A NOTE ON PRIMITIVE EXTENSIONS OF RANK 3 OF ALTERNATING GROUPS

By

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1. T. Tsuzuku determined (the degrees of) the primitive extensions of rank 3 of symmetric groups ([4]). In this note we take up alternating groups instead of symmetric groups and prove the following theorem.

Theorem 1. *Let A_n be the alternating group of degree n. If A_n has a primitive extension of rank* 3, *then* $n=1$, 3, 5 *or* 7.

Since our proof is quite simlar to Tsuzuku's paper [4], we use the same notations as those in [4] and give a proof in outline only.

S_n: The symmetric group of degree n.

A_n: The alternating group of degree n (on a set Γ).

G: A primitive extension of rank 3 of A_n on a set $\Omega=\{0, 1, 2, \cdots, n, \tilde{1}, \tilde{2}, \cdots, \tilde{m}\}$ which consists of $1+n+m$ letters.

H: The stabilizer G_0 of a letter, say 0, of Ω. The orbits of H are denoted by $\Delta_0=\{0\}$, $\Delta_1=\{1, 2, \cdots, n\}$ and $\Delta_2=\{\tilde{1}, \tilde{2}, \cdots, \tilde{m}\}$ and the group (H, Δ_1) is isomorphic to (A_n, Γ).

L: The stabilizer of the subset $\{0, \tilde{1}\}$ of Ω.

$|X|$: The number of elements in a set X.

2. *Proof of Theorem 1.* Clearly A_2 does not have a primitive extension of rank 3 and so $n\neq 2$. In the following we assume that $n\neq 1$, 3, 5 and 7. By assumption, the group (A_n, Γ) is isomorphic to (H, Δ_1) and $|L|$ is equal to $\frac{n!}{2m}$. According to a theorem of Manning ([4], 2. Prop. 1), $|L|$ is divisible by $\frac{(n-2)!}{2}$ and $\frac{(n-1)!}{2}>|L|\geqq\frac{(n-2)!}{2}$.

I. The case $|L|>\frac{(n-2)!}{2}$ *and L is transitive on* Δ_1.

If L is a primitive subgroup of (H, Δ_1), then, in the same way as 3. I in [4], $2n(n-1)>\left[\frac{n+1}{2}\right]!$ and so we have $n=10$, 9, 8, 6, or 4. In case $n=10$, 9 or 8, by a theorem of Jordan ([5], th. 13. 9), L is either A_n or S_n and this is a contradiction. For the cases $n=6$ or 4, and also for the case

L is imprimitive on Δ_1, in the same way as 3. I in [4], we obtain contradictions.

II. The case $|L|>\frac{(n-2)!}{2}$ *and L is intransitive on* Δ_1.

Since L is a subgroup of $S_r\times S_{n-r}\cap A_n$ with a positive integer r, $\frac{(n-2)!}{2}$ must be a proper divisor of $r!(n-r)!$. Hence we have the following cases (we may assume $r\leqq n-r$): $r=1$, 2 or 3.

(i) $r=1$: Since L is a subgroup of $S_1\times S_{n-1}\cap A_n=S_1\times A_{n-1}$, we may regard L as a subgroup of A_{n-1} (and so S_{n-1}) and we have $|L|=\frac{(n-2)!}{2}t$ where t is a proper divisor of $n-1$. If $t\geqq 3$, then $\frac{(n-1)!}{2}>|L|>(n-2)!$ and so $2<$the index of L in $S_{n-1}<n-1$, which is impossible. If $t=2$, then the index of L in A_{n-1} is equal to $\frac{n-1}{2}$, which is a contradiction.

(ii) $r=2$: In this case, we have a relation $\frac{(n-2)!}{2}<|L|\leqq|S_2\times S_{n-2}\cap A_n|=(n-2)!$ and hence $L=S_2\times S_{n-2}\cap A_n$. In the following, for a group Y and a subgroup X of Y, let $\eta|_X$ be the restriction to X of a character η of Y and let θ^Y be the character of Y induced by a character θ of X. Moreover, let 1_X be the principal character of X. Here we remark that, if $Y=X_1X_2$ where X_1, X_2 are subgroups of a group Y and θ is a character of X_1, then $\theta^Y|_{X_2}=(\theta|_{X_1\cap X_2})^{X_2}$. By the structure of G, 1_H^G is equal to $1_G+\varphi_1+\varphi_2$, where $\varphi_i(i=1, 2)$ is an irreducible character of G with degree f_i, and $1_{H|H}^G$ is equal to $1_H+1_{A_{n-1}}^{A_n}+1_L^{A_n}$. Since $S_n=(S_2\times S_{n-2})\ A_n$,

$$\begin{aligned}1_L^{A_n}&=1_{S_2\times S_{n-2}\cap A_n}^{A_n}=1_{S_2\times S_{n-2}|A_n}^{S_n}\\&=1_{A_n}+\chi_{0|A_n}^{0\cdots0}+\chi_{00|A_n}^{0\cdots0}.\end{aligned}$$

(cf. [4], 2. Prop. 5 or [3], 3. 3)

Similarly

$$1_{A_{n-1}}^{A_n}=1_{S_{n-1}|A_n}^{S_n}=1_{A_n}+\chi_{0|A_n}^{0\cdots0}.$$

Therefore we have

$$1_{H|H}^G=3\ 1_{A_n}+2\chi_{0|A_n}^{0\cdots0}+\chi_{00|A_n}^{0\cdots0}.$$

If $n\geqq 5$, then all the characters appeared in the right-hand side are irreducible and hence we can obtain a contradiction in the same way as 3. II. (ii) in [4]. If $n=4$, then $\chi_{00|A_4}^{00}$ is decomposed into two irreducible characters each of which has degree 1 and so $1_{H|H}^G$ is decomposed into 7 irreducible characters which have degrees 1, 1, 1, 1, 1, 3 and 3 respectively. But f_1 and f_2 must be partial sums of these integers, which is impossible by a theorem of Frame ([4], 2. Prop. 2).

(iii) $r=3$: In this case $n=8$ or 6 and hence $|L|=3!\,5!$ or $3!\,3!$ respectively. This is impossible since L is a subgroup of $S_3\times S_5\cap A_8$ or $S_3\times S_3\cap A_6$ respectively.

III. The case $|L|=\dfrac{(n-2)!}{2}$. As in 3. III of [4], we have $n=57$. Since L is intransitive on Δ_1 and $|L|=\dfrac{55!}{2}$, L must be $S_1\times S_1\times A_{55}$. On the other hand, since $S_{57}=(S_1\times S_1\times S_{55})A_{57}$ and $(S_1\times S_1\times S_{55})\cap A_{57}=S_1\times S_1\times A_{55}$, we have

$$\begin{aligned} 1_L^{A_{57}} &= 1_{S_1\times S_1\times S_{55}}^{S_{57}}\Big|_{A_{57}} \\ &= 1_{A_{57}}+2\chi^{0\cdots 0}_{0|A_{57}}+\chi^{0\cdots 0}_{00|A_{57}}+\chi^{\substack{0\cdots 0\\ 0}}_{0|A_{57}}. \quad \text{(cf. [3], 3. 3)} \end{aligned}$$

In the same way as 3. III in [4], decomposing $1_{A_{57}}+1_{A_{56}}^{A_{57}}+1_L^{A_{57}}$ into irreducible characters and considering these degrees, we have a desired contradiction.

Thus Theorem 1 is proved.

3. Now we can easily obtain

Theorem 2. *Let G be a primitive extension with degree t of rank* 3 *of* A_n. *Then, one of the following holds.*

(i) $n=1$, $t=3$ *and* G *is isomorphic to the cyclic group of order* 3.

(ii) $n=3$, $t=7$ *and* G *is isomorphic to the Frobenius group of order* 21.

(iii) $n=5$, $t=16$ *and* G *is isomorphic to the semidirect product* A_5N, *where* N *is an elementary abelian group of order* 16. (*see* 4. (iv) *in* [4])

(iv) $n=7$, $t=50$ *and* G *is isomorphic to* $U_3(5)$ (*the* 3-*dimensional projective special unitary group over the finite field consisting of* 5^2 *elements.*

In fact, in the case $n=7$, m must be $7\cdot 2$, $7\cdot 3$ or $7\cdot 6$ by a theorem of Manning ([4], 2. Prop. 1). But it is impossible that m is equal to $7\cdot 2$ by a theorem of Wielandt ([5], Th. 31. 2.) According to a theorem of Higman ([2], 3. Th.), m cannot be $7\cdot 3$ and hence m must be $7\cdot 6$ and G is isomorphic to $U_3(5)$ (see also (6.1) Th. in [2]).

Remark. By a theorem of Higman ([2], 3. Th.), The primitive extensions of rank 3 of S_n are exhausted by the groups listed in 4 of [4].

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