# PARTIALLY ORDERED ABELIAN SEMIGROUPS

## I. ON THE EXTENSION OF THE STRONG PARTIAL ORDER DEFINED ON ABELIAN SEMIGROUPS

By

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**Definition 1.** A set $S$ is said to be a *partially ordered abelian semigroup* (p. o. semigroup), when in $S$ are satisfied the following conditions:

I) $S$ is an abelian semigroup under the multiplication, that is:
1) A single-valued product $ab$ is defined in $S$ for any pair $a$, $b$ of $S$,
2) $ab = ba$ for any $a,b$ of $S$,
3) $(ab)c = a(bc)$ for any $a$, $b$, $c$ of $S$.

II) $S$ is a partially ordered set under the relation $\geqq$, that is:
1) $a \geqq a$,
2) $a \geqq b$, $b \geqq a$ imply $a = b$,
3) $a \geqq b$, $b \geqq c$ imply $a \geqq c$.

III) Homogeneity: $a \geqq b$ implies $ac \geqq bc$ for any $c$ of $S$.

A partial order which satisfies the condition III) is called a *partial order defined on an abelian semigroup.*

If $S$ is an abelian group, then $S$ is said to be a *partially ordered abelian group* (p. o. group).

Moreover, if a partial order defined on an abelian semigroup (group) $S$ is a linear order, then $S$ is said to be a *linearly ordered abelian semigroup* (*group*) (l. o. semigroup (l. o. group)).

We write $a > b$ for $a \geqq b$ and $a \neq b$.

**Definition 2.** A partial order defined on an abelian semigroup $S$ (or a p. o. semigroup $S$) is called *strong,* when the following condition is satisfied: $ac \geqq bc$ implies $a \geqq b$.

**Theorem 1.** A partial order defined on an abelian group $G$ is always strong.

*Proof.* Since $G$ is a group, there exists an inverse element $c^{-1}$ of $c$. By the homogeneity $ac \geqq bc$ implies $(ac)c^{-1} \geqq (bc)c^{-1}$. Therefore $a \geqq b$.

**Theorem 2.** In the strong p. o. semigroup $S$ the following properties are held:

1) $ac = bc$ implies $a = b$ (product cancellation law).
2) $ac > bc$ implies $a > b$ (order cancellation law).
3) $a > b$ implies $ac > bc$ for any $c$ of $S$.

*Proof.* 1): If $ac = bc$, or, what is the same, if $ac \geqq bc$ and $bc \geqq ac$, then $a \geqq b$ and $b \geqq a$, that is, $a = b$.

2): If $ac > bc$ implies $a = b$, then $ac = bc$, which is absurd.

3): If $a > b$ implies $ac = bc$ for some $c$ of $S$, then by 1) we have $a = b$ which contradicts the hypothesis $a > b$.

**Theorem 3.** In the l. o. semigroup $S$ the following properties are held:

1) $ac > bc$ implies $a > b$,
2) $a^n > b^n$ for some positive integer $n$ implies $a > b$.

*Proof.* 1): If, under the hypothesis $ac > bc$, $a \not> b$, then by the linearity of $S$, $b \geqq a$. By the homogeneity we have $bc \geqq ac$, this contradicts the hypothesis. 2): Similarly, if $a^n > b^n$ implies $a \not> b$, then we have $b^n \geqq a^n$.

**Theorem 4.** In the l. o. semigroup $S$ the following conditions are equivalent to each other:

1) $ac \geqq bc$ implies $a \geqq b$ (strong),
2) $ac = bc$ implies $a = b$,
3) $a > b$ implies $ac > bc$ for all $c$ of $S$.

*Proof.* 1) $\rightarrow$ 2): See Theorem 2, 1). 2) $\rightarrow$ 3): Suppose that $a > b$ implies $ac = bc$ for some $c$ of $S$. By 2) we have $a = b$. 3) $\rightarrow$ 1): Suppose that $ac \geqq bc$ implies $a \ngeqq b$. By the linearity we have $b > a$, therefore we have $bc > ac$ by 3).

**Definition 3.** Two p. o. semigroups $S$ and $S'$ will be called *order-isomorphic* if there exists an algebraic isomorphism $x \longleftrightarrow x'$ between them which preserves order: if $a \longleftrightarrow a'$, $b \longleftrightarrow b'$, then $a \geqq b$ if and only if $a' \geqq b'$.

A p. o. semigroup $S$ will be said to be *order-embedded* in a p. o. semigroup $S'$, if there exists an order-isomorphism of $S$ into $S'$.

**Theorem 5.** A p. o. semigroup $S$ can be order-embedded in a p. o. group if and only if $S$ is strong.

*Proof.* Necessity: By Theorem 1.

Sufficiency: By Theorem 2, the product cancellation law is held in

$S$. Let $G$ be the set of all symbols $(a, a')$, $a, a' \in S$. We introduce the equality of the elements of $G$ as follows: $(a, a')$ is equal to $(b, b')$ if and only if $ab' = a'b$. As we can then prove, the above-defined equality fulfils the equivalence relation. In particular $(ax, a'x) = (a, a')$ for any $x$ of $S$. Next, we define the multiplication of the elements in $G$ as follows: $(a, a')(b, b') = (ab, a'b')$. If $(a, a') = (c, c')$ and $(b, b') = (d, d')$, then $(ab, a'b') = (cd, c'd')$. One can easily verify the commutative and associative laws of multiplication. Moreover, $(x, x)$ is the unit element of $G$ and $(a', a)$ is an inverse element of $(a, a')$. Therefore $G$ is an abelian group under the multiplication introduced above.

Now let us define an order in $G$ as follows: $(a, a') \geqq (b, b')$ if and only if $ab' \geqq a'b$ in $S$. By the strongness of $S$ it follows immediately that if $(a, a') = (c, c')$, $(b, b') = (d, d')$ and $(a, a') \geqq (b, b')$, then $(c, c') \geqq (d, d')$. Moreover, it is easy to see that the above-defined order $\geqq$ fulfils the conditions II) 1), 2), 3) and III). Therefore $G$ becomes a p. o. group. The correspondence $a \longleftrightarrow (ax, x)$ is the order-isomorphism of $S$ into $G$.

Such an obtained group $G = Q(S)$, which is the minimal p. o. group containing $S$ and uniquely determined by $S$ apart from its order-isomorphism, will be called the *quotient group* of the p. o. semigroup $S$.

*Corollary.* A l. o. semigroup $S$ can be order-embedded in a l. o. group if and only if $S$ is strong.

**Theorem 6.** Let $S$ be a p. o. semigroup with the unit element $e$. $e \geqq a$ for any $a$ of $S$ if and only if $a \geqq ab$ for any $a, b$ of $S$.

*Proof.* Necessity: $e \geqq b$ for any $b$ of $S$ implies $ae = a \geqq ab$ for any $a, b$ of $S$.

Sufficiency: If $a \geqq ab$ for any $a, b$ of $S$, then we put $a = e$. Thus we have $e \geqq b$ for any $b$ of $S$. Moreover, if $S$ has the zero element, i. e., the element 0 such that $0a = 0$ for any $a$ of $S$, then $a \geqq 0$ for any $a$ of $S$.

*Corollary.* Let $S$ be a p. o. semigroup order-embedded in a p. o. group $G$. $e \geqq a$ for any $a$ of $S$, where $e$ is the unit element of $G$, if and only if $a \geqq ab$ for any $a, b$ of $S$.

**Theorem 7.** Let $S$ be a strong p. o. semigroup, $G$ be the quotient group of $S$ and $e$ the unit element of $G$. $e \geqq a$ for any $a$ of $S$ and $e > a$ $(a \in G)$ implies $a \in S$ if and only if $a \geqq ab$ for any $a, b$ of $S$ and if $a > b$, then there exists an element $c$ of $S$ such that $b = ac$.

*Proof.* Necessity: By Corollary of Theorem 6, $a \geqq ab$ for any $a, b$ of $S$. If $a > b$, then $e > a^{-1}b$, and hence $a^{-1}b = c \in S$. Therefore $b = ac$.

Sufficiency: It is clear that $e \geqq a$ for any $a$ of $S$. Moreover, let $x$ be any element of $G$ such that $e > x$. We can put $x = a^{-1}b$, $a, b \in S$. Thus we obtain $a > b$. Hence there exists an element $c$ of $S$ such that $b = ac$, therefore $x = a^{-1}b = c \in S$.

**Definiton 4.** Let $S$ be a p.o.semigroup. An element $a$ of $S$ is called *positive* or *negative*, when $a^2 \geqq a$ or $a \geqq a^2$ respectively. In a p. o. group these coincide with the usual definition.

A partial order defined on $S$ is called *directed*, when to any $a$, $b$ of $S$ there exists an element $c$ of $S$ such that $a \geqq c$ and $b \geqq c$.

**Theorem 8.** Let $G$ be a p. o. group and $S$ be the p. o. semigroup of all negative elements of $G$. Then $G = Q(S)$ if and only if $G$ is directed.[1]

*Proof.* Necessity: By Theorem 7, $a \geqq ab$ for any $a$, $b$ of $S$. Therefore $S$ is directed. Let $x$, $y$ be any elements of $G$. One can write $x = ac^{-1}$, $y = bc^{-1}$, $a, b, c \in S$. Since $S$ is directed, there exists an element $d$ of $S$ such that $a \geqq d$ and $b \geqq d$. And hence if we put $z = dc^{-1}$, we have $x \geqq z$ and $y \geqq z$. Therefore $G$ is directed.

Sufficiency: Let $x$ be any element of $G$. If $a$ be chosen such that $x \geqq a$ and $e \geqq a$ ($e$ is the unit element of $G$), then

$$x = a\,(ax^{-1})^{-1}, \quad e \geqq a, \quad e \geqq ax^{-1}.$$

**Definition 5.** An element of a semigroup $S$ is said to be of *infinite order* if all its powers are different. If there exists a positive integer $n$ such that $a^i \neq a^j$ for $1 \leqq i < j \leqq n$ and $a^n = a^k$ for all integers $k \geqq n$, then $a$ is called *quasi-idempotent* and such positive integer $n$ is called the *length* of $a$. If the length of $a$ is 1 then $a$ is idempotent in the usual sense.

**Theorem 9.** An element of a l. o. semigroup $S$ is of infinite order or quasi-idempotent.

*Proof.* Let $a$ be not of infinite order. There exist positive integers $n$, $m$ such that $a^n = a^m$, $m > n$, and $n$ is the least. Since $S$ is a l. o. semigroup,

$$a > a^2 > \cdots > a^{n-1} > a^n \geqq a^{n+1} \geqq \cdots \geqq a^m = a^n \text{ (or its dual).}$$

Therefore $a^n = a^k$ for all $k \geqq n$.

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(1) Cf. A. H. Clifford: Partially ordered abelian groups, Ann. Math., vol. 41 (1940), pp. 465–473.

**Theorem 10.** Let $S$ be a strong l. o. semigroup. Then $a^n = b^n$ implies $a = b$. And if there exists a quasi-idempotent element $e$, then $e$ is the unit element.

*Proof.* Since $S$ is strong, $a > b$ implies $a^2 > ab > b^2$. Hence for all positive integers $n$, $a^n > b^n$. Next, the length of $e$ must be 1. Hence $e^2 = e$. For every $x$ of $S$, $ex = e^2x$ and hence $x = ex$, that is, $e$ is the unit element. Therefore $S$ has at most one quasi-idempotent element.

**Definition 6.** A partial order defined on an abelian semigroup $S$ is called *normal*, when the following condition is satisfied:(2)

$$a^n \geqq b^n \text{ for some positive integer } n \text{ implies } a \geqq b.$$

**Theorem 11.** A strong l. o. semigroup $S$ is always normal.

*Proof.* Suppose that $a \ngeqq b$. Then we have, by the linearity of $S$, $b > a$, which implies $b^n > a^n$ for every positive integer $n$.

*Corollary.* A l. o. group $G$ is always normal.

**Theorem 12.** In the normal p. o. semigroup the following properties are held: 1) $a^n > b^n$ implies $a > b$, 2) $a^n = b^n$ implies $a = b$.

*Proof.* 1): By the normality, $a^n > b^n$ implies $a \geqq b$. If $a = b$, then we have $a^n = b^n$. 2): The normality means that if $a^n = b^n$, or what is the same $a^n \geqq b^n$ and $b^n \geqq a^n$, then $a \geqq b$ as well as $b \geqq a$, that is, $a = b$.

*Corollary.* An element of a normal p. o. group has an infinite order, except the unit element.

**Definition 7.** Suppose that two partial orders $P$ and $Q$ are defined on the same semigroup $S$ and that the relation $a > b$ in $P$ implies $a > b$ in $Q$; then $Q$ will be called an *extension* of $P$. An extension which defines a linear order on $S$ will be called a *linear extension.*

In the set $\mathfrak{P}$ of all partial orders defined on the same semigroup $S$, we put $Q \succ P$ if and only if $Q$ is an extension of $P$. Then $\mathfrak{P}$ is a partially ordered set under this relation $\succ$.

**Theorem 13.** Let $P$ be a strong partial order defined on an abelian semigroup $S$ and $x$ and $y$ are any two elements non-comparable in $P$. Then there exists an extension $Q$, which is strong, of $P$ such that $x > y$ in $Q$ if and only if $P$ is normal.(3)

*Proof.* Sufficiency: Let $P$ be a normal strong partial order defined

(2) Cf. L. Fuchs: On the extension of the partial order of groups, Amer. Journ. Math., vol. 72 (1950), pp. 191–194.

(3) Cf. L. Fuchs: l. c.

on $S$ and the elements $x$ and $y$ are not comparable in $P$. Let us define a relation $Q$ as follows:

We put $a > b$ in $Q$ if and only if $a \neq b$ and there are two non-negative integers $n$, $m$, such that not both zero and

$$(\S) \qquad a^n y^m \geqq b^n x^m \quad \text{in} \quad P,$$

where if $m = 0$ or $n = 0$ (§) means that $a^n \geqq b^n$ or $y^m \geqq x^m$ in $P$ respectively.

First, we note that $n$ is never zero, for otherwise we should have $y^m \geqq x^m$ in $P$, whence (by the normality) we have $y \geqq x$ in $P$ against the hypothesis.

i) We begin with verifying that $a > b$ and $b > a$ in $Q$ are contradictory. Suppose that $a > b$ and $b > a$, namely $a^n y^m \geqq b^n x^m$ and $b^i y^j \geqq a^i x^j$ in $P$ for some non-negative integers $n, m, i, j$. By multiplying $i$ times the first, $n$ times the second inequality, one obtains $(ab)^{ni} y^{mi+nj} \geqq (ab)^{ni} x^{mi+nj}$ in $P$. By the strongness of $P$ we have $y^{mi+nj} \geqq x^{mi+nj}$ in $P$. If $mi + nj$ does not vanish, by the normality we have $y \geqq x$, this contradicts the hypothesis. On the other hand, if $mi + nj$ is zero, i. e., both $m$ and $j$ vanish, then $a^n \geqq b^n$ and $b^i \geqq a^i$ in $P$. Therefore we have $a \geqq b$ and $b \geqq a$ in $P$, that is, $a = b$ which is absurd.

ii) We show the transitivity of $Q$. Assume that $a > b$ and $b > c$ in $Q$, i. e., for some non-negative integers $n, m, i, j$, $a^n y^m \geqq b^n x^m$ and $b^i y^j \geqq c^i x^j$ in $P$. By multiplying as in i) we get $a^{ni} y^{mi+nj} \geqq c^{ni} x^{mi+nj}$ in $P$. Here $ni$ is not zero, and $a = c$ is by i) impossible, so that $a > c$ in $Q$.

iii) We prove next the homogeneity of $Q$. $a \neq b$ implies $ac \neq bc$ for any $c$ of $S$, since $P$ is strong. Hence if $a > b$ in $Q$, namely, if $a \neq b$ and $a^n y^m \geqq b^n x^m$ in $P$ for some $n, m$, then $ac \neq bc$ and $(ac)^n y^m \geqq (bc)^n x^m$ in $P$. Therefore $a > b$ implies $ac > bc$ in $Q$ for any $c$ of $S$.

iv) $Q$ is an extension of $P$, for if $a > b$ in $P$, then $ay^0 > bx^0$ in $P$, therefore $a > b$ in $Q$.

v) It is clear that $x > y$ in $Q$. In fact, $xy \geqq yx$ in $P$.

vi) We may prove the normality and the strongness of $Q$. Indeed, supposing $a^n > b^n$ in $Q$ for some positive integer $n$, i.e., $(a^n)^i y^j \geqq (b^n)^i x^j$ in $P$, we see at once that $a > b$ in $Q$. Suppose that $ac > bc$ in $Q$, i.e., $(ac)^n y^m \geqq (bc)^n x^m$ in $P$ for some $n, m$. Then by the strongness of $P$ we are led to the result $a > b$ in $Q$.

Necessity: Let us assume that there exist elements $a$ and $b$ such that $a^n \geqq b^n$ and $a \ngeqq b$ in $P$. Then $a$ and $b$ can not be comparable in $P$ by the strongness of $P$. And hence there exists a strong extension

$Q$ of $P$ in which $b > a$. This is however absurd, since by the strongness of $Q$ this would imply $b^n > a^n$ in $Q$, contrary to the hypothesis $a^n \geqq b^n$ in $P$.

**Definition 8.** If $P_1, P_2, \cdots, P_\alpha, \cdots$ is a well-ordered chain of partial orders defined on the same abelian semigroup $S$ such that each of them is some extension of the preceding ones, then the *union* of the chain may be defined to be a partial order $P$ defined on $S$ such that $a \geqq b$ in $P$ if and only if $a \geqq b$ in $P_\alpha$ holds for some one and hence for all subsequent subscripts $\alpha$.

It is easy to see that $P$ is normal or strong if all $P_\alpha$ are normal or strong respectively.

**Theorem 14.** For every normal strong partial order $P$ defined on an abelian semigroup $S$ and every two elements $x, y$ non-comparable in $P$, there exists a normal strong linear extension $L_{xy}$ with the property that $x > y$ in $L_{xy}$.

*Proof.* By Theorem 13 there exists a normal strong extension $Q$ of $P$ such that $x > y$ in $Q$. Let $\mathfrak{P}'$ be a set of all normal strong partial orders defined on $S$ which are extensions of $Q$. $\mathfrak{P}'$ is a partially ordered set as a subset of $\mathfrak{P}$ in Definition 7. By ZORN's lemma there exists a maximal linearly ordered subset $\mathfrak{P}^*$ of $\mathfrak{P}'$. Let $L_{xy}$ be an union of $\mathfrak{P}^*$. Then $L_{xy}$ is a maximal order, that is, order which has no proper extension. By Theorem 13 this can happen only in case any two elements are comparable in $L_{xy}$, that is to say, $L_{xy}$ is linear. Moreover, $L_{xy}$ is strong and normal, and $x > y$ in $L_{xy}$.

**Theorem 15.** A strong linear order may be defined on an abelian semigroup $S$ if and only if in $S$ are satisfied the following conditons: 1) $ax = bx$ implies $a = b$, 2) $a^n = b^n$ for some positive integer $n$ implies $a = b$.

*Proof.* The necessity is obvious by Theorems 2 and 12. If we consider a vacuous partial order $P$ of $S$ in the sense of TUKEY, then $P$ is the partial order defined on $S$. And conditions 1) and 2) say that $P$ is strong and normal. Therefore, by Theorem 14 for any $x, y$ of $S$ there exists a strong linear extension $L_{xy}$ of $P$ in which $x > y$.

*Corollary.* A linear order may be defined on an abelian group if and only if all its elements, except the unit element, are of infinite order.[4]

(4) F. LEVI: Arithmetische Gesetze im Gebiete diskreter Gruppen, Rendiconti Palermo, vol. 35 pp. (1913), 225–236.

G. BIRKHOFF: Lattice Theory, second edition, Theorem 14, p. 224.

**Definition 9.** Let $\mathfrak{S} = \{P_\alpha\}$ be any set of partial orders, each defined on the same abelian semigroup $S$. We define the new partial order $P$ on $S$ as follows: For any two elements $a$, $b$ we put $a \geqq b$ in $P$ if and only if $a \geqq b$ in every $P_\alpha$ of the set $\mathfrak{S}$. Indeed, $P$ is again a partial order defined on $S$, moreover $P$ is normal or strong if all $P_\alpha$ of $\mathfrak{S}$ are normal or strong respectively. The partial order $P$ is said to be the *product* of the $P_\alpha$ or to be *realized* by the set $\mathfrak{S}$ of partial orders, written $P = \Pi P_\alpha$.

Let $\mathfrak{G} = \{G_\alpha\}$ be a set of l. o. groups and $G$ the (restricted or complete) direct product of $G_\alpha$. Then one can introduce a partial order defined on $G$ as usual, so that $G$ becomes a p. o. group. We shall call $G$ a *vector-group*. It is clear that a vector-group is always strong and normal.

**Theorem 16.** A strong partial order $P$ defined on an abelian semigroup $S$ may be realized by a certain set of strong linear orders if and only if $P$ is normal.

*Proof.* The necessity is obvious, since by Theorem 11 a strong linear order, and hence every product of strong linear orders, is normal. On the other hand, if $P$ is not linear, then there exist to any pair of elements $x$, $y$ non-comparable in $P$ the corresponding linear extensions $L_{xy}$ and $L_{yx}$ described in Theorem 14. It is easy to see that these linear orders realize $P$.

**Theorem 17.** A p. o. semigroup $S$ can be order-embedded in a vector-group if and only if $S$ is normal and strong.

*Proof.* Let $P$ be a partial order defined on $S$. If $P$ is strong and normal, then by Theorem 16 $P$ is realized by a certain set of strong linear orders, which are extensions of $P$, defined on the semigroup $S$; $P = \Pi P_\alpha$. Let $S_\alpha$ be the strong l. o. semigroup when we consider that $P_\alpha$ is the strong linear order defined on $S$. And let $G_\alpha$ be the quotient group of $S_\alpha$. $G_\alpha$ is a l. o. group. Then $S$ is order-embedded in the direct product $G$ of $G_\alpha$. The necessity is obvious.

*Corollary.* A p. o. group $G$ can be order-embedded in a vector-group if and only if $G$ is normal[5].

**Theorem 18.** Let $\mathfrak{T} = \{S_\alpha\}$ be a set of strong l. o. semigroups and $S$ the (restricted or complete) direct product of $S_\alpha$. Then one can introduce a linear order defined on $S$, so that $S$ becomes a strong l. o.

(5) A. H. Clifford: l.c., Theorem 1.

semigroup.[6]

*Proof.* We may consider that the $S_\alpha$ are well-ordered. Elements of $S$ are then given by their components: $x = \{x_\alpha\}$, $x_\alpha \in S_\alpha$.

Let us define a relation $P$ in $S$ as follows:

We put $x > y$ in $P$ if and only if $x \neq y$ and

$$x_\alpha = y_\alpha \quad \text{for all} \quad \alpha < \beta \quad \text{and} \quad x_\beta > y_\beta .$$

We see readily that $P$ is a strong linear order defined on $S$.

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(6) Cf. K. Iwasawa: On linearly ordered groups, Journ. Math. Soc. Japan, vol. 1 (1948), pp. 1–9.