STRONGLY π-REGULAR RINGS

By

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ARENS-KAPLANSKY [1] and KAPLANSKY [3] investigated, as generalizations of algebraic algebras and rings with minimum condition, following two types of rings: one is a π-regular ring, that is, a ring in which for every element a there exists an element x and a positive integer n such that $a^n x a^n = a^n$, and the other is a ring in which for every a there exists an x and an n such that $a^{n+1}x = a^n$ — this we shall call a right π-regular ring. The present note is devoted mainly to study the latter more precisely. Apparently, the two notions of π-regularity and right π-regularity are different ones in general. However we can prove, among others, that under the assumption that a ring is of bounded index (of nilpotency) it is π-regular if and only if it is right π-regular. Moreover, we shall show, in this case, that we may find, for every a, an element z such that $az = za$ and $a^{n+1}z = a^n$, where n is the least upper bound of all indices of nilpotency in the ring. This is obviously a stronger result than a theorem of KAPLANSKY (2) as well as that of GERTSCHIKOFF (3), both of which are stated in section 8 of KAPLANSKY [3].

1. Strong regularity. Let A be a ring. Let a be an element of A. a is called *regular* (in A) if there exists an element x of A such that $axa = a$, while a is said to be *right* (or *left*) *regular* if there exists x such that $a^2 x = a$ (or $xa^2 = a$). Further, we call a *strongly regular* if it is both right regular and left regular.

Lemma 1. *Let a be a strongly regular element of A. Then there exists one and only one element z such that $az = za$, $a^2 z (= za^2) = a$ and $az^2 (= z^2 a) = z$, and in particular a is regular. For any element x such that $a^2 x = a$, z coincides with ax^2. Moreover, z commutes with every element which is commutative with a.*

Proof. Let x, y be two elements such that $a^2 x = a$, $ya^2 = a$. Then

$$ax = ya^2 x = ya, \tag{1}$$

so that

(2) $$ax^2 = yax = y^2a.$$

From (1) we have also

(3) $$axa = ya^2 = a = a^2x = aya.$$

Now put $z = ax^2$. It follows then from (1), (2), (3) that $az = ayax = ax = ya = yaxa = za$, $a^2z = aza = axa = a$, $az^2 = yaz = yax = z$, as desired.

Suppose next z' be any element which satisfies the same equalities as z: $az' = z'a$, $a^2z' = a$, $az'^2 = z'$. Then, by replacing x, y in (2) by z, z' respectively, we get $z = az^2 = z'^2a = z'$, showing the uniqueness of z.

For the proof of the last assertion, let c be any element such that $ac = ca$. Then we have first $zac = zca = zca^2z = za^2cz = acz = caz$, i.e., c commutes with $za = az$. It follows from this now $zc = z^2ac = zcza = zacz = czaz = cz$, and this completes our proof.

Lemma 2. *Let a be a right regular element of A and let $a^2x = a$. Then, for any positive integer n, we have*

$$(a - ax^na^n)^r = \begin{cases} a^r - ax^{n-r+1}a^n, & r = 1, 2, \cdots, n, \\ 0, & r = n+1. \end{cases}$$

Proof. Since the assertion is valid for $r = 1$, we may proceed by induction on r (for fixed n). Suppose $r \leqq n$ and our lemma holds for r:

$$(a - ax^na^n)^r = a^r - ax^{n-r+1}a^n.$$

Right-multiplying by $a - ax^na^n$ and using the relation $a^{n+1}x^n = a$, which follows immediately from $a^2x = a$, we have $(a - ax^na^n)^{r+1} = a^{r+1} - a^{r+1}x^na^n - ax^{n-r+1}a^{n+1} + ax^{n-r+1}a^{n+1}x^na^n = a^{r+1} - a^{r+1}x^na^n$. But when $r < n$ $a^{r+1}x^n = a^{r+1}x^rx^{n-r} = ax^{n-r}$, while when $r = n$ $a^{r+1}x^n = a^{n+1}x^n = a$. This completes our induction.

Now A is called a ring of *bounded index* if indices of nilpotency of all nilpotent elements of A are bounded; and, in this case, the least upper bound of all indices of nilpotency is called the *index* of A. (Cf. Jacobson [2], Kaplansky [3].) We can now prove the fundamental

Theorem 1. *Let A be a ring of bounded index. Then every right regular element of A is (left whence) strongly regular.*

Proof. Let n be the index of A. Let A be any right regular element of A: $a^2x = a$. Then, since $(a - ax^na^n)^{n+1} = 0$ by Lemma 2, we must have $(a - ax^na^n)^n = 0$. On the other hand, $(a - ax^na^n)^n = a^n - axa^n$ by the same lemma, and we obtain $a^n - axa^n = 0$. Apply furthermore Lemma 2 to $n+1$ instead of n. Then $(a - ax^{n+1}a^{n+1})^{n+1} = a^{n+1} - axa^{n+1} = a(a^n - axa^n)$

$=0$, and so $(a-ax^{n+1}a^{n+1})^n=0$. But $(a-ax^{n+1}a^{n+1})^n=a^n-ax^2a^{n+1}$ again by Lemma 2. Hence it follows $a^n=ax^2a^{n+1}$. Right-multiply now by ax^n and make use of the relation $a^{n+1}x^n=a$. Then we find finally $a=ax^2a^2$, which shows the left regularity of a.

In connection with the preceding theorem, we want to add the following theorem, although we shall not need it later:

Theorem 2. *Let a be a right regular element of A. Then a is strongly regular if and only if $r(a^2)=r(a)$, where $r(\ )$ denotes the set of all right annihilators.*

Proof. The "only if" part is easy to see. So we have only to prove the "if" part. The right regularity of a implies $a^2A=aA$. The mapping $u\to au\ (u\in aA)$ gives therefore an operator-homomorphism of the right ideal aA onto itself. Moreover, this is an isomorphism because the kernel is zero by the assumption $r(a^2)=r(a)$. Let φ be the inverse mapping of it; φ is also an operator-isomorphism of aA onto itself. Since $a\in(a^2A=)aA$ we have in particular $\varphi a^2=a$. From this it follows $(\varphi^2a)a^2=\varphi(\varphi a^2)a=\varphi a^2=a$, showing the left regularity of a.

Remark. Von Neumann called A a *regular* ring if every element of A is regular, while Arens-Kaplansky [1] defined A to be a *strongly regular* ring when every element is right regular. However, it was shown in above paper that if A is strongly regular then every element of A is indeed strongly regular; this follows also from our Theorem 1 directly, since a strongly regular ring A has evidently no non-zero nilpotent element. This fact justifies our definition of strong regularity for elements.

2. Strong π-regularity. Let us call an element a of A *π-regular, right π-regular*, or *left π-regular* if a suitable power of a is regular, right regular, or left regular respectively. Furthermore we call a *strongly π-regular* if it is both right π-regular and left π-regular. Now it can readily be seen that a power a^n of a is right (or left) regular if and only if there exists an element x such that $a^{n+1}x=a^n$ (or $xa^{n+1}=a^n$). On the other hand, we have

Lemma 3. *Let x, y satisfy $a^{n+1}x=a^n$, $ya^{m+1}=a^m$ for some n, m. Then they satisfy $a^{m+1}x=a^m$, $ya^{n+1}=a^n$ too.*

Proof. When $m\geqq n$ $a^{m+1}x=a^m$ follows immediately from $a^{n+1}x=a^n$. Suppose now $m<n$. Then $a^m=ya^{m+1}$ implies $a^m(=y^2a^{m+2}=\cdots)=y^{n-m}a^n$, and so we obtain $a^{m+1}x=y^{n-m}a^{n+1}x=y^{n-m}a^n=a^m$. Similarly, we can verify the validity of $ya^{n+1}=a^n$.

Now we prove

Theorem 3. *Let a be a strongly π-regular element of A. Suppose that a^n is right regular. Then a^n is in fact strongly regular, and moreover there exists an element z such that $az=za$ and $a^{n+1}z=a^n$.*

Proof. That a^n is strongly regular is an immediate consequence of Lemma 3. Now from Lemma 1 it follows that there exists an element z such that $a^{2n}z=a^n$ and z commutes with every element which is commutative with a^n; however the latter condition implies, since a is commutative with a^n, that $az=za$. Denoting $a^{n-1}z$ again by z, z is evidently the desired element.

Corollary. *Strongly π-regular element is π-regular.*

Now we define the *index* of a strongly π-regular element a as the least integer n such that a^n is right regular. By Lemma 3, the index n is characterized also as the least integer such that a^n is left regular. It is to be noted further that every nilpotent element is strongly π-regular and its index of nilpotency coincides with the index in the sense defined above, as can be seen quite easily. Furthermore we have

Lemma 4. *Let a be a strongly π-regular element of index n, and z an element such that $az=za$ and $a^{n+1}z=a^n$ (as in Theorem 3). Then $a-a^2z$ is a nilpotent element of index n.*

Proof. Since $az=za$ we have the following binomial expansion:

$$(a-a^2z)^n = a^n - \binom{n}{1}a^{n+1}z + \binom{n}{2}a^{n+2}z^2 - \cdots + (-1)^n a^{2n}z^n.$$

But $a^n=a^{n+1}z$ implies $a^n=a^{n+2}z^2=\cdots=a^{2n}z^n$. Hence we get

$$(a-a^2z)^n = a^n - \binom{n}{1}a^n + \binom{n}{2}a^n - \cdots + (-1)^n a^n = (a-a)^n = 0.$$

On the other hand, $(a-a^2z)^{n-1}$ is, again by a binomial expansion, say, expressible in a form $a^{n-1}-a^nx$ with some x; but this is certainly not zero because a is of index n. Thus, the index of $a-a^2z$ is exactly n.

We now obtain from Theorems 1, 3 and Lemma 4 immediately the following

Theorem 4. *Let A be a ring of bounded index (of nilpotency). Then every right π-regular element of A is strongly π-regular and its index does not exceed the index of A.*

Above results show us in fact the appropriateness of our definition of index for strongly π-regular elements. This is strengthened further by the following

Remark. Suppose that A is (not necessarily finite dimensional) algebra over a field K. Let a be an algebraic element of A, and $\mu(\lambda)$ the minimum polynomial of a (without constant term). JACOBSON [2] defined the index of a as the largest integer r such that λ^r divides $\mu(\lambda)$. Now we want to show that *a is then strongly π-regular and (the Jacobson index) r coincides with the index in our sense.* For the proof, we may assume, since λ^r divides exactly $\mu(\lambda)$, that $\mu(\lambda)$ is of the form $\lambda^r+\alpha_1\lambda^{r+1}+\alpha_2\lambda^{r+2}+\cdots$ (with $\alpha_1, \alpha_2, \cdots$ in K). It follows then $\alpha_1\lambda\mu(\lambda)=\alpha_1\lambda^{r+1}+\alpha_1^2\lambda^{r+2}+\cdots$, and so we have $\mu(\lambda)-\alpha_1\lambda\mu(\lambda)=\lambda^r+(\alpha_2-\alpha_1^2)\lambda^{r+2}+\cdots=\lambda^r-\lambda^{r+1}\nu(\lambda)$, where $\nu(\lambda)=(\alpha_1^2-\alpha_2)\lambda+\cdots$ is also a polynomial. Since now $\mu(\lambda)$ has a for a root, so does $\mu(\lambda)-\alpha_1\lambda\mu(\lambda)$ too, i.e., we have $a^r=a^{r+1}\nu(a)$, which shows the strong π-regularity of a. Let n be the index of a (as strongly π-regular element). Then $n\leqq r$, and moreover we have from Lemma 3 that $a^n=a^{n+1}\nu(a)$, that is, a is a root of the polynomial $\lambda^n-\lambda^{n+1}\nu(\lambda)$. Since $\mu(\lambda)$ is the minimum polynomial of a, the latter must be divisible by $\mu(\lambda)$, and this implies in particular that $n\geqq r$, proving our assertion.

Now we say that a ring A is *π-regular, right π-regular, left π-regular, or strongly π-regular* if so is every element of A respectively. (Cf. KAPLANSKY [3].) Evidently A is strongly π-regular if and only if it is both right π-regular and left π-regular. Moreover, strong π-regularity of A implies π-regularity of A, according to Corollary of Theorem 3. However, the converse is also true provided A is assumed to be of bounded index. Namely, we have

Theorem 5. *Under the assumption that A is of bounded index, the following four conditions are equivalent to each other:*

i) *A is π-regular,*
ii) *A is right π-regular,*
iii) *A is left π-regular,*
iv) *A is strongly π-regular.*

Proof. That ii) implies iv) is a direct consequence of Theorem 4. By right-left symmetry, iii) implies also iv). Therefore we have only to prove that ii) follows from i).

Suppose that A is a π-regular ring of index n. Let a be an element of A. Then a^n is π-regular, that is, there exists an integer $n'(\geqq 1)$ such that $a^{nn'}$ is regular. Put $r=nn'$. Then $r\geqq n$ and there exists an element x such that $a^rxa^r=a^r$. Write $e=a^rx$. Then e is an idempotent and satisfies $eA=a^rA$. Similarly, the π-regularity of, say, a^{r+1} implies the existence of an integer s and an idempotent f such that $s>r$ and fA

$=a^sA$. Since then $a^rA\supset a^sA$, fA is necessarily a direct summand right subideal of eA. Hence we can construct, as is well-known, two orthogonal idempotents f_1 and g such that $e=f_1+g$ and $f_1A=fA(=a^sA)$. Now take any primitive ideal P of A. By KAPLANSKY [3, Theorem 2.3] the residue class ring $\bar{A}=A/P$ is a (full) matrix ring over a division ring of degree at most n. Denote by $\bar{a}$ the residue class of a modulo P, and consider the chain of right ideals $\bar{A}\supset\bar{a}\bar{A}\supset\bar{a}^2\bar{A}\supset\cdots$. It follows then, since the degree of the simple ring $\bar{A}$ is equal to the composition length for right ideals of $\bar{A}$, that $\bar{a}^n\bar{A}=\bar{a}^{n+1}\bar{A}=\cdots$, and we have in particular $\bar{a}^r\bar{A}=\bar{a}^s\bar{A}$. Write further by $\bar{e}$, $\bar{f}_1$, $\bar{g}$ the residue classes of e, f_1, g modulo P respectively. Then $\bar{e}\bar{A}=\bar{a}^r\bar{A}$, $\bar{f}_1\bar{A}=\bar{a}^s\bar{A}$ whence $\bar{e}\bar{A}=\bar{f}_1\bar{A}$. On the other hand, $\bar{e}=\bar{f}_1+\bar{g}$ and $\bar{f}_1$, $\bar{g}$ are orthogonal dempotents; hence $\bar{e}\bar{A}$ is the direct sum of $\bar{f}_1\bar{A}$ and $\bar{g}\bar{A}$: $\bar{e}\bar{A}=\bar{f}_1\bar{A}\oplus\bar{g}\bar{A}$. This implies evidently that $\bar{g}\bar{A}=0$, i.e., $\bar{g}=0$ or $g\in P$. This is the case for every primitive ideal P, and so g must lie in the intersection of all P's i.e. the (Jacobson) radical of A. If we observe however that 0 is the only quasi-regular idempotent, it follows indeed $g=0$, and this shows that $a^rA(=eA=f_1A)=a^sA$ whence $a^rA=a^{r+1}A$. The latter equality implies, since $a^r=a^rxa^r$ is in a^rA, the right π-regularity of a. Thus, the proof of our theorem is concluded.

Remark. The radical of a π-regular ring as well as that of a right π-regular ring is always a nil-ideal, as was shown in KAPLANSKY [3, section 2] and ARENS-KAPLANSKY [1, Theorem 3.1]; the assumption in the latter that (the right π-regular ring) A is of bounded index being superfluous for proving our assertion.

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