2$ and $M$ be $E^1$. If $f$ is a suitable coordinate on $M$, we have $g=1$ and $\psi_\lambda=\partial f/\partial u^\lambda$. If $\mu$ is a coclosed mapping, $f$ must satisfy $\Delta f=0$ where $\Delta$ is the Laplacian in $N$. Hence $f(u)$ is constant and $\mu$ is a trivial coclosed mapping.

EXAMPLE 4. Let $N$ be the same as in Example 3, but $M$ be $S^1$. Let us consider a covering of $M$ by coordinate neighborhoods $U_\alpha$, $\alpha\in\{\alpha\}$, and let $f_\alpha$ be the local coordinate on the coordinate neighborhood $U_\alpha$, for each $\alpha\in\{\alpha\}$, such that $g=1$ and $f_\alpha(u)=f_\beta(u)+\text{const}$ on $U_\alpha\cap U_\beta$ where $\alpha$, $\beta\in\{\alpha\}$.

$\mu$ is a coclosed mapping if $f_\alpha$ satisfies $\Delta f_\alpha=0$ in $U_\alpha$. An example of such coclosed mappings which is not trivial is the projection $\pi$: $S^1\times S^1\to S^1$.

Let us consider covariant differentiation of the connecting tensor $B_\lambda^h$.

Let $\left\{ {h \atop ji} \right\}$ and $'\left\{ {\kappa \atop \mu\lambda} \right\}$ be the Christoffel symbols respectively in $(M, g)$ and in $(N, a)$. The van der Waerden-Bortolotti operator $\nabla_\mu$ is a differential operator such that

$$\nabla_\mu T_i^h = B_\mu^j\left(\partial_j T_i^h + \left\{ {h \atop jk} \right\} T_i^k - \left\{ {k \atop ji} \right\} T_k^h\right),$$

$$\nabla_\mu T_\lambda^h = B_\mu^j\left(\partial_j T_\lambda^h + \left\{ {h \atop jk} \right\} T_\lambda^k\right) - '\left\{ {\omega \atop \mu\lambda} \right\} T_\omega^h$$

or

$$\nabla_\mu T_\lambda^\kappa = \partial_\mu T_\lambda^\kappa + \left\{ {\kappa \atop \mu\omega} \right\} T_\lambda^\omega - \left\{ {\omega \atop \mu\lambda} \right\} T_\omega^\kappa$$

according as $T$ is a tensor in $M$, a connecting tensor or a tensor in $N$. We consider also ordinary covariant differentiation in $M$ and in $N$ and denote it by $\nabla_j$ or $'\nabla_\mu$, so that we have

$$\nabla_\mu T_i^h = B_\mu^j \nabla_j T_i^h, \qquad '\nabla_\mu T_\lambda^\kappa = \nabla_\mu T_\lambda^\kappa.$$

This operator $'\nabla_\mu$ is also used when $T$ is a connecting tensor, for example, in the following way,

$$'\nabla_\mu T_\lambda^h = B_\mu^j \partial_j T_\lambda^h - '\left\{ {\omega \atop \mu\lambda} \right\} T_\omega^h.$$

From the connecting tensor $B_\lambda^h$ we define a connecting tensor $H_{\mu\lambda}{}^h$ as follows:

$$H_{\mu\lambda}{}^h = \nabla_\mu B_\lambda^h = '\nabla_\mu B_\lambda^h + B_{\mu\lambda}^{ji}\left\{ {h \atop ji} \right\}. \tag{1. 4}$$

A mapping $\mu$: $N\to M$ where $H_{\mu\lambda}{}^h$ vanishes identically is called a totally geodesic mapping.

We prove the following proposition.

PROPOSITION 1. 2. *A totally geodesic mapping is a coclosed mapping.*

PROOF. Since $H_{\mu\lambda}{}^h=0$ we have

$$'\nabla^\kappa B_\lambda^h = -a^{\mu\kappa} B_{\mu\lambda}^{ji}\left\{ {h \atop ji} \right\}.$$

We get by straightforward calculation

$$
\begin{aligned}
\nabla^{\lambda_1}\phi_{\lambda_1\lambda_2\cdots\lambda_m}
&= {}'\nabla^{\lambda_1}(B^{1\cdots\cdots m}_{[\lambda_1\cdots\lambda_m]}\sqrt{g}\,) \\
&= {}'\nabla^{\lambda_1}(B^{[1\cdots\cdots m]}_{[\lambda_1\cdots\lambda_m]}\sqrt{g}\,) \\
&= m({}'\nabla^{\lambda_1}B^{[1}_{[\lambda_1})B^{2\cdots\cdots m]}_{\lambda_2\cdots\lambda_m]}\sqrt{g} \\
&\quad +\frac{1}{2}B^{[1\cdots\cdots m]}_{[\lambda_1\cdots\lambda_m]}\sqrt{g}\,a^{\mu\lambda_1}\partial_\mu \log g \\
&= -ma^{\mu\lambda_1}B^{j\,i}_{\mu[\lambda_1}\left\{ {[1 \atop |ji|} \right\}B^{2\cdots\cdots m]}_{\lambda_2\cdots\lambda_m]}\sqrt{g} \\
&\quad +\frac{1}{2}B^{[1\cdots\cdots m]}_{[\lambda_1\cdots\lambda_m]}\sqrt{g}\,a^{\mu\lambda_1}B^j_\mu\partial_j \log g \\
&= -a^{\mu\lambda_1}B^j_\mu\left\{ {k \atop jk} \right\}B^{1\cdots\cdots m}_{[\lambda_1\cdots\lambda_m]}\sqrt{g} \\
&\quad +\frac{1}{2}a^{\mu\lambda_1}B^j_\mu\partial_j \log g\, B^{1\cdots\cdots m}_{[\lambda_1\cdots\lambda_m]}\sqrt{g} \\
&= 0\,.
\end{aligned}
$$

It has been proved by Eells and Sampson [1] that, if the following two conditions are satisfied, namely,

a) the Ricci curvature of $N$ is non-negative

b) $M$ has non-positive sectional curvature,

then any harmonic mapping $N \to M$ is a totally geodesic mapping. Then we have the following corollary.

COROLLARY. *A harmonic mapping $N \to M$ is a coclosed mapping if* a) *and* b) *are satisfied.*

Since we have

$$\partial_{[\mu}(B^{1\cdots\cdots m}_{\lambda_1\cdots\lambda_m]}\sqrt{g}\,) = (\partial_{[\mu}B^{1\cdots\cdots m}_{\lambda_1\cdots\lambda_m]})\sqrt{g} + B^{1\cdots\cdots m}_{[\lambda_1\cdots\lambda_m}B^i_{\mu]}\partial_i\sqrt{g} = 0$$

because of

$$\partial_{[\mu}B^{1\cdots\cdots m}_{\lambda_1\cdots\lambda_m]} = 0\,, \qquad B^{1\cdots\cdots m}_{[\lambda_1\cdots\lambda_m}B^i_{\mu]} = 0\,,$$

the $m$-form $\phi_{\lambda\cdots\lambda_m}$ defined by (1.3) is always a closed form. If $\mu$ is a coclosed mapping, the $m$-form (1.3) is a harmonic form. Since we have

$$\int_N \phi_{\lambda_1\cdots\lambda_m}\phi^{\lambda_1\cdots\lambda_m}\eta(a) > 0$$

if $\mu$ is a non-trivial coclosed mapping, we have the following proposition.

PROPOSITION 1.3. *If there exists a non-trivial coclosed mapping $\mu$: $N \to M$, the Betti numbers of $N$ satisfy $B_m(N) \neq 0$, $B_{n-m}(N) \neq 0$.*

Let $K_{\nu\mu\lambda\kappa}$ denote the curvature tensor of $N$ and $K_{\mu\lambda}$ denote the Ricci tensor of $N$. When $\xi_{\lambda_1\cdots\lambda_p}$ is a skew tensor field, let us define a quadratic form $F(\xi_{\lambda_1\cdots\lambda_p})$ by

$$F(\xi_{\lambda_1\cdots\lambda_p}) = K_\nu{}^\mu \xi^{\nu\lambda_2\cdots\lambda_p}\xi_{\mu\lambda_2\cdots\lambda_p} + \frac{p-1}{2} K_{\omega\nu}{}^{\mu\kappa}\xi^{\omega\nu\lambda_3\cdots\lambda_p}\xi_{\mu\kappa\lambda_3\cdots\lambda_p}.$$

The following theorem has been proved by Yano [3].

*In a compact Riemannian manifold N, there exists on harmonic tensor field of order p which satisfies*

$$F(\xi_{\lambda_1\cdots\lambda_p}) \geqq 0$$

*unless we have*

$$\nabla_\mu \xi_{\lambda_1\cdots\lambda_p} = 0$$

*and then automatically*

$$F(\xi_{\lambda_1\cdots\lambda_p}) = 0\,.$$

Thus we have also the following theorem which has been also proved by Lichnerowicz, Mogi, Tomonaga and Yano [3].

*If the quadratic form $F(\xi_{\lambda_1\cdots\lambda_p})$ is prositive definite and N is a compact Riemannian manifold, there exists no harmonic tensor of order p other than the trivial one.*

From these theorems we obtain the following theorems.

THEOREM 1.4. *If the quadratic form $F(\xi_{\lambda_1\cdots\lambda_m})$ is positive definite, there exists no non-trivial coclosed mapping $\mu$: $N\rightarrow M$ whatever M may be.*

THEOREM 1.5. *If the quadratic form $F(\xi_{\lambda_1\cdots\lambda_m})$ is non-negative, there exists no non-trivial coclosed mapping $\mu$ :$N\rightarrow M$ other than those which satisfy*

$$F(\phi_{\lambda_1\cdots\lambda_m}) = 0$$

*and*

$$\nabla_\mu \phi_{\lambda_1\cdots\lambda_m} = 0$$

*and then automatically*

$$\phi_{\lambda_1\cdots\lambda_m}\phi^{\lambda_1\cdots\lambda_m} = \text{const.}$$

*If such a non-trivial coclosed mapping exists, this mapping has no singular point.*

## § 2. A local property of a coclosed mapping

In §§ 2 and 3 we consider only surjective coclosed mappings.

We assume $m<n$ and consider a mapping $\mu: N \to M$ and a neighborhood in $N$ where the rank of $d\mu$ is $m$. Let the indices such as $\alpha, \beta, \gamma, \cdots$ run the range $\{m+1, \cdots, n\}$ and let $C^\alpha_\lambda$ be such that the matrix of order $n$ where the elements are $B^h_\lambda$ and $C^\alpha_\lambda$ has the inverse matrix $(B^\kappa_i, C^\kappa_\beta)$ so that

$$B^\lambda_i B^h_\lambda = \delta^h_i, \quad C^\lambda_\beta C^\alpha_\lambda = \delta^\alpha_\beta, \quad B^\lambda_i C^\alpha_\lambda = 0, \quad C^\lambda_\beta B^h_\lambda = 0.$$

If $\mu: N \to M$ is a coclosed mapping, we have

(2.1) $$'\nabla^{\lambda_1}(B^{[1\cdots\cdots m]}_{[\lambda_1\cdots\lambda_m]}\sqrt{g}) = 0.$$

Using (1.4) we can write (2.1) in the form

$$m a^{\mu\lambda_1} H_{\mu[\lambda_1}{}^{[1} B^{2\cdots\cdots m]}_{\lambda_2\cdots\lambda_m]}\sqrt{g}$$
$$- m a^{\mu\lambda_1} B^{ji}_{\mu[\lambda_1}\left\{ {[1 \atop |j\,i|} \right\} B^{2\cdots\cdots m]}_{\lambda_2\cdots\lambda_m]}\sqrt{g}$$
$$+\frac{1}{2} a^{\mu\lambda_1} B^j_\mu \partial_j \log g\, B^{[1\cdots\cdots m]}_{[\lambda_1\cdots\lambda_m]}\sqrt{g} = 0.$$

As the second and the third terms in the left-hand member cancel one another, we get

(2.2) $$H^{\lambda_1}{}_{[\lambda_1}{}^{[1} B^{2\cdots\cdots m]}_{\lambda_2\cdots\lambda_m]} = 0$$

where $H^\mu{}_\lambda{}^h = a^{\nu\mu} H_{\nu\lambda}{}^h$.

We get from (2.2)

(2.3) $$H^\omega{}_\omega{}^1 B^{[2\cdots\cdots m]}_{\lambda_2\cdots\lambda_m} - \sum_{r=2}^{m} H^\omega{}_\omega{}^r B^{[2\cdots 1\cdots m]}_{\lambda_2\cdots\cdots\lambda_m} - \sum_{s=2}^{m} H^\omega{}_{\lambda_s}{}^1 B^{[2\cdots\cdots\cdots m]}_{\lambda_2\cdots\omega\cdots\lambda_m}$$
$$+ \sum_{s=2}^{m}\sum_{r=2}^{m} H^\omega{}_{\lambda_s}{}^r B^{[2\cdots 1\cdots m]}_{\lambda_2\cdots\omega\cdots\lambda_m} = 0.$$

As the transvection of the left-hand member of (2.3) and $(m-1)!\, B^{\lambda_2\cdots\lambda_m}_{2\cdots m}$ is

$$H^\omega{}_\omega{}^1 - \sum_{s=2}^{m} H^\omega{}_\lambda{}^1 B^\lambda_s B^s_\omega + \sum_{s=2}^{m} H^\omega{}_\lambda{}^s B^\lambda_s B^1_\omega,$$

we get

$$H^\omega{}_\omega{}^1 + (H^\omega{}_\nu{}^k B^1_\omega - H^\omega{}_\nu{}^1 B^k_\omega) B^\nu_k = 0.$$

As the transvection of the left-hand member of (2.3) and $(m-1)\, B^{\lambda_2\cdots\lambda_m}_{13\cdots m}$ is

$$-H^\omega{}_\omega{}^2 - H^\omega{}_\lambda{}^1 B^\lambda_1 B^2_\omega + H^\omega{}_\lambda{}^2 B^\lambda_1 B^1_\omega$$
$$+ \sum_{s=3}^{m} (H^\omega{}_\lambda{}^2 B^\lambda_s B^s_\omega - H^\omega{}_\lambda{}^s B^\lambda_s B^2_\omega),$$

we get

$$H^\omega{}_\omega{}^2 + (H^\omega{}_\nu{}^k B^2_\omega - H^\omega{}_\nu{}^2 B^k_\omega) B^\nu_k = 0.$$

In such a way we get

$$H^{\omega}{}_{\omega}{}^{h}+(H^{\omega}{}_{\nu}{}^{k}B^{h}_{\omega}-H^{\omega}{}_{\nu}{}^{h}B^{k}_{\omega})B^{\nu}_{k}=0\,. \tag{2.4}$$

As the transvection of the left-hand member of (2.3) and $C^{\lambda_2}_{\alpha}B^{\lambda_3\cdots\lambda_m}_{3\cdots m}$ is

$$-H^{\omega}{}_{\lambda}{}^{1}C^{\lambda}_{\alpha}B^{2}_{\omega}+H^{\omega}{}_{\lambda}{}^{2}C^{\lambda}_{\alpha}B^{1}_{\omega}\,,$$

we get

$$(H^{\omega}{}_{\nu}{}^{2}B^{1}_{\omega}-H^{\omega}{}_{\nu}{}^{1}B^{2}_{\omega})C^{\nu}_{\alpha}=0\,.$$

Similarly we get

$$(H^{\omega}{}_{\nu}{}^{i}B^{h}_{\omega}-H^{\omega}{}_{\nu}{}^{h}B^{i}_{\omega})C^{\nu}_{\alpha}=0\,. \tag{2.5}$$

Since the transvection of the left-hand member of (2.3) and $C^{\lambda_2\lambda_3}_{\beta\ \alpha}$ identically vanishes, we can coclude that (2.4) and (2.5) together are equivalent to (2.2).

If we define $P_{k}{}^{ih}$ by

$$P_{k}{}^{ih}=B^{\nu}_{k}(H^{\omega}{}_{\nu}{}^{i}B^{h}_{\omega}-H^{\omega}{}_{\nu}{}^{h}B^{i}_{\omega})\,, \tag{2.6}$$

we get

$$H^{\omega}{}_{\nu}{}^{i}B^{h}_{\omega}-H^{\omega}{}_{\nu}{}^{h}B^{i}_{\omega}=B^{k}_{\nu}P_{k}{}^{ih} \tag{2.7}$$

from (2.5) and

$$H^{\omega}{}_{\omega}{}^{h}=-P_{k}{}^{kh} \tag{2.8}$$

from (2.4).

Conversely, if there exist $P_{k}{}^{ih}$ which satisfy (2.7) and (2.8), then we get (2.4) and (2.5). Thus we obtain the following theorem.

THEOREM 2.1. *Let $\mu$: $N\to M$ be a surjective mapping where rank $d\mu=m<n$. Then a necessary and sufficient condition in order that $\mu$ be a coclosed mapping is that* (2.7) *and* (2.8) *be compatible.*

## § 3. Geometric interpretation of coclosed mappings

Let us first consider a geometric interpretation of the condition (2.7).

Let $Q$ be a point of $N$ and let $P=\mu(Q)$. Then $\mu^{-1}(P)$ is a submanifold of $N$ which contains $Q$. The subspace of $N_Q$ perpendicular to the tangent space $(\mu^{-1}(P))_Q$ is spanned by the $m$ vectors $B^{1}_{\mu}a^{\mu\kappa},\cdots,B^{m}_{\mu}a^{\mu\kappa}$. Thus we have on $N$ an $m$-dimensional distribution $\mathscr{M}$ which is perpendicular to the congruence of submanifolds $\mu^{-1}(P)$ where $P$ moves freely in $M$.

$\mathscr{M}$ is involutive if and only if there exist $C_{h}{}^{ji}$ such that

$$B^{j}_{\omega}a^{\omega\lambda}\partial_{\lambda}(B^{i}_{\nu}a^{\nu\kappa})-B^{i}_{\omega}a^{\omega\lambda}\partial_{\lambda}(B^{j}_{\nu}a^{\nu\kappa})=C_{h}{}^{ji}B^{h}_{\mu}a^{\mu\kappa}\,. \tag{3.1}$$

(3. 1) is equivalent to

$$B^j_\omega a^{\omega\lambda\prime}\nabla_\lambda(B^i_\nu a^{\nu\kappa})-B^i_\omega a^{\omega\lambda\prime}\nabla_\lambda(B^j_\nu a^{\nu\kappa})=C_h{}^{ji}B^h_\mu a^{\mu\kappa}$$

and, as we have

$$'\nabla_\lambda(B^i_\nu a^{\nu\kappa})=('\nabla_\lambda B^i_\nu)a^{\nu\kappa}=\left(H_{\lambda\nu}{}^i-\left\{ {i \atop lk} \right\}B^{lk}_{\lambda\nu}\right)a^{\nu\kappa},$$

we immediately see that (3. 1) is equivalent to (2. 7).

Thus we have the following lemma.

LEMMA 3. 1. *Let $\mu$: $N\to M$ be a surjective mapping where $n>m$ and let the rank of $d\mu$ be $m$ everywhere. Then the $m$-dimensional distribution $\mathcal{M}$ which is perpendicular to the congruence of submanifolds $\mu^{-1}(P)$, $P\in M$, is involutive if $\mu$ is coclosed.*

Assuming that the conditions stated in the beginning part of Lemma 3. 1 and the condition (2. 7) are satisfied, let us now examine the condition (2. 8).

Let $P\in M$ and $Q\in N$ be such that $P=\mu(Q)$. As $\mathcal{M}$ is involutive we can take suitable neighborhoods $V$ and $U=\mu(V)$ respectively of $Q$ and $P$ and choose local coordinates $(u^1,\cdots,u^m,\ u^{m+1},\cdots,u^n)$ and $(x^1,\cdots,x^m)$ respectively in $V$ and $U$ in such a way that $\mu$ is represented by

$$\mu:\ (x^1,\cdots,x^m,\ y^{m+1},\cdots,y^n)\to(x^1,\cdots,x^m) \tag{3. 2}$$

in $V\times U$ and the submanifolds of $V$ represented by $u^h=$const and the submanifolds of $V$ represented by $u^\alpha=$const intersect perpendicularly.

Using such local coordinates we have

$$(a_{\mu\lambda})=\begin{pmatrix} a_{ji} & 0 \\ 0 & a_{\beta\alpha}\end{pmatrix},\qquad (a^{\mu\lambda})=\begin{pmatrix} a^{ji} & 0 \\ 0 & a^{\beta\alpha}\end{pmatrix} \tag{3. 3}$$

where $a_{ji}a^{jh}=\delta^h_i$, $a_{\nu\beta}a^{\nu\alpha}=\delta^\alpha_\beta$. Since we have $B^h_\lambda=\delta^h_\lambda$ because of (3. 2), we get

$$H_{\mu\lambda}{}^h=\left\{ {h \atop ji} \right\}\delta^j_\mu\delta^i_\lambda-\frac{1}{2}a^{lh}(\partial_\mu a_{\lambda l}+\partial_\lambda a_{\mu l}-\partial_l a_{\mu\lambda}) \tag{3. 4}$$

and

$$H^\omega{}_\nu{}^i B^h_\omega-H^\omega{}_\nu{}^h B^i_\omega$$
$$=\left[a^{kh}\left\{ {i \atop kj} \right\}-a^{ki}\left\{ {h \atop kj} \right\}+a^{li}a^{kh}(\partial_l a_{jk}-\partial_k a_{jl})\right]\delta^j_\nu,$$

hence

$$P_j{}^{ih}=a^{kh}\left\{ {i \atop kj} \right\}-a^{ki}\left\{ {h \atop kj} \right\}+a^{li}a^{kh}(\partial_l a_{kj}-\partial_k a_{lj}). \tag{3. 5}$$

Now we get from (3.4)

$$H^{\omega}{}_{\omega}{}^{h} = a^{ji}\left\{ {h \atop ji} \right\} - \frac{1}{2} a^{lh}(\partial_{\mu} a_{\lambda l} + \partial_{\lambda} a_{\mu l} - \partial_{l} a_{\mu\lambda}) a^{\mu\lambda}$$

$$= a^{ji}\left\{ {h \atop ji} \right\} - \frac{1}{2} a^{lh}(\partial_{j} a_{il} + \partial_{i} a_{jl} - \partial_{l} a_{ji}) a^{ji}$$

$$+ \frac{1}{2} a^{lh} a^{\beta\alpha} \partial_{l} a_{\beta\alpha}$$

and from (3.5)

$$P_{k}{}^{kh} = a^{kh}\left\{ {i \atop ki} \right\} - a^{kj}\left\{ {h \atop kj} \right\} + a^{li} a^{kh}(\partial_{l} a_{ik} - \partial_{k} a_{il}) .$$

Thus (2.8) is equivalent to

$$a^{kh}\left\{ {i \atop ki} \right\} - \frac{1}{2} a^{kh} a^{ji} \partial_{k} a_{ji} + \frac{1}{2} a^{kh} a^{\beta\alpha} \partial_{k} a_{\beta\alpha} = 0 ,$$

that is, to

$$g^{ji} \partial_{k} g_{ji} + a^{\beta\alpha} \partial_{k} a_{\beta\alpha} - a^{ji} \partial_{k} a_{ji} = 0 .$$

This means that

$$\frac{\det(g_{ji}) \det(a_{\beta\alpha})}{\det(a_{ji})} \tag{3.6}$$

depends only upon the variables $y^{m+1}, \cdots, y^{n}$.

This proves the following theorem.

THEOREM 3.2. *Let $\mu$: $N \to M$ be a surjective mapping where $n > m$ and let the rank of $d\mu$ be $m$ everywhere. Then a necessary and sufficient condition in order that $\mu$ be a coclosed mapping is that the following conditions be satisfied.*

(i) *$N$ is covered by a set of coordinate neighborhoods in each of which, $V$, there exist local coordinates $u^{\kappa}$ such that the system of subspaces represented by $u^{h}=$const and the system of subspaces represented by $u^{\alpha}=$const (namely, $y^{\alpha}=$const) intersect orthogonally and that $u^{h}=x^{h}$ represents the mapping which is the restriction of $\mu$ in $V$, $x^{h}$ being the local coordinates in $\mu(V)$.*

(ii) *The metric tensors of $N$ and $M$ are such that the function* (3.6) *does not depend upon the variables $x^{1}, \cdots, x^{m}$ (namely $u^{1}, \cdots, u^{m}$) in the coordinate neighborhood considered in* (i).

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