ON THE THEORY OF SCHLICHT FUNCTIONS

By

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Let us denote by k a certain positive integer. If

$$f(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$$

is regular and schlicht in the unit circle, then we may say, for convenience, that $f(z)$ is a function of the class k. If $f(z)$ is a function of the class k and maps the unit circle on a starshaped domain with centre at the origin, then we may say, for brevity, that $f(z)$ is a starshaped function of the class k. Similarly, if $f(z)$ is a function of the class k and maps the unit circle on a convex domain, we may call $f(z)$ a convex function of the class k. Our main object in this paper is to state some theorems concerning a starshaped resp. convex function of the class k[1], which are known when $k = 1$ or 2. First we will obtain some results on the coefficients, using an easy lemma. Next, some extensions of STROHHÄCKER's theorems[2] will be mentioned. Applying the above results, we can extend SZEGÖ's theorem[3] on the polynomial sections of a starshaped resp. convex function.

It is well known that under the condition that $f(z)$ is regular in D and that $f'(z)$ never vanishes there, we cannot necessarily assert $f(z)$ to be schlicht in D. In § V, imposing a further condition on

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(1) Recently Mr. CHEN has obtained some results concerning a (schlicht) function of the class k. See Kien Kwong CHEN: Proc. Imp. Acad. Japan, 1933, vol. 9, p. 465–467.

(2) E. STROHHÄCKER: Math. Zeits., Bd. 37, 1933, p. 350–380.

(3) G. SZEGÖ: Math. Ann., Bd. 100, 1928, p. 188–211. See also S. TAKAHASHI: Proc. Physico-Math. Soc. Japan, 3rd ser., vol. 16, 1934, p. 7–15; L. BIEBERBACH: Bulletin, Calcutta Math. Soc., vol. 20, 1930, p. 17–20; and also A. KOBORI: Memoires, College of Science, Kyoto Imp. Univ., ser. A, vol. 16, 1933, p. 127–135.

$f(z)$, we will study the univalency (*Schlichtheit*) of $f(z)$. In the last paragraph we will state a theorem on a schlicht meromorphic function in the unit circle, which is of some interest in itself.

§I. AN IMPORTANT LEMMA.

Suppose that

$$(1)\qquad \Phi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$$

is a starshaped function of the class k. If we put $\zeta = z^k$ and define a function

$$f(\zeta) = [\Phi(z)]^k,$$

then $f(\zeta)$ is regular and starshaped(1) for $|\zeta| < 1$. For the proof, since $f(\zeta) = \zeta + \dots$ is regular and does not vanish for $0 < |\zeta| < 1$, we have only to show(2)

$$\Re\left(\zeta \frac{f'(\zeta)}{f(\zeta)}\right) > 0 \quad \text{for} \quad |\zeta| < 1.$$

But this inequality follows from

$$\Re\left(z \frac{\Phi'(z)}{\Phi(z)}\right) > 0 \quad \text{for} \quad |z| < 1,$$

since we easily get

$$\zeta \frac{f'(\zeta)}{f(\zeta)} = z \frac{\Phi'(z)}{\Phi(z)}, \quad (\zeta = z^k).$$

Thus we obtain

(1) We say that $f(z)$ is starshaped for $|z| < \rho$, if $f(z)$ is regular and schlicht and maps $|z| < \rho$ on a starshaped domain with centre at the origin.

(2) Here we use a well known theorem: Suppose that $f(z) = z + \dots$ is regular in the unit circle. Then $f(z)$ is starshaped for $|z| < 1$, if and only if

$$f(z) \neq 0 \quad \text{for} \quad 0 < |z| < 1 \quad \text{and} \quad \Re\left(z \frac{f'(z)}{f(z)}\right) > 0 \quad \text{for} \quad |z| < 1.$$

Lemma. *If $\Phi(z)$ is a starshaped function of the class k, then*

$$f(\zeta) = [\Phi(z)]^k, \quad (\zeta = z^k) \tag{2}$$

is a starshaped function of the class 1.

§ II. ON THE COEFFICIENTS OF A STARSHAPED RESP. CONVEX FUNCTION.

In this paragraph we will enunciate a theorem on the coefficients of a starshaped resp. convex function of the class k.

Theorem 1. *If* $\Phi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$ *is a starshaped function of the class k, then*

$$|c_n| \leqq \frac{1}{n!}\,\frac{2}{k}\left(\frac{2}{k}+1\right)\left(\frac{2}{k}+2\right)\dots\left(\frac{2}{k}+(n-1)\right) \tag{3}$$

$$(n = 1, 2, 3, \dots).$$

The right hand side of (3) *cannot be replaced by any smaller number. This extremal case can be given by the function*

$$\Phi_0(z) = \frac{z}{(1-z^k)^{\frac{2}{k}}} = z\, e^{-\frac{2}{k}\log(1-z^k)}, \tag{4}$$

taking a branch of log *such that* $\log 1 = 0$.

Proof. Consider the function $f(\zeta) = [\Phi(z)]^k$, $(\zeta = z^k)$. Since the function $\zeta\dfrac{f'(\zeta)}{f(\zeta)}$ is regular for $|\zeta| < 1$, this can be expanded in a power series:

$$\zeta\frac{f'(\zeta)}{f(\zeta)} = 1 + k\zeta\,\frac{c_1 + 2c_2\zeta + \dots + nc_n\zeta^{n-1} + \dots}{1 + c_1\zeta + c_2\zeta^2 + \dots + c_n\zeta^n + \dots}$$

$$= 1 + b_1\zeta + b_2\zeta^2 + \dots + b_n\zeta^n + \dots$$

Using CARATHÉODORY's theorem[1] and the lemma, we obtain

$$|b_n| \leqq 2 \qquad (n=1, 2, 3, \ldots\ldots),$$

whence follows that

$$\zeta \frac{f'(\zeta)}{f(\zeta)} \ll 1+\frac{2\zeta}{1-\zeta},^{(2)}$$

that is,

$$\frac{c_1+2c_2\zeta+\ldots\ldots+nc_n\zeta^{n-1}+\ldots\ldots}{1+c_1\zeta+c_2\zeta^2+\ldots\ldots+c_n\zeta^n+\ldots\ldots} \ll \frac{2}{k}\frac{1}{1-\zeta},$$

consequently

$$\log(1+c_1\zeta+c_2\zeta^2+\ldots\ldots+c_n\zeta^n+\ldots\ldots) \ll -\frac{2}{k}\log(1-\zeta),$$

taking a branch of log such that $\log 1 = 0$. From

$$1+c_1\zeta+c_2\zeta^2+\ldots\ldots+c_n\zeta^n+\ldots\ldots \ll \frac{1}{(1-\zeta)^{\frac{2}{k}}},$$

follows the inequality (3). Our theorem is completely proved, considering the function

$$\Phi_0(z) = \frac{z}{(1-z^k)^{\frac{2}{k}}} = z+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{2}{k}\left(\frac{2}{k}+1\right)\ldots\left(\frac{2}{k}+(n-1)\right)z^{nk+1}$$

which is a starshaped function of the class k, because

$$\Re\left(z\frac{\Phi_0'(z)}{\Phi_0(z)}\right) = \Re\left(\frac{1+z^k}{1-z^k}\right) > 0 \quad \text{for} \quad |z|<1 .$$

Remark. $w=\Phi_0(z)$ maps the unit circle on a whole w-plane cut from $w=\frac{1}{\sqrt[k]{4}}e^{\frac{i(2\nu+1)\pi}{k}}$ to $w=\infty$ along each ray which starts from $w=0$ and passes through $w=\frac{1}{\sqrt[k]{4}}e^{\frac{i(2\nu+1)\pi}{k}}$, where $\nu=0, 1, 2, \ldots\ldots, k-1$.

(1) CARATHÉODORY's theorem states that if $\varphi(\zeta)=1+\sum_1^{\infty} b_n\zeta^n$ is regular and $\Re(\varphi(\zeta))>0$ for $|\zeta|<1$, then $|b_n|\leqq 2$.

(2) Let $A(z)=\sum_0^{\infty} a_n z^n$, $B(z)=\sum_0^{\infty} b_n z^n$ be two power series and let all the coefficients b_n of $B(z)$ be non-negative. Then $A(z)\ll B(z)$ means that $|a_n|\leqq b_n$ for every n.

Theorem 2. *Suppose that* $\varphi(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_n z^{nk+1} + \dots$ *is a convex function of the class* k. *Then*

$$(5)\qquad |c_n| \leqq \frac{1}{nk+1}\frac{1}{n!}\frac{2}{k}\Big(\frac{2}{k}+1\Big)\Big(\frac{2}{k}+2\Big)\dots\Big(\frac{2}{k}+(n-1)\Big)$$

$$(n = 1, 2, 3, \dots).$$

The right hand side of (5) *cannot be replaced by any smaller number. This extremal case can be given by the function*

$$(6)\qquad \varphi_0(z) = \int_0^z \frac{dz}{(1-z^k)^{\frac{2}{k}}}.$$

Proof. By ALEXANDER's theorem(1)

$$\Phi(z) = z\varphi'(z) = z + \sum_{n=1}^{\infty}(nk+1)c_n z^{nk+1}$$

is a starshaped function of the class k. Hence, applying theorem 1, we have

$$(nk+1)|c_n| \leqq \frac{1}{n!}\frac{2}{k}\Big(\frac{2}{k}+1\Big)\Big(\frac{2}{k}+2\Big)\dots\Big(\frac{2}{k}+(n-1)\Big).$$

§ III. SOME EXTENSIONS OF STROHHÄCKER'S THEOREMS.

Here some extensions of STROHHÄCKER's theorems will be mentioned.

Theorem 3. *If* $\Phi(z)$ *is a starshaped function of the class* k *and if* $|z_0| \leqq r < 1$, *then the point* $\frac{\Phi(z_0)}{z_0}$ *belongs to the closed domain* D, *which is the image of the circle* $|\zeta| \leqq r^k$ *by* $\frac{1}{(1-\zeta)^{\frac{2}{k}}}$.

Proof. Since, by the lemma, the function $f(\zeta) = [\Phi(z)]^k$, $(\zeta = z^k)$ is a starshaped function of the class 1, the point $\frac{f(\zeta_0)}{\zeta_0} = \left[\frac{\Phi(z_0)}{z_0}\right]^k$, $(\zeta_0 = z_0^k)$

(1) The following theorem is due to Mr. J. ALEXANDER. Let $\varphi(z) = z + \dots$ be regular in the unit circle. Then $\varphi(z)$ maps $|z| < 1$ on a convex domain, if and only if $\Phi(z) = z\varphi'(z)$ maps $|z| < 1$ on a starshaped domain with centre at the origin. Cf. J. ALEXANDER: Ann. of Math., 2. ser., vol. 17, 1915–16, p. 12–22.

belongs, by STROHHÄCKER's theorem[1], to the domain D', which is the image of the circle $|\zeta| \leqq r^k$ by $\frac{1}{(1-\zeta)^2}$. Therefore the point $\frac{\Phi(z_0)}{z_0}$ belongs to the domain D, which is the image of the circle $|\zeta| \leqq r^k$ by $\frac{1}{(1-\zeta)^{\frac{2}{k}}}$.

Remark. D cannot be replaced by any smaller domain, for the function $\Phi_0(z)$ gives the extremal case; more precisely, the set of values taken by $\frac{\Phi_0(z)}{z}$ for $|z| \leqq r$ is identical with the domain D.

As an immediate result of the above theorem, we have

Theorem 4. *If $\Phi(z)$ is a starshaped function of the class k, then*

$$\frac{1}{(1+r^k)^{\frac{2}{k}}} \leqq \left|\frac{\Phi(z)}{z}\right| \leqq \frac{1}{(1-r^k)^{\frac{2}{k}}} \qquad (|z| \leqq r < 1). \tag{7}$$

Proof. The circle $|\zeta| \leqq r^k$ can be mapped by $s = \frac{1}{1-\zeta}$ on a circle K, which has the segment $\frac{1}{1+r^k} \cdots \frac{1}{1-r^k}$ as diameter. Since the domain D (in theorem 3) is the image of K by $w = s^{\frac{2}{k}}$, it is clear that D lies on the ring: $\frac{1}{(1+r^k)^{\frac{2}{k}}} \leqq |w| \leqq \frac{1}{(1-r^k)^{\frac{2}{k}}}$. Therefore our assertion is true, by theorem 3.

Remark. The limits of (7) can be also attained by the function $\Phi_0(z)$.

Theorem 5. *If $\Phi(z)$ is a starshaped function of the class k, and if $|z_0| \leqq r < 1$, then the point $z_0 \frac{\Phi'(z_0)}{\Phi(z_0)}$ lies on the circular domain which has the segment $\frac{1-r^k}{1+r^k} \cdots \frac{1+r^k}{1-r^k}$ as diameter.*

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(1) Mr. STROHHÄCKER has proved that if $f(z)$ is starshaped for $|z|<1$ and if $|z_0| \leqq r < 1$, then the point $\frac{f(z_0)}{z_0}$ lies in the closed domain, which is the image of the circle $|z| \leqq r$ by $\frac{1}{(1-z)^2}$. Cf. E. STROHHÄCKER: loc. cit.

Proof. Since

$$g(z) \equiv z\frac{\Phi'(z)}{\Phi(z)} = 1+kc_1 z^k + \dots .$$

is regular and $\Re(g(z)) > 0$ for $|z| < 1$, the function

$$\psi(z) \equiv \frac{1}{z^k}\frac{1-g(z)}{1+g(z)}$$

is regular and $|\psi(z)| < 1$ for $|z| < 1$. Hence

$$\left|\frac{1-g(z)}{1+g(z)}\right| < |z|^k \qquad (|z| < 1),$$

whence follows our assertion.

Theorem 6. *Let $\Phi(z)$ be a starshaped function of the class k. Then we have*

$$\text{(8)} \qquad \frac{1-r^k}{1+r^k} \leqq \left| z\frac{\Phi'(z)}{\Phi(z)} \right| \leqq \frac{1+r^k}{1-r^k} \qquad (|z| \leqq r < 1)$$

$$\text{(9)} \qquad \frac{1-r^k}{1+r^k} \leqq \Re\left(z\frac{\Phi'(z)}{\Phi(z)}\right) \leqq \frac{1+r^k}{1-r^k} \qquad (|z| \leqq r < 1).$$

Proof. This theorem comes directly from the above.

Remark. The limits of (8) and (9) can be also attained by the function $\Phi_0(z) = \dfrac{z}{(1-z^k)^{\frac{2}{k}}}$, for $z\dfrac{\Phi_0'(z)}{\Phi_0(z)} = \dfrac{1+z^k}{1-z^k}$.

Theorem 7. *If $\varphi(z)$ is a convex function of the class k, and if $|z_0| \leqq r < 1$, then the point $\varphi'(z_0)$ belongs to D, where D denotes the same domain as in theorem 3.*

Proof. Since $\Phi(z) = z\varphi'(z)$ is a starshaped function of the class k, $\dfrac{\Phi(z_0)}{z_0} = \varphi'(z_0)$ lies, by theorem 3, on the domain D.

§IV. AN EXTENSION OF SZEGÖ'S THEOREM.

Mr. G. SZEGÖ[1] has proved that if

$$f(z)=z+c_1z^2+c_2z^3+\dots+c_nz^{n+1}+\dots$$

is regular and starshaped for $|z|<1$, then every section

$$f_n(z)=z+c_1z^2+c_2z^3+\dots+c_nz^{n+1}$$

is starshaped for $|z|<\frac{1}{4}$. Recently Mr. S. TAKAHASHI[2] has shown that if $f(z)$ is an odd function, then SZEGÖ's theorem can be mentioned in the form: Every section $f_n(z)$ is starshaped for $|z|<\frac{1}{\sqrt{3}}$. Now we state a theorem which contains SZEGÖ-TAKAHASHI's theorem as its special case.

Theorem 8. *Let k be any positive integer and*

$$\Phi(z)=z+c_1z^{k+1}+c_2z^{2k+1}+\dots+c_nz^{nk+1}+\dots$$

be a starshaped function of the class k. Then every section

$$z+c_1z^{k+1}+c_2z^{2k+1}+\dots+c_nz^{nk+1}$$

is starshaped for

$$|z|<\sqrt[k]{\frac{k}{2(k+1)}},$$

where the number $\sqrt[k]{\frac{k}{2(k+1)}}$ cannot be replaced by any greater one. And the extremal case can be given by the function $\Phi_0(z)$ of (4).

Proof. For the proof we apply the elegant method which was used by Mr. G. SZEGÖ in the case $k=1$. We put, for $n\geqq 2$,

(1) G. SZEGÖ: loc. cit.
(2) S. TAKAHASHI: loc. cit.

$$s_n(z) = z + c_1 z^{k+1} + c_2 z^{2k+1} + \dots + c_{n-1} z^{(n-1)k+1},$$

$$\rho_n(z) = c_n z^{nk+1} + c_{n+1} z^{(n+1)k+1} + \dots; \text{ that is, } \Phi(z) = s_n(z) + \rho_n(z),$$

and

$$R = \sqrt[k]{\frac{k}{2(k+1)}}.$$

In order that $s_n(z)$ should be starshaped for $|z| < R$, it is sufficient that for $0 < |z| < R$

(10) $$s_n(z) \neq 0$$

and

(11) $$\Re\left(z\frac{s_n'(z)}{s_n(z)}\right) = \Re\left(z\frac{\Phi'(z) - \rho_n'(z)}{\Phi(z) - \rho_n(z)}\right)$$

$$= \Re\left(z\frac{\Phi'(z)}{\Phi(z)}\right) + \Re\left(z\frac{\rho_n(z)\frac{\Phi'(z)}{\Phi(z)} - \rho_n'(z)}{\Phi(z) - \rho_n(z)}\right) > 0.$$

Hence, for our object, it will suffice to show that for $|z| = R$

(12) $$|\Phi(z)| > |\rho_n(z)|$$

and

(13) $$\frac{1-|z|^k}{1+|z|^k} \geqq |z| \frac{|\rho_n(z)|\left|\frac{\Phi'(z)}{\Phi(z)}\right| + |\rho_n'(z)|}{|\Phi(z)| - |\rho_n(z)|}$$

$$= \frac{\left|\frac{\rho_n(z)}{z}\right| \left|z\frac{\Phi'(z)}{\Phi(z)}\right| + |\rho_n'(z)|}{\left|\frac{\Phi(z)}{z}\right| - \left|\frac{\rho_n(z)}{z}\right|}.$$

In fact $\frac{\rho_n(z)}{\Phi(z)}$ is regular for $|z| < 1$, since $\Phi(z) \neq 0$ for $0 < |z| < 1$. Hence, if we have (12) for $|z| = R$, then the same inequality holds for $0<|z|<R$, whence follows (10) for $0<|z|<R$. Consequently $z\frac{s_n'(z)}{s_n(z)}$ is regular for $|z|\leqq R$ and if $\Re\left(z\frac{s_n'(z)}{s_n(z)}\right)\geqq 0$ for $|z|=R$, then $\Re\left(z\frac{s_n'(z)}{s_n(z)}\right)>0$ for $|z|<R$. But the inequality $\Re\left(z\frac{s_n'(z)}{s_n(z)}\right)\geqq 0$ for $|z|=R$ follows from (13) for $|z|=R$, by (9) and (12).

We put

$$\rho=R^k=\frac{k}{2(k+1)} \quad \text{and} \quad a=\frac{2}{k}. \tag{14}$$

Thus we have obtained the following inequalities:

$$|c_n|\leqq\frac{a(a+1)(a+2)\ldots.(a+n-1)}{n!} \tag{3}$$

$$\left|\frac{\Phi(z)}{z}\right|\geqq\frac{1}{(1+\rho)^{\frac{2}{k}}} \qquad (|z|=R), \tag{7}$$

$$\left|z\frac{\Phi'(z)}{\Phi(z)}\right|\leqq\frac{1+\rho}{1-\rho} \qquad (|z|=R). \tag{8}$$

On the other hand,

$$\begin{aligned}\left|\frac{\rho_n(z)}{z}\right| &\leqq\sum_{\nu=n}^{\infty}|c_\nu|\rho^\nu \\ &\leqq\sum_{\nu=n}^{\infty}\frac{a(a+1)\ldots.(a+\nu-1)}{\nu!}\rho^\nu \qquad (|z|=R).\end{aligned} \tag{15}$$

Similarly,

$$|\rho_n'(z)|\leqq\sum_{\nu=n}^{\infty}\frac{(\nu k+1)a(a+1)\ldots.(a+\nu-1)}{\nu!}\rho^\nu \qquad (|z|=R). \tag{16}$$

I. Let us consider the case where $n\geqq 4$. First we prove the inequality (12) for $|z|=R$. By (7) and (15),

$$(17)\qquad \left|\frac{\Phi(z)}{z}\right|-\left|\frac{p_n(z)}{z}\right| \geqq \frac{1}{(1+\rho)^a}-\sum_{\nu=n}^{\infty}\frac{a(a+1)\ldots\ldots(a+\nu-1)}{\nu\,!}\rho^\nu$$

$$\geqq \frac{1}{(1+\rho)^a}-\sum_{\nu=4}^{\infty}\frac{a(a+1)\ldots\ldots(a+\nu-1)}{\nu\,!}\rho^\nu$$

$$=\left\{\frac{1}{(1-\rho)^a}-\sum_{\nu=4}^{\infty}\frac{a(a+1)\ldots\ldots(a+\nu-1)}{\nu\,!}\rho^\nu\right\}^{(1)}$$

$$-\left\{\frac{1}{(1-\rho)^a}-\frac{1}{(1+\rho)^a}\right\}$$

$$=\frac{(3k+4)(k+2)}{3(k+1)^2}-\left\{\left(\frac{2k+2}{k+2}\right)^{\frac{2}{k}}-\left(\frac{2k+2}{3k+2}\right)^{\frac{2}{k}}\right\}\equiv A(k).$$

Since $\frac{(3k+4)(k+2)}{3(k+1)^2}$ and $\left(\frac{2k+2}{k+2}\right)^{\frac{2}{k}}-\left(\frac{2k+2}{3k+2}\right)^{\frac{2}{k}}$ are decreasing functions of k in the interval $(1, \infty)$, it is clear that for $k\geqq 2$

$$\frac{(3k+4)(k+2)}{3(k+1)^2}>\lim_{k\to\infty}\frac{(3k+4)(k+2)}{3(k+1)^2}=1$$

and

$$\left(\frac{2k+2}{k+2}\right)^{\frac{2}{k}}-\left(\frac{2k+2}{3k+2}\right)^{\frac{2}{k}}<\frac{2\cdot2+2}{2+2}-\frac{2\cdot2+2}{3\cdot2+2}=\frac{3}{4}\,.$$

Hence

$$A(k)>1-\frac{3}{4}=\frac{1}{4}\qquad\text{for}\qquad k\geqq 2\,.$$

On the other hand,

$$A(1)=\frac{7\cdot3}{3\cdot2^2}-\left(\frac{16}{15}\right)^2>0\,.$$

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(1) $\frac{1}{(1-\rho)^a}=1+\sum_{\nu=1}^{\infty}\frac{a(a+1)\ldots\ldots(a+\nu-1)}{\nu\,!}\rho^\nu$

$=1+a\rho+\frac{1}{2!}a(a+1)\rho^2+\frac{1}{3!}a(a+1)(a+2)\rho^3+\sum_{\nu=4}^{\infty}\frac{a(a+1)\ldots\ldots(a+\nu-1)}{\nu\,!}\rho^\nu\,.$

Here we prove the inequality (13) for $|z| = R$. For this purpose, it will suffice to show that, by (7), (8), (15) and (16),

$$(18)\quad \frac{1-\rho}{1+\rho} \geqq \frac{\dfrac{1+\rho}{1-\rho}\sum_{\nu=n}^{\infty}\dfrac{a(a+1)\ldots(a+\nu-1)}{\nu!}\rho^{\nu}+\sum_{\nu=n}^{\infty}(\nu k+1)\dfrac{a(a+1)\ldots(a+\nu-1)}{\nu!}\rho^{\nu}}{\dfrac{1}{(1+\rho)^{\frac{2}{k}}}-\sum_{\nu=n}^{\infty}\dfrac{a(a+1)\ldots(a+\nu-1)}{\nu!}\rho^{\nu}}$$

or that

$$(19)\quad \frac{1-\rho}{1+\rho} \geqq \frac{\dfrac{1+\rho}{1-\rho}\sum_{\nu=4}^{\infty}\dfrac{a(a+1)\ldots(a+\nu-1)}{\nu!}\rho^{\nu}+\sum_{\nu=4}^{\infty}(\nu k+1)\dfrac{a(a+1)\ldots(a+\nu-1)}{\nu!}\rho^{\nu}}{\dfrac{1}{(1+\rho)^{\frac{2}{k}}}-\sum_{\nu=4}^{\infty}\dfrac{a(a+1)\ldots(a+\nu-1)}{\nu!}\rho^{\nu}},$$

since the right hand side of (18) increases as n decreases for $4 \leqq n < \infty$. In (17) we have shown that

$$\frac{1}{(1+\rho)^{\frac{2}{k}}} > \sum_{\nu=4}^{\infty}\frac{a(a+1)\ldots\ldots(a+\nu-1)}{\nu!}\rho^{\nu}.$$

Hence (19) is equivalent to

$$(20)\quad \frac{1-\rho}{1+\rho}\cdot\frac{1}{(1+\rho)^{\frac{2}{k}}} \geqq \sum_{\nu=4}^{\infty}\left\{\frac{1+\rho}{1-\rho}+\frac{1-\rho}{1+\rho}+\nu k+1\right\}\frac{a(a+1)\ldots(a+\nu-1)}{\nu!}\rho^{\nu}.$$

At first we prove (20) for $k \geqq 2$. Obviously $\rho = \dfrac{k}{2(k+1)}$ is an increasing function of k for $2 \leqq k < \infty$ and the values taken by ρ belongs to the interval $\frac{1}{3} \leqq \rho < \frac{1}{2}$. k and a can be written, as functions of ρ, such that $k = \dfrac{2\rho}{1-2\rho}$ and $a = \dfrac{1}{\rho}-2$ for $\frac{1}{3} \leqq \rho < \frac{1}{2}$. Hence, for the proof of (20), it suffices to show that for $\frac{1}{3} \leqq \rho < \frac{1}{2}$

(21) $$\frac{1-\rho}{1+\rho}\cdot\frac{1}{(1+\rho)^{\frac{1}{\rho}-2}} \geqq \sum_{\nu=4}^{\infty}\left\{\frac{1+\rho}{1-\rho}+\frac{1-\rho}{1+\rho}+\frac{2\nu\rho}{1-2\rho}+1\right\} \times \frac{(1-2\rho)(1-\rho)(1+\rho)\ldots\ldots(1+\rho(\nu-3))}{\nu!}$$

or

(22) $$\frac{1}{(1+\rho)^{\frac{1}{\rho}-2}} \geqq \sum_{\nu=4}^{\infty}\left\{\frac{3+2(\nu-3)\rho+\rho^2-2(\nu+1)\rho^3}{1-\rho}\right\} \times \frac{(1+\rho)\ldots\ldots(1+\rho(\nu-3))}{\nu!}\,.$$

Put

$$Q(\rho) = \frac{3+2(\nu-3)\rho+\rho^2-2(\nu+1)\rho^3}{1-\rho} \qquad \left(\begin{matrix}\nu\geqq 4\\ \frac{1}{3}\leqq\rho\leqq\frac{1}{2}\end{matrix}\right).$$

Then

$$(1-\rho)^2Q'(\rho) = (2\nu-3)+2\rho-(6\nu+7)\rho^2+4(\nu+1)\rho^3>0$$

for $\frac{1}{3}\leqq\rho\leqq\frac{1}{2}$.(1) Consequently the right hand side of (22) (say $\equiv B(\rho)$) is an increasing function of ρ for $\frac{1}{3}\leqq\rho\leqq\frac{1}{2}$. Hence, for $\frac{1}{3}\leqq\rho<\frac{1}{2}$

(23) $$B(\rho)<B\left(\frac{1}{2}\right)=\sum_{\nu=4}^{\infty}\frac{3\nu}{2}\cdot\frac{\left(1+\frac{1}{2}\right)\ldots\left(1+\frac{\nu-3}{2}\right)}{\nu!} = 3\sum_{\nu=4}^{\infty}\frac{1}{2^{\nu-1}}=\frac{3}{4}\,.$$

On the other hand,

(24) $$\frac{1}{(1+\rho)^{\frac{1}{\rho}-2}} > \frac{1}{\left(1+\frac{1}{3}\right)^{3-2}} = \frac{3}{4} \qquad \text{for} \quad \frac{1}{3}\leqq\rho<\frac{1}{2}\,,$$

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(1) Put $\lambda(\rho)=(1-\rho)^2Q'(\rho)$. Then $\lambda'(\rho)=2(1-\rho)(1-6(\nu+1)\rho)<0$ for $\frac{1}{3}\leqq\rho\leqq\frac{1}{2}$. Consequently $\lambda(\rho)\geqq\lambda\left(\frac{1}{2}\right)=\nu-3-\frac{1}{4}\geqq\frac{3}{4}>0$ for $\frac{1}{3}\leqq\rho\leqq\frac{1}{2}$.

because $\frac{1}{(1+\rho)^{\frac{1}{\rho}-2}}$ is increasing in the interval $\frac{1}{3}\leqq\rho<\frac{1}{2}$. From (23) and (24) follows the required inequality (22). Thus, for $k\geqq 2$ our proof of (20) is completed.

But there remains the case $k=1$. If $k=1$, then, by (14), $\rho=\frac{1}{4}$ and $a=2$; in this case,

$$\text{The left hand side of (20)} = \frac{1-\frac{1}{4}}{1+\frac{1}{4}}\frac{1}{\left(1+\frac{1}{4}\right)^2}=\frac{48}{125}>\frac{1}{4}$$

$$\text{The right hand side of (20)} = \sum_{\nu=4}^{\infty}\left\{\frac{1+\frac{1}{4}}{1-\frac{1}{4}}+\frac{1-\frac{1}{4}}{1+\frac{1}{4}}+\nu+1\right\}(\nu+1)\left(\frac{1}{4}\right)$$

$$=\frac{49}{15}\sum_{\nu=4}^{\infty}(\nu+1)\left(\frac{1}{4}\right)^{\nu}+\sum_{\nu=4}^{\infty}\nu(\nu+1)\left(\frac{1}{4}\right)^{\nu} \text{ (1)}$$

$$=\frac{49}{15}\cdot\frac{1}{36}+\frac{53}{27\cdot 16}<\frac{1}{4}.$$

Therefore the inequality (20) is also true for $k=1$.

Thus it is completely proved that *every section*

$$s_n(z)=z+c_1z^{k+1}+c_2z^{2k+1}+\ldots\ldots+c_{n-1}z^{(n-1)k+1}$$

is starshaped for

$$|z|<R=\sqrt[k]{\frac{k}{2(k+1)}},$$

as long as n is greater than or equal to 4.

II. Let us consider the section $s_n(z)$ when $n=2$; that is, $s_2(z)=z+c_1z^{k+1}$. If $c_1=0$, then $s_2(z)=z$. Next suppose that $c_1\neq 0$.

---

(1) Here we use the equalities: $\sum_4^{\infty}(\nu+1)\rho^{\nu}=\frac{1}{(1-\rho)^2}-(1+2\rho+3\rho^2+4\rho^3)$, $\sum_4^{\infty}\nu(\nu+1)\rho^{\nu}=\frac{2\rho}{(1-\rho)^3}-(2\rho+6\rho^2+12\rho^3)$.

$$w = z\frac{s_2'(z)}{s_2(z)} = \frac{1+(k+1)c_1 z^k}{1+c_1 z^k} = \frac{1+(k+1)\xi}{1+\xi},$$

putting $\xi = c_1 z^k$.

The circle $|\xi| \leqq \rho(<1)$ can be mapped by $\frac{1+(k+1)\xi}{1+\xi}$ on a circle K having the segment $\frac{1-(k+1)\rho}{1-\rho} \ldots \frac{1+(k+1)\rho}{1+\rho}$ as diameter, hence K lies on the half plane: $\Re(w) > 0$, provided that $\rho < \frac{1}{k+1}$. Therefore, $\Re\left(z\frac{s_2'(z)}{s_2(z)}\right) > 0$ for $|z| < \sqrt[k]{\frac{1}{|c_1|(k+1)}}$. Since $|c_1| \leqq \frac{2}{k}$, by theorem 1, it is easy to see that $s_2(z)$ is starshaped for $|z| < R = \sqrt[k]{\frac{k}{2(k+1)}}$.

Next consider the section $s_2(z)$ of $\Phi_0(z) = \frac{z}{(1-z^k)^{\frac{2}{k}}} = z + \frac{2}{k}z^{k+1} + \ldots$. Then $s_2(z) = z + \frac{2}{k}z^{k+1}$, $s_2'(z) = 1 + \frac{2}{k}(k+1)z^k$. The section $s_2(z) = z + \frac{2}{k}z^{k+1}$ cannot be starshaped for $|z| < R'$, if $R' > R$, because $s_2'(z)$ has zero-points on the circumference $|z| = R = \sqrt[k]{\frac{k}{2(k+1)}}$. Thus it is proved that *every section* $s_2(z)$ *is starshaped for* $|z| < R = \sqrt[k]{\frac{k}{2(k+1)}}$ *and* R *cannot be replaced by any greater number, provided that* k *is fixed, and the extremal case can be given by the section* $s_2(z)$ *of* $\Phi_0(z)$.

III. Lastly we must prove that

$$s_3(z) = z + c_1 z^{k+1} + c_2 z^{2k+1}$$

is starshaped for $|z| < R = \sqrt[k]{\frac{k}{2(k+1)}}$. We will prove for $|z| \leqq R$

$$\Re\frac{1+(k+1)c_1 z^k + (2k+1)c_2 z^{2k}}{1+c_1 z^k + c_2 z^{2k}} > 0. \tag{25}$$

The denominator does not vanish there, because

$$|c_1 z^k + c_2 z^{2k}| \leqq \frac{2}{k}R^k + \frac{1}{2}\frac{2}{k}\left(\frac{2}{k}+1\right)R^{2k} = \frac{5k+6}{4(k+1)^2} < 1.$$

Hence we can assume $|z| = R$. And, further, it suffices to show this inequality for $z = R$. (Consider $\bar{\varepsilon}\Phi(\varepsilon z)$ with proper ε, $|\varepsilon| = 1$). Hence our assertion can be stated as follows:

$$(26) \qquad \Re \frac{1+(k+1)c_1\dfrac{k}{2(k+1)}+(2k+1)c_2\Big(\dfrac{k}{2(k+1)}\Big)^2}{1+c_1\dfrac{k}{2(k+1)}+c_2\Big(\dfrac{k}{2(k+1)}\Big)^2} > 0 .$$

f we put $\zeta = z^k$, the function

$$(27) \qquad F(\zeta) = z\frac{\Phi'(z)}{\Phi(z)} = 1+k\zeta\frac{c_1+2c_2\zeta+\dots}{1+c_1\zeta+c_2\zeta^2+\dots}$$

$$= 1+2C_1\zeta+2C_2\zeta^2+\dots$$

has for $|\zeta|<1$ a positive real part, so that by CARATHÉODORY-TOEPLITZ's theorem we have

$$(28) \qquad |C_1| \leqq 1, \qquad |C_2-C_1^2| \leqq 1-|C_1|^2,$$

where

$$c_1 = \frac{2}{k}C_1, \qquad c_2 = \frac{2}{k^2}C_1^2+\frac{1}{k}C_2 .$$

Hence the inequality (26) can be written as follows:

$$(29) \qquad \Re \frac{1+C_1+\dfrac{(2k+1)(k+2)}{4(k+1)^2}C_1^2+\dfrac{k(2k+1)}{4(k+1)^2}\cdot\eta\cdot(1-|C_1|^2)}{1+\dfrac{1}{k+1}C_1+\dfrac{k+2}{4(k+1)^2}C_1^2+\dfrac{k}{4(k+1)^2}\cdot\eta\cdot(1-|C_1|^2)} > 0 .$$

$$(|\eta| \leqq 1).$$

When C_1 is fixed, the fraction of (29) can be considered as a regular function of η for $|\eta| \leqq 1$, because the denominator never vanishes there.[(1)] Hence we can assume $|\eta| = 1$. Consequently, if we put, for the sake of simplicity,

(1) $\left|\dfrac{1}{k+1}C_1+\dfrac{k+2}{4(k+1)^2}C_1{}^2+\dfrac{k}{4(k+1)^2}\cdot\eta\cdot(1-|C_1|^2)\right| \leqq \dfrac{1}{k+1}+\dfrac{k+2}{4(k+1)^2}+\dfrac{k}{4(k+1)^2}$

$= \dfrac{3}{2(k+1)} \leqq \dfrac{3}{4} < 1 .$

$$(2k+1)w_1 = 1+C_1+\frac{(2k+1)(k+2)}{4(k+1)^2}C_1^2 ,$$

(30) $$w_2 = 1+\frac{C_1}{k+1}+\frac{k+2}{4(k+1)^2}C_1^2 ,$$

$$Z = \frac{k}{4(k+1)^2}\eta(1-|C_1|^2) \qquad (|\eta| = 1) ,$$

we have only to show that

$$\Re\frac{w_1+Z}{w_2+Z} > 0$$

or

$$\Re(w_1+Z)(\overline{w_2}+\overline{Z}) = \Re w_1\overline{w_2}+\Re(w_1+w_2)\overline{Z}+|Z|^2 > 0 ,$$

or

$$\Re w_1\overline{w_2}+|Z|^2 > |Z|\ |w_1+w_2| ,$$

which is equivalent to

(31) $$\left(\frac{|w_1+w_2|}{2}-|Z|\right)^2 > \frac{|w_1-w_2|^2}{4} .$$

Since

$$|w_1+w_2| = \left|\frac{2k+2}{2k+1}+\frac{3k+2}{(2k+1)(k+1)}C_1+\frac{k+2}{2(k+1)^2}C_1^2\right|$$

$$\geqq \frac{2k+2}{2k+1}-\frac{3k+2}{(2k+1)(k+1)}-\frac{k+2}{2(k+1)^2}$$

$$= \frac{2(k^2-1)+k}{2(k+1)^2} \geqq \frac{k}{2(k+1)^4} ,$$

(31) is equivalent to

(32) $$|w_1+w_2|-|w_1-w_2| > 2|Z| .$$

If we put $C_1 = \zeta$, then (32) can be written such that

(33) $$\left|2(k+1)^2+(3k+2)\zeta+\frac{(k+2)(2k+1)}{2(k+1)}\zeta^2\right| -k\left|2(k+1)+\zeta\right|$$

$$> \frac{k(2k+1)}{2(k+1)}(1-|\zeta|^2) \qquad (|\zeta| \leqq 1) .$$

We write

$$\zeta = -2(k+1)+\zeta_1 = -2(k+1)+r\,e^{i\varphi}\;;$$

it is geometrically clear that $2k+1 \leqq r \leqq 2k+3$ and

$$(34)\qquad -\varphi_0(r) \leqq \varphi \leqq \varphi_0(r)\;,$$

if r is fixed, where $\varphi_0(r)$ can be determined from the equation $|-2(k+1)+r\,e^{i\varphi}| = 1$ such that $0 < \varphi_0(r) < \frac{\pi}{2}$. The point $-2(k+1)+r\,e^{i\varphi_0(r)}$ lies on the circumference $|\zeta| = 1$. We obtain

$$(35)\qquad \cos\varphi_0(r) = \frac{r^2+(4(k+1)^2-1)}{4(k+1)r}\;.$$

Now (33) can be written in the form:

$$(33)'\qquad |\alpha-\beta\zeta_1+\gamma\zeta_1^2| > \delta+Q\;,$$

where

$$Q = kr-\frac{k(2k+1)}{2(k+1)}\left|\zeta_1-2(k+1)\right|^2$$
$$= -2k(k+1)(2k+1)+k\big(1+2(2k+1)\cos\varphi\big)r-\frac{k(2k+1)}{2(k+1)}r^2$$

and

$$\alpha = 2(k+1)^2(2k+1)\;,\qquad \beta = 4k^2+7k+2\;,$$
$$\gamma = \frac{(k+2)(2k+1)}{2(k+1)}\;,\qquad \delta = \frac{k(2k+1)}{2(k+1)}\;.$$

Considering (34) and (35), we have

$$(36)\qquad Q \geqq -2k(k+1)(2k+1)+k\big(1+2(2k+1)\cos\varphi_0(r)\big)r-\frac{k(2k+1)}{2(k+1)}r^2$$
$$= -\delta+kr\;.$$

From (33)'

$$(37)\qquad \left\{(\alpha+\gamma r^2)\cos\varphi-\beta r\right\}^2+(\alpha-\gamma r^2)^2(1-\cos^2\varphi)-(\delta+Q)^2 > 0\;.$$

Put in (37)

$$\cos\varphi = \frac{Q+2k(k+1)(2k+1)-kr+\delta r^2}{2k(2k+1)r},$$

and consider the left hand side $f(r, Q)$ of (37) as a function of r and Q, where r varies in the interval $2k+1 \leqq r \leqq 2k+3$ and Q in a certain interval $Q_1(r) \leqq Q \leqq Q_2(r)$, $Q_1(r)$ being equal to $-\delta+kr$. We show that $f(r, Q)$ is an increasing function of Q for $Q_1(r) \leqq Q$, when r is fixed in the interval $2k+1 \leqq r \leqq 2k+3$.

$$\frac{\partial f(r, Q)}{\partial Q} = \frac{\alpha+\gamma r^2}{k(2k+1)r}\{(\alpha+\gamma r^2)\cos\varphi-\beta r\} - \frac{(\alpha-\gamma r^2)^2}{k(2k+1)r}\cos\varphi - 2(\delta+Q)$$

$$= \frac{4\alpha\gamma}{k(2k+1)} r\cos\varphi - \frac{\beta(\alpha+\gamma r^2)}{k(2k+1)} - 2(\delta+Q)$$

$$= \frac{4\alpha\gamma}{k(2k+1)} \cdot \frac{Q+2k(k+1)(2k+1)-kr+\delta r^2}{2k(2k+1)} - \frac{\beta(\alpha+\gamma r^2)}{k(2k+1)}$$

$$-2(\delta+Q),$$

$$\frac{\partial^2 f(r, Q)}{\partial Q^2} = \frac{2\alpha\gamma}{k^2(2k+1)^2} - 2 = \frac{6k+4}{k^2} > 0.$$

Hence $\frac{\partial f(r, Q)}{\partial Q}$ is positive for $Q_1(r) \leqq Q$, if it is positive for $Q = Q_1(r)$. However, when $Q = Q_1(r)$, by an easy calculation, we get

$$\frac{\partial f(r, Q)}{\partial Q} = \frac{1}{k}(6k^3+14k^2+9k+2) - 2kr - \frac{k+2}{2(k+1)} r^2, \tag{38}$$

which is positive for $2k+1 \leqq r \leqq 2k+3$, as it is true(1) for $r = 2k+3$.

Therefore it is shown that $f(r, Q)$ is minimum for $Q = Q_1(r)$, that is, $\cos\varphi = \cos\varphi_0(r)$ or $|\zeta| = 1$, $\zeta = e^{i\theta}$. Then our assertion can be enunciated in the form:

$$\left|2(k+1)^2 e^{-i\theta} + (3k+2) + \frac{(k+2)(2k+1)}{2(k+1)} e^{i\theta}\right| > k\left|2(k+1) + e^{i\theta}\right| \tag{33$''$}$$

---

(1) $\frac{\partial f(r, Q)}{\partial Q} = \frac{(k+2)^2}{2k(k+1)} > 0$ when $Q = Q_1(r)$ and $r = 2k+3$.

or

$$|ae^{-i\theta}+\beta+\gamma e^{i\theta}|^2-k^2|\delta+e^{i\theta}|^2>0\ ,$$

putting

$$\alpha=2(k+1)^2,\ \beta=3k+2,\ \gamma=\frac{(k+2)(2k+1)}{2(k+1)},\ \delta=2(k+1),$$

or

$$(33)''' \qquad \left\{\beta^2-k^2+(\alpha-\gamma)^2-k^2\delta^2\right\}+2\left\{\beta(\alpha+\gamma)-k^2\delta\right\}\cos\theta +4\alpha\gamma\cos^2\theta>0\ .$$

For the proof of (33)''', we have only to obtain[1]

$$(39) \qquad 4\alpha\gamma\left\{\beta^2-k^2+(\alpha-\gamma)^2-k^2\delta^2\right\}-\left\{\beta(\alpha+\gamma)-k^2\delta\right\}^2>0\ ,$$

since $\alpha\gamma>0$. Denoting by D the left hand side of (39), we easily have

$$D=\frac{(k+2)(2k+1)^2(16k^5+80k^4+147k^3+126k^2+52k+8)}{4(k+1)^2}>0\ .$$

(Q E.D.)

Thus our therem is completely proved.

Applying ALEXANDER's theorem[2] and theorem 8, we obtain at once

Theorem 9. *Let*

$$\varphi(z)=z+c_1z^{k+1}+c_2z^{2k+1}+\dots+c_nz^{nk+1}+\dots$$

be a convex function of the class k. Then every section

$$z+c_1z^{k+1}+c_2z^{2k+1}+\dots+c_nz^{nk+1}$$

is convex[3] *for*

$$|z|<\sqrt[k]{\frac{k}{2(k+1)}}\ ,$$

---

(1) Clearly $a+2bx+cx^2$ is always positive, provided that $c>0$ and $ca-b^2>0$.
(2) J. ALEXANDER: loc. cit.
(3) If $f(z)$ is regular for $|z|<\rho$ and maps $|z|<\rho$ on a convex domain, then we say that $f(z)$ is convex for $|z|<\rho$.

where $\sqrt[k]{\frac{k}{2(k+1)}}$ *cannot be replaced by any greater number. The extremal case can be given by the function* $\varphi_0(z)$ *of* (6).

Remark. When $k=1$ and $k=2$, this theorem reduces to SZEGÖ-TAKAHASHI's.[1]

Recently Mr. A. KOBORI[2] has given a complement to SZEGÖ's theorem: Let $f(z)=z+c_1z^2+c_2z^3+\dots+c_nz^{n+1}+\dots$ be regular and starshaped for $|z|<1$. Then every function $g(z)=z+b_1z^2+b_2z^3+\dots+b_nz^{n+1}+\dots$, with $|b_n|\leqq|c_n|$ $(n=1,2,3,\dots)$, is starshaped for $|z|<R=0,1646\dots$, where R is the root between 0 and 1 of the equation $2(1-r)^3=1+r$.

KOBORI's result can be extended in the following form:

Theorem 10. *Let*

$$\Phi(z)=z+c_1z^{k+1}+c_2z^{2k+1}+\dots+c_nz^{nk+1}+\dots$$

be a starshaped function of the class k. Then every function

$$G(z)=z+b_1z^{k+1}+b_2z^{2k+1}+\dots+b_nz^{nk+1}+\dots,$$

such that

$$|b_n|\leqq|c_n| \qquad (n=1,2,3,\dots),$$

is starshaped for $|z|<R$, *where R is the root between 0 and 1 of the equation*

$$2^k(1-r^k)^{k+2}=(1+r^k)^k.$$

Proof. It is well known that if $\psi(z)=z+\sum_1^\infty a_nz^{n+1}$ is regular for $|z|<1$ and if $\sum_1^\infty(n+1)|a_n|r^n<1$, then $\psi(z)$ is starshaped for $|z|<r<1$. We have, applying theorem 1,

(1) G. SZEGÖ: loc. cit., S. TAKAHASHI: loc. cit.

(2) A. KOBORI: loc. cit. And see also L. BIERBERBACH: loc. cit.

$$\sum_1^\infty (nk+1)\,|\,b_n\,|\,r^{nk} \leqq \sum_1^\infty (nk+1)\,|\,c_n\,|\,r^{nk}$$

$$\leqq \sum_1^\infty (nk+1)\frac{1}{n!}\,\frac{2}{k}\left(\frac{2}{k}+1\right)\ldots\left(\frac{2}{k}+(n-1)\right)r^{nk}$$

$$= \Phi_0'(r)-1 = \frac{1+r^k}{(1-r^k)^{\frac{2}{k}+1}}-1\,.$$

Hence, if we denote by R the root between 0 and 1 of the equation $\Phi_0'(r) = 2$ that is: $2^k(1-r^k)^{k+2} = (1+r^k)^k$, then every function $G(z)$ is starshaped for $|\,z\,| < R$.

Remark. The limit R can be attained by the function

$$G_0(z) = 2z-\Phi_0(z) = 2z-\frac{z}{(1-z^k)^{\frac{2}{k}}}$$

for $G_0'(z) = 2-\Phi_0'(z)$ has zero-points on the circumference $|\,z\,| = R$.

As an immediate result of the above,

Theorem 11.(1) *Let*

$$\varphi(z) = z+c_1 z^{k+1}+c_2 z^{2k+1}+\ldots\ldots+c_n z^{nk+1}+\ldots\ldots,$$

be a convex function of the class k. Then every function

$$g(z) = z+b_1 z^{k+1}+b_2 z^{2k+1}+\ldots\ldots+b_n z^{nk+1}+\ldots\ldots,$$

such that

$$|\,b_n\,| \leqq |\,c_n\,| \qquad (n = 1, 2, 3, \ldots\ldots),$$

is convex for $|\,z\,| < R$, where R denotes the same quantity in theorem 10.

Remark. It is evident that the limit R can be attained by the function

$$g_0(z) = \int_0^z \frac{G_0(z)}{z}\,dz = \int_0^z \left(2-\frac{\Phi_0(z)}{z}\right)dz = 2z-\varphi_0(z)\,.$$

(1) In the case $k = 1$, this theorem reduces also to KOBORI's. See, loc. cit.

§ V. ON THE UNIVALENCY OF A FUNCTION WITH A NON-VANISHING DERIVATIVE.

Supposing that $f(z)$ is a function which is regular in a certain domain D and whose derivative $f'(z)$ never vanishes there, we cannot necessarily assert $f(z)$ to be schlicht in D. For example, $f(z) = e^z$ has a non-vanishing derivative but is not schlicht in a circle of radius greater than π. Here we shall give some results on the univalency (*Schlichtheit*) of a function $f(z)$ with a non-vanishing derivative.

Theorem 12.(1) *Let $f(z)$ be regular in a convex domain D. Then $f(z)$ is schlicht, if the value-set of $f'(z)$ in D lies in a half-plane Ω not containing the origin in its interior.*

Proof. Let z_1 and z_2 $(z_1 \neq z_2)$ be two arbitrary points in D. Since D is a convex domain, the segment $\overline{z_1 z_2}$, joining z_1 and z_2, lies in D. Denote by M the set of values taken by $f'(z)$ on the segment $\overline{z_1 z_2}$, then it is clear that M is a bounded closed set lying in Ω. Hence we can describe a circle C which contains M in its interior and lies in Ω. If we denote by a the centre of C, then we have

$$|f'(z)-a| < |a| ,$$

for every z on the segment $\overline{z_1 z_2}$.

Now

$$f(z_2)-f(z_1) = \int_{z_1}^{z_2} f'(z)\,dz = \int_{z_1}^{z_2} a\,dz - \int_{z_1}^{z_2} (a-f')\,dz$$

$$= a(z_2-z_1) - \int_{z_1}^{z_2} (a-f')\,dz .$$

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(1) When I read this paper at the annual meeting of the Physico-Math. Soc. of Japan, held in April 1934, Prof. KAKEYA kindly remarked to me that this theorem can be easily proved by a geometrical consideration.

Taking the segment z_1z_2 as the path of integration,

$$\left|\int_{z_1}^{z_2}(a-f')\,dz\right| \leqq \int_0^{|z_1-z_2|}|a-f'|\,ds < |a|\int_0^{|z_1-z_2|}ds = |a|\,|z_1-z_2|\,.$$

Hence

$$\left|f(z_2)-f(z_1)\right| \geqq |a|\,|z_1-z_2| - \left|\int_{z_1}^{z_2}(a-f')\,dz\right|$$

$$> |a|\,|z_1-z_2| - |a|\,|z_1-z_2| = 0\,. \qquad \text{(Q.E.D.)}$$

Remark. For this proof we owe much to Prof. K. KUNUGUI.

As an immediate result

Theorem 13. *If* $f(z)=z+\ldots.$ *is regular and*

$$R(f'(z))>0 \quad \text{or} \quad |f'(z)-1|<1$$

for $|z|<R$, *then* $f(z)$ *is schlicht for* $|z|<R$.

Next an application of theorem 13 will be enunciated:

Theorem 14.[(1)] *If* $f(z)=c_0z+\frac{c_1}{2}z^2+\ldots.$ (c_0, c_1 *given*, $c_0\neq 0$) *is regular for* $|z|<1$ *and if* $|f'(z)|<1$ *for* $|z|<1$, *then* $f(z)$ *is schlicht for*

$$(40) \qquad |z|<R=\frac{1}{2}\left[-|a_0|(1-|c_0|)+\sqrt{|a_0|^2(1-|c_0|)^2+4|c_0|}\right],$$

where $a_0=\dfrac{c_1}{1-|c_0|^2}$. *This limit can be attained by the function*

$$(41) \qquad f_0(z)=\int_0^z \frac{c_0+(c_0\bar{a}_0e^{i\theta}+a_0)z+e^{i\theta}z^2}{1+(\bar{a}_0e^{i\theta}+\bar{c}_0a_0)z+\bar{c}_0e^{i\theta}z^2}\,dz\,,$$

where $\theta\equiv 2\arg c_1-\arg c_0+\pi \pmod{2\pi}$.

(1) This is an extension of a theorem which I have already proved. See K. NOSHIRO: Journ. Fac. Sci. Hokkaido Imperial Univ., ser. I, vol. 1, 1932, p. 157–161, esp. p. 160. And also K. NOSHIRO: loc. cit., vol. 2, 1934, p. 89–101, esp. p. 98; Here I considered the case when $f(z)=c_0z+c_1z^2+\ldots.$ (c_0, c_1 given, $c_0\neq 0$) is regular and $|f(z)|<1$ for $|z|<1$.

Proof. It is clear that

$$g(z) \equiv \frac{f'(z)-c_0}{1-\bar{c}_0 f'(z)} = \frac{c_1}{1-|c_0|^2} z + \dots . \tag{42}$$

is regular and $|g(z)| < 1$ for $|z| < 1$. Hence

$$|f'(z)-c_0| = \left| \frac{(1-|c_0|^2)\, g(z)}{1+\bar{c}_0\, g(z)} \right| \leq \frac{(1-|c_0|^2)\, |g(z)|}{1-|c_0|\, |g(z)|}, \tag{43}$$

Hence, by theorem 13, $f(z)$ is schlicht for $|z| < R$, provided that

$$|g(z)| < |c_0| \qquad \text{for} \qquad |z| < R. \tag{44}$$

Using a known inequality $|g(z)| \leq r\dfrac{|a_0|+r}{1+|a_0|r}$ for $|z| \leq r$, it is seen that the inequality (44) holds good, because R is the root between 0 and 1 of the equation $r\dfrac{|a_0|+r}{1+|a_0|r} = |c_0|$. Thus our theorem is proved, considering the function $f_0(z)$ whose derivative vanishes at a point $z_0 = Re^{i\lambda_0}$, where $\lambda_0 \equiv \arg c_0 - \arg c_1 + \pi \pmod{2\pi}$.

§ VI. A THEOREM ON A SCHLICHT MEROMORPHIC FUNCTION IN THE UNIT CIRCLE.

We will here enunciate a theorem analogous to FEJÉR's[1] on a schlicht bounded function.

Theorem 15. *Suppose that*

$$f(z) = \frac{1}{z} + c_2 z + c_3 z^2 + \dots + c_n z^{n-1} + \dots$$

is meromorphic and schlicht for $|z| < 1$. *Then*

$$|1 + c_2 + c_3 + \dots + c_n| \leq 2 + \sqrt{\frac{241}{432}} = 2{,}7469\dots \qquad (n = 2, 3, \dots).$$

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1) Mr. FEJÉR has proved that if $f(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n + \dots$ is regular-schlicht and bounded in the unit circle: $|f(z)| < 1$, then

$$|c_0 + c_1 + \dots + c_n| \leq 1 + \frac{1}{\sqrt{2}} \qquad (n = 0, 1, 2, \dots).$$

Cf. Acta Mathematica, Bd. 49, 1926, p. 183–190; Acta, Szeged, Bd. 4, 1928, p. 14–24.

Proof. We use two known results:

(45) $$\sum_{2}^{\infty}(k-1)\,|c_k|^2 \leqq 1 \qquad \text{(BIEBERBACH)}$$

$$|f(z)| \leqq |z| + \frac{1}{|z|} \qquad \text{(LÖWNER)}$$

Since

$$F(z) = zf(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots \qquad (c_0 = 1,\ c_1 = 0)$$

is regular and bounded in the unit circle: $|F(z)| \leqq |z|^2 + 1 < 2$, we have by LANDAU's theorem(1)

(46) $$\left|\frac{(n+1)c_0 + nc_1 + \ldots\ldots + 2c_{n-1} + c_n}{n+1}\right| \leqq 2\,.$$

On the other hand, using BIEBERBACH's result,

(47) $$|c_1 + 2c_2 + \ldots + nc_n| \leqq 2|c_2| + \ldots + n|c_n|$$

$$= \frac{2}{\sqrt{1}}\cdot\sqrt{1}\,|c_2| + \ldots + \frac{n}{\sqrt{n-1}}\cdot\sqrt{n-1}\,|c_n|$$

$$\leqq \sqrt{\sum_{2}^{n}(k-1)\,|c_k|^2}\sqrt{\sum_{2}^{n}\frac{k^2}{k-1}} \leqq \sqrt{\sum_{2}^{n}\frac{k^2}{k-1}}\,.$$

By (46) and (47),

$$|c_0 + c_1 + c_2 + \ldots + c_n| = \left|\frac{(n+1)c_0 + nc_1 + \ldots + c_n}{n+1} + \frac{c_1 + 2c_2 + \ldots + nc_n}{n+1}\right|$$

$$\leqq \left|\frac{(n+1)c_0 + nc_1 + \ldots + c_n}{n+1}\right|$$

$$+ \frac{1}{n+1}\,|c_1 + 2c_2 + \ldots + nc_n|$$

$$\leqq 2 + \sqrt{\frac{1}{(n+1)^2}\sum_{1}^{n-1}\frac{(k+1)^2}{k}}\,.$$

(1) E. LANDAU: Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, 2. Aufl., p. 22.

Here remark that

$$(48)\qquad \phi(n)=\frac{1}{(n+1)^2}\sum_1^{n-1}\frac{(k+1)^2}{k}=\frac{(n-1)(n+4)}{2(n+1)^2}+\frac{1}{(n+1)^2}\sum_1^{n-1}\frac{1}{k}$$

decreases as n increases for $n\geqq 5$. To prove that $\phi(n+1)<\phi(n)$ for $n\geqq 5$, it suffices to show

$$(49)\qquad \sum_1^{n-1}\frac{1}{k}>\frac{2+20n+11n^2-n^3}{2n(2n+3)}\qquad \text{for}\qquad n\geqq 5\,.$$

But this inequality (49) holds good, because, for $n\geqq 5$,

$$\sum_1^{n-1}\frac{1}{k}\geqq\frac{25}{12}\qquad\text{and}\qquad\frac{2+20n+11n^2-n^3}{2n(2n+3)}<2\,.$$

Consequently it is seen that

$$\max_{n\geq 2}\phi(n)=\max_{5\geq n\geq 2}\phi(n)\,,$$

whence it follows that

$$\max_{n\geq 2}\phi(n)=\frac{241}{432}\,.$$

Thus we obtain

$$|1+c_2+c_3+\ldots\ldots+c_n|\leqq 2+\sqrt{\frac{241}{432}}=2{,}7469\ldots\ldots$$

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After I completed this paper, Mr. K. JOH kindly wrote to me to say that theorems 1 and 2 had been obtained by Mr. G. GOLUSIN in Recueil Math. Moscou **36**, p. 152–172.