REMARK ON THE STRUCTURE OF LIE AND JORDAN RINGS

By

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The purpose of this paper[1] is to investigate the form of multiplication of Lie and Jordan rings when we seek for their model in a non-commutative ring.

Suppose L is the module with operator in which a multiplication is defined and the following conditions are satisfied:

$$(1)\quad a\times(b+c)=a\times b+a\times c,\qquad (b+c)\times a=b\times a+c\times a$$

$$(2)\quad \lambda(a\times b)=(\lambda a)\times b=a\times(\lambda b),\qquad a,b,c\in L,\qquad \lambda\in P,$$

where P is a operator domain.

Suppose that P is the field whose characteristic is $m\,(0<m\leqq\infty)$ (in our case we say the characteristic of a field is ∞ when it is 0 in usual sense), and S is the non-commutative ring in which L is mapped as the module by an operator-homomorphism φ.

Now we assume that $a\times b$ is mapped on the polynomial with coefficients in P of degree $n\,(<m)$ with respect to x, y which are the images of a, b by the homomorphism φ respectively:

$$\varphi(a\times b)=\sum\xi^{i_1,\cdots,i_r}_{j_1,\cdots,j_r}x^{i_1}y^{j_1}\cdots x^{i_r}y^{j_r}+C,$$

$$\xi\in P,\quad x,y,C\in S,\quad i_1+\cdots+i_r\leqq n,\quad i_k\geqq 0,$$

$$j_1+\cdots+j_r\leqq n,\quad j_k\geqq 0,\quad i_1+\cdots+i_r+j_1+\cdots+j_r\neq 0,$$

where $\sum$ covers $i_1,\cdots,i_r,j_1,\cdots,j_r$ satisfying the above conditions.

Being $0\times b=a\times 0=0\times 0=0$, it follows immediately that C and the sum of all terms with $i_1+\cdots+i_r=0$ or $j_1+\cdots+j_r=0$ vanish. Therefore we can assume from the beginning that

$$\varphi(a\times b)=\sum\xi^{i_1,\cdots,i_r}_{j_1,\cdots,j_r}x^{i_1}y^{j_1}\cdots x^{i_r}y^{j_r},\qquad \begin{matrix}i_1+\cdots+i_r>0\\ j_1+\cdots+j_r>0\end{matrix}.$$

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Now for an arbitrary element λ of P it holds from (2)

$$\varphi(a \times (\lambda b)) = \sum \xi^{i_1, \cdots, i_r}_{j_1, \cdots, j_r} x^{i_1} (\lambda y)^{j_1} \cdots x^{i_r} (\lambda y)^{j_r},$$

$$\varphi(\lambda (a \times b)) = \lambda \sum \xi^{i_1, \cdots, i_r}_{j_1, \cdots, j_r} x^{i_1} y^{j_1} \cdots x^{i_r} y^{j_r},$$

whence

$$\sum \lambda^{j_1 + \cdots + j_r} \xi^{i_1, \cdots, i_r}_{j_1, \cdots, j_r} x^{i_1} y^{j_1} \cdots x^{i_r} y^{j_r} = \sum \lambda \xi^{i_1, \cdots, i_r}_{j_1, \cdots, j_r} x^{i_1} y^{j_1} \cdots x^{i_r} y^{j_r}.$$

Arranging the expansion with respect to λ, we have $\sum_{i=1}^{n} \lambda^i A_i = 0$ where A_i is a polynomial in x, y not containing λ. If $\lambda \neq 0$ it follows:

$$\sum_{i=0}^{n-1} \lambda^i A_{i+1} = 0. \tag{3}$$

Since there exist at least n elements different from 0 in P, we have from (3):

$$\begin{aligned} &A_1 + \lambda_1 A_2 + \cdots + \lambda_1^{n-1} A_n = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \qquad\qquad \lambda_i \neq 0 \quad (i = 1, 2, \cdots, n) \\ &A_1 + \lambda_n A_2 + \cdots + \lambda_n^{n-1} A_n = 0. \end{aligned}$$

Since

$$\begin{vmatrix} 1, & \lambda_1, & \cdots, & \lambda_1^{n-1} \\ 1, & \lambda_2, & \cdots, & \lambda_2^{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \\ 1, & \lambda_n, & \cdots, & \lambda_n^{n-1} \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4) \cdots (\lambda_{n-1} - \lambda_n) \neq 0,$$

then $A_i = 0 \, (i = 1, \cdots, n)$. Hence $j_1 + \cdots + j_r = 1$. Similarly we have $i_1 + \cdots + i_r = 1$.

Therefore changing the coefficients, we can put

$$\varphi(a \times b) = \xi xy + \xi' yx. \tag{4}$$

Now if we add to L the next condition (5) besides (1), (2):

(5) $$a \times b = b \times a,$$

it follows:

$$\xi xy + \xi' yx = \xi yx + \xi' xy, \qquad (\xi - \xi')(xy - yx) = 0.$$

Since S is non-commutative there exist at least two elements x, y such that $xy \neq yx$, therefore $\xi = \xi'$, i.e., (4) will be written as $\varphi(a \times b) = \xi(xy + yx)$, and the module L is represented as a Jordan ring.

When there exists an element x in S such that $x^2 \neq 0$ and if we adopt besides (1), (2) the next condition (6) instead of (5):

(6) $$a \times a = 0,$$

we have from (4) $(\xi + \xi')x^2 = 0$. As $x^2 \neq 0$, we have $\xi' = -\xi$, i.e. (4) will be written as $\varphi(a \times b) = \xi(xy - yx)$, and in this case the polynomial $a \times (b \times c) + b \times (c \times a) + c \times (a \times b)$ is mapped on 0, i.e. the module L is represented as a Lie ring.

When for all elements x of S $x^2 = 0$, then

$$0 = (x+y)^2 = x^2 + xy + yx + y^2 = xy + yx, \qquad yx = -xy,$$

therefore we can describe (4) as $\varphi(a \times b) = \lambda xy$ and the module L is represented as an associative ring.