Totally ordered linear space structures and separation theorem

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Kazuo IWATA

Introduction. On the geometric form of the Hahn-Banach theorem[1] (e.g. [16, Ch. 2]) it seems to the author that "only if" part of the theorem has not yet been fully discussed. In this note, he deals with this nature in a real spase and considers some related problems.

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Let E be a real linear space with some non-zero vectors.

DEFINITION 1. a) The algebraic dual of E is denoted by E^*.

b) For brevity, a finite system in E is called positively dependent if it is linearly dependent with positive coefficients, otherwise, positively independent. An arbitrary system A in E is called positively independent if every finite system with members in A is positively independent. As a convention, the empty set is regarded as positively independent.

c) If on E there is defined a binary relation, $\mathscr{T}$, such that for all $x, y, z \in E$:

1) reflexive, 2) asymmetric, 3) transitive, 4) comparable,

5) $x\mathscr{T}z$ implies $x+z\mathscr{T}y+z$, 6) $x\mathscr{T}y$ implies $\alpha x\mathscr{T}\alpha y$ for $\alpha \geqslant 0$;

then E is called a totally ordered linear space (abbreviated t.o.l.s.) with respect to $\mathscr{T}$, and denoted by $(E, \mathscr{T})$. The order isomorphism between these t.o.l.s. structures of given E by the identity map is written by "=". From now on, we shall write $x \leqslant y(\mathscr{T})$ (resp. $x<y(\mathscr{T})$) instead of $x\mathscr{T}y$ (resp. instead of $x\mathscr{T}y$ and $x \neq y$). "$x<y(\mathscr{T})$" is read "y is greater than x with respect

1) Formulated in more detail, on the present subject, the space does not need to be topologized. As a geometric form of the Hahn-Banach theorem, in this note, the following is chosen: Let E be a real linear space, K a convex set in E which is absorbing at all its points, and let M be a linear subspace of E not meeting K. Then there exists a maximal subspace which contains M and does not meet K.

As well known, this is an essential part in the proof of the Hahn-Banach separation theorem (i.e., so called Mazur's theorem) in a locally convex space.

to $\mathscr{T}$". Let us denote by $(E, \mathscr{T})^+$ the set of all positive (i.e., greater than 0 with respect to $\mathscr{T}$) vectors in $(E, \mathscr{T})$, a priori, which is a salient, without vertex 0, convex cone in E; and at the same time a maximal element (by set inclusion) of all such.

d) When only a single total ordering $\mathscr{T}$ on E is considered, for convenience, we define $|x|(x \in E)$ to mean $\max\{x, -x\}$ with respect to $\mathscr{T}$. With this notation, $u \in E$ is called an order unit of $(E, \mathscr{T})$ if for each $x \in E$ there exists a real number $\alpha_x > 0$ for which $|x| < \alpha_x u(\mathscr{T})$ holds.

e) In this note, a subset B of E is called absorbing at b if for each vector $x \in E$ there exists a real number $\beta_x > 0$ such that $b + \beta_x x \in B$, a priori, $b \in B$.

We first give some lemmas. Lemma 1 owes much to [15, p. 418].

LEMMA 1. *Let A be a subset of E. A necessary and sufficient condition that there exists a t.o.l.s. $(E, \mathscr{T})$ with $A \subset (E, \mathscr{T})^+$ is that A is positively independent.*

PROOF. Necessity of the condition is evident. (Sufficiency) By virtue of Kuratowski-Zorn lemma, there exists a maximal positively independent subset P which contains A. Then P has the following properties:

1) $P \ni p, q$ implies $P \ni p+q$,
2) $P \ni p$ and $\alpha > 0$ imply $P \ni \alpha p$,
3) $P \cup (-P) \cup \{0\} = E$,
4) $P \cap (-P) = \emptyset$.

Therefore, by defining $x < y(\mathscr{T})$ to mean $y - x \in P$, we obtain a t.o.l.s. $(E, \mathscr{T})$ as desired.

LEMMA 2. *Let a t.o.l.s. $(E, \mathscr{T})$ and an $f \in E^*$ be given. If we define on E a new binary relation $\mathscr{T}_1$ as follows:*

i) *when $f(x-y)=0$, let the relation $\mathscr{T}_1$ between x and y be $\mathscr{T}$,*

ii) *when $f(x-y) \neq 0$, let $x < y(\mathscr{T}_1)$ if and only if $f(y-x) > 0$;*

then E is a t.o.l.s. with respect to new $\mathscr{T}_1$.

PROOF. The new relation $\mathscr{T}_1$ defined above still fulfills the ordering conditions of t.o.l.s.

Such an ordering $\mathscr{T}_1$ on E induced by f from $\mathscr{T}$ is denoted by $f(\mathscr{T})$.

LEMMA 3. (1) *Let a t.o.l.s. $(E, \mathscr{T})$ and a non-zero $f \in E^*$ be given, then $u \in E$ is an order unit of $(E, f(\mathscr{T}))$ if and only if $f(u) > 0$.*

(2) *Let a t.o.l.s. $(E, \mathscr{T})$ with order unit be given, then there exists a non-zero $f \in E^*$ which satisfies*

$$(E, \mathscr{T}) = \big(E, f(\mathscr{T})\big).$$

And two non-zero linear forms satisfy the above if and only if they are posittve scalar multiples each of the other.

PROOFS. Part (1) is an immediate consequence of Lemma 2. (2) To this end, let the set N consist of all $z \in E$ such that $|z|$ is not the order unit of $(E, \mathscr{T})$. Then, by the totally ordered character of $(E, \mathscr{T})$, N constitutes the linear subspace of E. Let an order unit u_0 of $(E, \mathscr{T})$ be fixed. Putting

$$\xi = \sup\left\{\alpha : \alpha u_0 \leqslant |u| (\mathscr{T})\right\},$$

where $|u|$ being an order unit of $(E, \mathscr{T})$, let us determine a real number $\xi > 0$ for $|u|$. Provided that $|u| - \xi u_0 \bar{\in} N$, there exists $\kappa > 0$ such that

$$u_0 < \kappa \left| |u| - \xi u_0 \right| (\mathscr{T}),$$

i.e., $\left(\xi + \frac{1}{\kappa}\right) u_0 < |u| (\mathscr{T})$ or $|u| < \left(\xi - \frac{1}{\kappa}\right) u_0 (\mathscr{T})$, a contradiction at any rate. This yields

$$|u| = z + \xi u_0 (z \in N, \xi > 0)$$

which implies that an arbitrary $x \in E$ is uniquely expressed in the form

$$x = z_x + \xi_x u_0 (z_x \in N)$$

since u_0 is an order unit of $(E, \mathscr{T})$ and N is linear. Hence defining f by $f(x) = \xi_x (x \in E)$, we get a non-zero $f \in E^*$ which establishes $(E, \mathscr{T}) = (E, f(\mathscr{T}))$. The remains are easy, and this completes the proof.

By Definition 1 (c), d) and e)) we have easily (as is seen in part in the proof of Krein's extension theorem[2], or as is seen in [13, p. 32]):

LEMMA 4. *Let $(E, \mathscr{T})$ be a t.o.l.s. and suppose that A is a subset of E such that $A \subset (E, \mathscr{T})^+$. A necessary and sufficient condition that $u \in E$ (resp. $a_0 \in A$, each $a \in A$) is an order unit of $(E, \mathscr{T})$ is that the subset $(E, \mathscr{T})^+$ is absorbing at u (resp. at a_0, at each $a \in A$).*

DEFINITION 2. In this note, a system A in E is said to lie (resp. lie semi-strictly, lie strictly) on one side of a maximal subspace $N(f) = \{x \in E : f(x) = 0\}$ ($f \in E^*$ being non-zero) if $0 \leqslant f(a)$ (resp. $0 \leqslant f(a)$ and not all zero, $0 < f(a)$) for each member a in A. But we restrict ourselves, here, to the case where a system A is positively independent unless finite system.

We now obtain the following

THEOREM 1. *Let A be a positively independent subset of E. A ne-*

2) Cf. [14, p. 136].

cessary and sufficient condition that A lies (resp. lies semi-strictly, lies strictly) on one side of a maximal subspace of E is that there exists a t.o.l.s. $(E, \mathscr{T})$, *with* $A \subset (E, \mathscr{T})^+$, *such that some* $u \in E$ (*resp.* $a_0 \in A$, *each* $a \in A$) *becomes an order unit of* $(E, \mathscr{T})$.

PROOF. We work with the semi-strict case, the remains are likewise obtained by Lemmas 1, 2 and 3. Of course, the result nearly follows when A is a system in E, although A is a subset in E in this theorem. (Necessity) Let the maximal subspace in hypothesis be $N(f)$, that is

$$0 \leqslant f(a), \quad 0 < f(a_0) \ (a, a_0 \in A).$$

Take $(E, \mathscr{T})$ such that $A \subset (E, \mathscr{T})^+$ by Lemma 1, and consider $(E, f(\mathscr{T}))$ by Lemma 2. Then our assertion is obtained by Lemma 3(1). (Sufficiency) Since $a_0 \in A$ is an order unit of $(E, \mathscr{T})$ and A lies in $(E, \mathscr{T})^+$, our assertion is nothing other than Lemma 3 itself.

REMARK 1. In the strict case above, the initial hypothesis is redundant. (The same is in Theorem 2.)

REMARK 2. As a result, the dense cone[3)] in the finite sequence space $\boldsymbol{R}^\infty$ does not lie on one side of any maximal subspace of $\boldsymbol{R}^\infty$. While, if E is an n-dimensional linear space $\boldsymbol{R}^n$, then a positively independent system A in $\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^1$) lies semi-strictly (resp. lies strictly) on one side of a maximal subspace of E.

In accordance with Theorem 1 and Lemma 4, we can state as

THEOREM 2. *Let A be a positively independent subset of E. A necessary and sufficient condition that A lies (resp. lies semi-strictly, lies strictly) on one side of a maximal subspace of E is that there exists a t.o.l.s.* $(E, \mathscr{T})$, *with* $A \subset (E, \mathscr{T})^+$, *such that the subset* $(E, \mathscr{T})^+$ *is absorbing at some point* $b \in E$ (*resp. at* $a_0 \in A$, *at each* $a \in A$).

We proceed to the form

THEOREM 3. *Let K be a convex set in E and M a linear subspace of E not meeting K. A necessary and sufficient condition that K lies (resp. lies semi-strictly, lies strictly) on one side of a maximal subspace H of E with* $H \supset M$ *is that there exists a t.o.l.s.* $(E/M, \mathscr{T})$, *with* $\varphi(K) \subset (E/M, \mathscr{T})^+$, *such that the subset* $(E/M, \mathscr{T})^+$ *is absorbing at some point* (*resp. at* $k_0 + M (k_0 \in K)$, *at each* $k + M (k \in K)$ *in* E/M, *where* φ *is the canonical mapping of E onto the quotient space* E/M.

PROOF. By hypothesis, the image $\varphi(K)$ is convex and does not contain the origin. That is, $\varphi(K)$ is a positively independent subset of E/M. More-

3) For the concept, cf. [13, p. 10].

over, under the postulate $f(x)=F(x+M)$ $(x\in E)$, the following assertions are equivalent:

1) in E, K lies (resp. lies semi-strictly, lies strictly) on one side of a maximal subspace $H=N(f)$ of E with $H\supset M$;

2) in E/M, the subset $\varphi(K)$ lies (resp. lies semi-strictly, lies strictly) on one side of a maximal subspace $N(F)$ of E/M.

Hence, the result follows immediately from Theorem 2.

REMARK 3. In particular, in the above, if K is absorbing at all its points in E, so is $\varphi(K)$ at all its points in E/M. Hence, a fortiori, the geometric form of the Hahn-Banach theorem follows from Lemma 1 and Theorem 3. For the author, above proof (self-contained) based on t.o.l.s. structures seems to be new.

In view of Theorem 2 we obtain the following theorem from which, a fortiori, Krein's extension theorem[4] follows:

THEOREM 4. *Let E be a partially ordered linear space*[5] *and C the set of all positive vectors in E. Let M be a linear subspace of E, f a linear form on M, and set $A=\{x\in M: f(x)>0\}$. Then we have*

(1) *f is positive*[6] *if and only if $A\cup C$ is positively independent.*

(2) *Suppose that f is positive and is not identically zero. A necessary and sufficient condition that f can be extended to a positive linear form F on E is that there exists a t.o.l.s. $(E, \mathscr{T})$, with $A\cup C\subset(E, \mathscr{T})^+$, such that the subset $(E, \mathscr{T})^+$ is absorbing at some point of M.*

PROOFS. (1) For the "if" part, it is easy whether $A=\emptyset$ or not. The converse is also easy whether f is identically zero or not. (2) (Necessity) It suffices to take $A\cup C$ as A in Theorem 2. (Sufficiency) According to Theorem 2, there exists $F_1\in E^*$ such that $(E, \mathscr{T})=(E, F_1(\mathscr{T}))$ for which $(E, \mathscr{T})^+$ *is absorbing* at $b\in E$ if and only if $F_1(b)>0$. On the other hand, since $A\subset(E, \mathscr{T})^+$, if $(E, \mathscr{T})^+$ is absorbing at $m\in M$, this reduces to $f(m)>0$. Hence by observing the above facts in the whole of M, we get a real number $\alpha>0$ which satisfies

$$\alpha F_1(x)=f(x)\ (x\in M).$$

Finally, since $C\subset(E, \mathscr{T})^+$, $\alpha F_1(x)$ $(x\in E)$ is an extension as desired.

By the way, in view of Theorem 2, we can prove the following (gener-

4) See [11, §8.3] or [16, Ch. 2, §3, n° 1]. Also loc. cit. 2).

5) That is, the ordering relation $\mathscr{P}$ on E satisfies all postulates of Definition 1 c) excepting perhaps 4).

6) This means that if $x\in M$ and $0\leqslant x$ $(\mathscr{P})$ (i. e., the induced structure) imply $f(x)\geqslant 0$.

alization of Stiemke-Carver-Dines theorem[7]) due to Ky Fan[8]. With regard to this form, the proof below is somewhat shorter and more direct.

COROLLARY. *Let A be a non-empty finite system in E. A necessary and sufficient condition that A does not lie strictly (resp. does not lie semi-strictly, does not lie) on one side of any maximal subspace of E is that A is positively dependent (resp. positively dependent with coefficients all not zero, positively dependent with coefficients all not zero and further the span of A is E).*

PROOF. Let $A=\{a_1, a_2, \cdots, a_n\}$. To prove the first assertion, we let A be positively independent. The set defined by

$$D=\left\{(\lambda_1, \lambda_2, \cdots, \lambda_n): \sum_{i=1}^{n} \lambda_i = 1, \ \lambda_i \geqslant 0 \ (i=1,2,\cdots,n)\right\}$$

which lies in the Euclidean n-space is a bounded closed set. Introducing an inner product on E, we consider the function r with domain D such that

$$r(\lambda_1, \lambda_2, \cdots, \lambda_n) = \|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n\| .$$

Then r is a continuous function on D, and hence we may put the minimum value of r on D be $2\varepsilon > 0$.

Now let us observe the following set-theoretic union of the spheres:

$$B=\bigcup_{i=1}^{n}\left\{a_i + x: \ \|x\| < \varepsilon, \ x \in E\right\}.$$

Consider the convex combinations of B:

$$\sum \alpha_t (a_t + x_t), \quad \text{where} \quad \sum \alpha_t = 1, \quad \alpha_t > 0 .$$

Then, since

$$\|\sum \alpha_t (a_t + x_t)\| \geqslant \|\sum \alpha_t a_t\| - \|\sum \alpha_t x_t\| \geqslant 2\varepsilon - \varepsilon(\sum \alpha_t) = \varepsilon > 0 ,$$

we see that B is positively independent. Whereas, a priori, B is absorbing at every point of A. Hence, a fortiori, the first assertion follows from Lemma 1 and Theorem 2. The remains of the proof follow from this by reductio ad absurdum and by use of quotient space.

Supplement to Corollary. The author dealt with this nature in [17, 18] too, with no use of separation theorem.

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7) Cf. [1–4].

8) For the first two, see [9, Part I. Corollaries 5, 4].

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