ON TRANSCENDENTAL POINTS IN PROPER SPACES OF DISCRETE SEMI-ORDERED LINEAR SPACES

By

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A cardinal number $\mathfrak{f}$ is said to be *singular*, if

1) $\mathfrak{f} >$ the countable density $\aleph_0$,
2) $\mathfrak{f} > \mathfrak{c}$ implies $\mathfrak{f} > 2^{\mathfrak{c}}$,
3) for any system of cardinal numbers $\mathfrak{c}_\lambda < \mathfrak{f}\ (\lambda \in \Lambda)$ with a density $< \mathfrak{f}$ we have $\sum_{\lambda \in \Lambda} \mathfrak{c}_\lambda < \mathfrak{f}$.

The existence of singular cardinal numbers is not known yet. It will be extraodinarily great, if exists.

A cardinal number $\mathfrak{c}$ is said to be *regular*, if there is no singular cardinal number $\leqq \mathfrak{c}$. The countable density $\aleph_0$ is naturally regular. If a cardinal nnmber $\mathfrak{c}$ is regular, then $2^{\mathfrak{c}}$ also is regular. For a system of regular cardinal numbers $\mathfrak{c}_\lambda\ (\lambda \in \Lambda)$, if the density of Λ is regular, then $\sum_{\lambda \in \Lambda} \mathfrak{c}_\lambda$ also is regular.

Let S be a set and R the totality of real functions on S. R is then obviously a discrete semi-ordered linear space.[1] The purpose of this paper is to prove: *If the density of S is regular, then for any positive linear functional Φ on R, we can find a finite number of elements $s_\nu \in S$ and positive numbers α_ν $(\nu = 1, 2, \cdots, \kappa)$ such that*

$$\Phi(\varphi) = \sum_{\nu=1}^{\kappa} \alpha_\nu \varphi(s_\nu) \qquad \text{for every} \quad \varphi \in R.$$

Let R be now an arbitrary linear space. We have defined a *strongest convex linear topology*[2] on R, of which the totality of convex vicinities in R is a basis. By virtue of the fact stated just above, we see easily that if the density of $\hat{R}$ is regular, then R is regular[3] (reflexive) by the strongest convex linear topology.

(1) I. HALPERIN and H. NAKANO: Discrete semi-ordered linear spaces, Canadian Jour. Math., 3 (1951), 293-298.

(2) H. NAKANO: Topology and topological spaces, Tokyo Math. Book Series III, (1951), §70.

(3) c. f. 2).

§ 1. Transcendental ideals of sets.

Let R be a set. A collection $\mathfrak{p}$ of subsets from R is said to be an *ideal*, if

1) $O \bar{\in} \mathfrak{p}$,
2) $X \supset Y \in \mathfrak{p}$ implies $X \in \mathfrak{p}$,
3) $X, Y \in \mathfrak{p}$ implies $XY \in \mathfrak{p}$.

An ideal $\mathfrak{p}$ is said to be *maximal,* if there is no other ideal including $\mathfrak{p}$. For a maximal ideal $\mathfrak{p}$, we see easily that for any set $X \bar{\in} \mathfrak{p}$ we can find $Y \in \mathfrak{p}$ such that $XY = O$.

Theorem 1.1. *Let $\mathfrak{p}$ be a maximal ideal, and Λ a set with a density $\mathfrak{c}$. If $X_\lambda \in \mathfrak{p}\ (\lambda \in \Lambda)$ implies $\prod_{\lambda\in\Lambda} X_\lambda \in \mathfrak{p}$, then for a set Γ with the density $2^{\mathfrak{c}}$ we also have that $X_\gamma \in \mathfrak{p}\ (\gamma \in \Gamma)$ implies $\prod_{\gamma\in\Gamma} X_\gamma \in \mathfrak{p}$.*

Proof. The collection of systems $(\varepsilon_\lambda)_{\lambda\in\Lambda}$ for $\varepsilon_\lambda = 0, 1$ has by definition the density $2^{\mathfrak{c}}$. Let $A_{(\varepsilon_\lambda)_{\lambda\in\Lambda}}$ for all $(\varepsilon_\lambda)_{\lambda\in\Lambda}$ be a partition of R, that is,

$$R = \sum_{(\varepsilon_\lambda)_{\lambda\in\Lambda}} A_{(\varepsilon_\lambda)_{\lambda\in\Lambda}},$$

$$A_{(\varepsilon_\lambda)_{\lambda\in\Lambda}} A_{(\delta_\lambda)_{\lambda\in\Lambda}} = O \qquad \text{for } (\varepsilon_\lambda)_{\lambda\in\Lambda} \neq (\delta_\lambda)_{\lambda\in\Lambda}.$$

For every finite number of elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots, \kappa)$, putting

$$Y_{\delta_{\lambda_1}, \delta_{\lambda_2}, \cdots, \delta_{\lambda_\kappa}} = \sum_{\varepsilon_{\lambda_\nu} = \delta_{\lambda_\nu} (\nu = 1, 2, \cdots, \kappa)} A_{(\varepsilon_\lambda)_{\lambda\in\Lambda}}$$

for $\delta_{\lambda_\nu} = 0, 1$ $(\nu = 1, 2, \cdots, \kappa)$, we have obviously

$$R = \sum_{\varepsilon_{\lambda_\nu} = 0, 1} Y_{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \cdots, \varepsilon_{\lambda_\kappa}} \qquad \text{for every } \lambda_1, \lambda_2, \cdots, \lambda_\kappa \in \Lambda,$$

and

$$Y_{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \cdots, \varepsilon_{\lambda_\kappa}} Y_{\varepsilon'_{\lambda_1}, \varepsilon'_{\lambda_2}, \cdots, \varepsilon'_{\lambda_\kappa}} = O,$$

if $\varepsilon_{\lambda_\nu} \neq \varepsilon'_{\lambda_\nu}$ for some ν. Thus for each finite number of elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots, \kappa)$ we can find uniquely $\delta_{\lambda_\nu} = 0, 1$ $(\nu = 1, 2, \cdots, \kappa)$ such that

$$Y_{\delta_{\lambda_1}, \delta_{\lambda_2}, \cdots, \delta_{\lambda_\kappa}} \in \mathfrak{p}.$$

As $Y_{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \cdots, \varepsilon_{\lambda_\kappa}} \supset Y_{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, \cdots, \varepsilon_{\lambda_\kappa}, \varepsilon_{\lambda_{\kappa+1}}}$, we see easily further that there exists uniquely $(\delta_\lambda)_{\lambda\in\Lambda}$ such that

$$Y_{\delta_{\lambda_1}, \delta_{\lambda_2}, \cdots, \delta_{\lambda_\kappa}} \in \mathfrak{p} \qquad \text{for every } \lambda_1, \lambda_2, \cdots, \lambda_\kappa \in \Lambda,$$

Then, as the totality of systems $\lambda_1, \lambda_2, \cdots, \lambda_\kappa \in \Lambda$ also has the density $\mathfrak{c}$, we have by assumption

$$A_{(\delta_\lambda)_{\lambda\in\Lambda}} = \prod_{\lambda_\nu\in\Lambda} Y_{\delta_{\lambda_1},\delta_{\lambda_2},\dots,\delta_{\lambda_\kappa}} \in \mathfrak{p}\,.$$

Therefore, for a set Γ with the density 2^c, if $\sum_{\gamma\in\Gamma} A_\gamma = R$, $A_\gamma A_{\gamma'} = O$ for $\gamma \neq \gamma'$, then there exists uniquely $\gamma\in\Gamma$ such that $A_\gamma\in\mathfrak{p}$. If $\sum_{\gamma\in\Gamma} B_\gamma = R$, then we can find by the transfinite induction subsets $A_\gamma \subset B_\gamma$ $(\gamma\in\Gamma)$ such that

$$\sum_{\gamma\in\Gamma} A_\gamma = R\,, \quad A_\gamma A_{\gamma'} = O \qquad \text{for} \quad \gamma \neq \gamma'\,,$$

and hence there exists $\gamma\in\Gamma$ such that $B_\gamma\in\mathfrak{p}$. If $X_\gamma\in\mathfrak{p}\,(\gamma\in\Gamma)$, then we have obviously

$$R = \sum_{\gamma\in\Gamma}(R - X_\gamma) + \prod_{\gamma\in\Gamma} X_\gamma\,, \quad R - X_\gamma \bar{\in}\, \mathfrak{p} \qquad \text{for every} \quad \gamma\in\Gamma\,,$$

and consequently $\prod_{\gamma\in\Gamma} X_\gamma\in\mathfrak{p}$, as proved just above.

Theorem 1.2. *Let* $\mathfrak{p}$ *be a maximal ideal and* Γ *a set for which* $X_\gamma\in\mathfrak{p}$ $(\gamma\in\Gamma)$ *implies* $\prod_{\gamma\in\Gamma} X_\gamma\in\mathfrak{p}$. *For a system of sets* Λ_γ $(\gamma\in\Gamma)$ *if* $X_\lambda\in\mathfrak{p}$ $(\lambda\in\Lambda_\gamma)$ *implies* $\prod_{\lambda\in\Lambda_\gamma} X_\lambda\in\mathfrak{p}$ *for every* $\gamma\in\Gamma$, *then* $X_\lambda\in\mathfrak{p}$ $(\lambda\in\sum_{\gamma\in\Gamma}\Lambda_\gamma)$ *implies* $\prod_{\lambda\in\sum_{\gamma\in\Gamma}\Lambda_\gamma} X_\lambda\in\mathfrak{p}$.

Proof. We have obviously by assumption that $X_\lambda\in\mathfrak{p}(\lambda\in\sum_{\gamma\in\Gamma}\Lambda_\gamma)$ implies $\prod_{\lambda\in\Lambda_\gamma} X_\lambda\in\mathfrak{p}$ for every $\gamma\in\Gamma$, and hence

$$\prod_{\lambda\in\sum_{\gamma\in\Gamma}\Lambda_\gamma} X_\lambda = \prod_{\gamma\in\Gamma}\Big(\prod_{\lambda\in\Lambda_\gamma} X_\lambda\Big)\in\mathfrak{p}\,.$$

A maximal ideal $\mathfrak{p}$ is said to be *transcendental*, if $X_\nu\in\mathfrak{p}\,(\nu=1,2,\cdots)$ implies $\prod_{\nu=1}^{\infty} X_\nu\in\mathfrak{p}$. With this definition, we conclude immediately by Theorems 1.1 and 1.2.

Theorem 1.3. *If the density of* R *is regular, then for every transcendental maximal ideal* $\mathfrak{p}$, *we have* $\prod_{X\in\mathfrak{p}} X\in\mathfrak{p}$, *and hence* $\prod_{X\in\mathfrak{p}} X$ *is composed only of a single element.*

§ 2. Transcendental points of discrete semi-ordered linear spaces

Let R be a discrete semi-ordered linear space[4] and $a_\lambda\in R\,(\lambda\in\Lambda)$ a basis of R, i. e., $a_\lambda \frown a_\rho = 0$ for $\lambda\neq\rho$ and for each positive element $x\in R$ we can find uniquely a system of real numbers $\xi_\lambda \geqq 0$ $(\lambda\in\Lambda)$ such that

(4) c. f. 1).

$$x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda .$$

For a positive element $x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda$, putting

$$\Lambda_x = \{\lambda : \xi_\lambda \neq 0\} ,$$

we see easily:

$$[a]x = \bigcup_{\lambda \in \Lambda_a} \xi_\lambda a_\lambda \qquad \text{for every positive element } x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda ;$$

$$\Lambda_a \subset \Lambda_b \qquad \text{if and only if} \quad [a] \leqq [b] ;$$

$$\Lambda_{a \smile b} = \Lambda_a \dotplus \Lambda_b , \quad \Lambda_{a \frown b} = \Lambda_a \Lambda_b .$$

Thus every projector $[a]$ may be represented by the set Λ_a. Therefore every point of the proper space of R may be considered as a maximal ideal of subsets from Λ. Furthermore, for a maximal ideal $\mathfrak{p}$ of subsets from Λ, if $\mathfrak{p} \ni \Lambda_a$ for a positive element $a \in R$, then $\mathfrak{p}$ is a point of the proper space of R and $\mathfrak{p} \in U_{[a]}$.

A point $\mathfrak{p}$ of the proper space of R is said to be *transcendental*[5], if $\mathfrak{p} \in U_{[a_\nu]}(\nu=1, 2, \cdots)$ implies $\mathfrak{p} \in U_{[b]} \subset \prod_{\nu=1}^{\infty} U_{[a_\nu]}$ for some $b \in R$. With this definition, it is evident that a point $\mathfrak{p}$ is transcendental if and only if $\mathfrak{p}$ is a transcendental maximal ideal.

For a positive element $a \in R$, the density of Λ_a is called the *dimension* of a.

Theorem 2.1. *If the dimension of every positive element of R is regular, then the proper space of R has no transcendental point up to isolated points.*

Proof. For a transcendental point $\mathfrak{p}$ of the proper space of R, we can find obviously a positive element $a \in R$ such that $\mathfrak{p} \in U_{[a]}$. Then we can consider $\mathfrak{p}$ as a transcendental maximal ideal of subsets from Λ_a. Therefore we can find by Theorem 1.3 $\lambda \in \Lambda$ such that $U_{[a_\lambda]}$ is composed only of the single point $\mathfrak{p}$, and hence $\mathfrak{p}$ is an isolated point.

From this Theorem 2.1 we conclude immediately

Theorem 2.2. *If the density of R is regular, then the proper space of R has no transcendental point up to isolated points.*

§ 3. Universally complete discrete semi-ordered linear spaces

Let a discrete semi-ordered linear space R be *universally complete*[6],

(5) H. Nakano: Ueber ein lineares Funktional auf dem teilweise geordneten Modul, Proc. Tokyo Acad., 18 (1942), 548-552.

(6) H. Nakano: Modern spectral theory, Tokyo Math. B. S., II (1950), § 34.

i.e., for every orthogonal system of positive elements $x_\gamma \in R\ (\gamma \in \Gamma)$ there exists $\bigcup_{\gamma \in \Gamma} x_\gamma$. R is then obviously *totally unbounded*[7], i.e., for an orthogonal sequence of positive elements $x_\nu \in R\ (\nu = 1, 2, \cdots)$, if $\bigcup_{\nu=1}^{\infty} x_\nu$ exists, then we can find a sequence of positive numbers $\alpha_\nu \uparrow_{\nu=1}^{\infty} + \infty$ for which $\bigcup_{\nu=1}^{\infty} \alpha_\nu x_\nu$ exists. Therefore every positive linear functional on R is continuous.[8]

Theorem 3.1. *If a discrete semi-ordered linear space R is universally complete and the density of R is regular, then for every positive linear functional Φ on R we can find a finite number of discrete positive elements $a_\nu \in R\ (\nu = 1, 2, \cdots, \kappa)$ such that*

$$\Phi(x) = \sum_{\nu=1} \Phi([a_\nu] x) \qquad \textit{for every} \quad x \in R.$$

Proof. Let $\mathfrak{C}_\Phi$ be the characteristic set[9] of Φ. If $\mathfrak{C}_\Phi$ contains infinite points, then we can find a sequence of positive elements $a_\nu \in R$ $(\nu = 1, 2, \cdots)$ such that $[a_\nu][a_\mu] = 0$ for $\nu \neq \mu$ and $U_{[a_\nu]}\, \mathfrak{C}_\Phi \neq 0$ for every $\nu = 1, 2, \cdots$. Then we have $\Phi(a_\nu) > 0$ for every $\nu = 1, 2, \cdots$, and hence, putting

$$a = \bigcup_{\nu=1}^{\infty} \frac{\nu}{\Phi(a_\nu)} a_\nu ,$$

we have

$$\Phi(a) \geqq \Phi\left(\frac{\nu}{\Phi(a_\nu)} a_\nu\right) = \nu \qquad \text{for every} \quad \nu = 1, 2, \cdots,$$

contradicting $\Phi(a) < +\infty$. Thus $\mathfrak{C}_\Phi$ is composed only of a finite number of points. Furthermore every point of $\mathfrak{C}_\Phi$ is transcendental. Because, if a point $\mathfrak{p} \in \mathfrak{C}_\Phi$ is not transcendental, then we can find by definition a sequence of projectors $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$, such that $U_{[p_\nu]} \ni \mathfrak{p}$, but $U_{[p_\nu]}$ does not contains any other point of $\mathfrak{C}_\Phi$ for every $\nu = 1, 2, \cdots$. As $\mathfrak{p} \in \mathfrak{C}_\Phi$, we can find a positive lement $a \in R$ such that $\Phi([p] a) > 0$ for $U_{[p]} \in \mathfrak{p}$. Then we have

$$\Phi([p_\nu] a) = \Phi([p_1] a) - \Phi(([p_1] - [p_\nu]) a) = \Phi([p_1] a),$$

because $U_{[p_1] - [p_\nu]} \mathfrak{C}_\Phi = O$. As Φ is continuous, we have hence

$$\Phi([p_1] a) = \lim_{\nu \to \infty} \Phi([p_\nu] a) = 0,$$

contradicting $\Phi([p_1] a) > 0$. Therefore $\mathfrak{C}_\Phi$ is composed only of a finite number of transcendental points. As the density of R is regular by

(7) H. Nakano: Modulared semi-ordered linear spaces, Tokyo Math. B. S., I (1950), §17.

(8) c. f. 7) Theorem 19.8.

(9) c. f. 7) § 20.

assumption, we see by Theorem 2.2 that $\mathfrak{E}_{\Phi}$ is composed only of a finite number of isolated points. Thus we can find a finite number of discrete positive elements $a_{\nu} \in R$ $(\nu=1, 2, \cdots, \kappa)$ such that

$$\Phi(x) = \sum_{\nu=1}^{\kappa} \Phi([a_{\nu}]x) \qquad \text{for every} \quad x \in R .$$

Recalling a theorem in an earlier paper,[10] we obtain immediately by this Theorem 3.1

Theorem 3.2. *Let R be a universally complete, discrete semi-ordered linear space with a regular density. For a positive linear functional Φ on R, if*

$$\mathrm{Min}\,\{\Phi(x),\ \Phi(y)\} = 0 \qquad \textit{for} \quad x \frown y = 0 ,$$

then there exists a positive discrete element $a \in R$ such that

$$\Phi(x) = \Phi([a]x) \qquad \textit{for every} \quad x \in R .$$ [11]

(10) c. f. 5) Satz 8.

(11) The same problem was considered by E. HEWITT, but he could not succeed to prove. E. HEWITT: Linear functionals on spaces of continuous functions, Fund. Math. 37 (1950), 161–189.