FOURIER SERIES IX: STRONG SUMMABILITY OF THE DERIVED FOURIER SERIES.

By

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1. Introduction. Let $f(x)$ be an integrable and periodic function with period 2π and its Fourier series and its conjugate be

$$(1.1) \qquad a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) ,$$

$$(1.2) \qquad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x) .$$

Further, let their termwise derived series be

$$(1.3) \qquad \sum_{n=1}^{\infty} n (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} nB_n(x) ,$$

$$(1.4) \qquad -\sum_{n=1}^{\infty} n (a_n \cos nx + b_n \sin nx) = -\sum_{n=1}^{\infty} nA_n(x) .$$

A series $\sum_{n=1}^{\infty} c_n$ is said to be summable H_k or strongly summable to s, if

$$\sum_{n=0}^{m} |s_n - s|^k = o(m) \qquad (m \to \infty) ,$$

where $s_n = \sum_{k=0}^{n} c_k$.

B. N. PRASAD and U. N. SINGH [7] have found a criteria for H_1 summability of the derived Fourier series which reads as follows:

Theorem A. *If $f(t)$ is a continuous function of bounded variation and if for some $\alpha > 1$,*

$$(1.5) \qquad G(t) = \int_0^t |dg(u)| = o\left\{ t \Big/ \left(\log \frac{1}{t} \right)^{\alpha} \right\} , \qquad \text{as} \quad t \to 0 ,$$

where $g(u) = g_x(u) = f(x+u) - f(x-u) - 2us$,
then

$$\sum_{n=1}^{m} |\tau_n(x) - s| = o(m) ,$$

where $\tau_n(x)$ is the n-th partial sum of the series (1.3). *That is, the derived Fourier series of $f(x)$ is summable H_1 to s at a point x.*

One of us [6] (cf. [4]) generalized Theorem A in the following form:

Theorem 1. *Under the assumption of Theorem A, we have*

$$\sum_{n=1}^{m} |\tau_n(x)-s|^k = o(m), \qquad \text{for} \quad k>0. \tag{1.6}$$

In the case $0<k<2$ of Theorem 1, we can generalize in the following form [4]:

Theorem 2. *If $f(t)$ is a continuous function of bounded variation and if for some $\beta>1/2$*

$$\int_0^t |dg(u)| = O\left\{t\Big/\left(\log\frac{1}{t}\right)^{\beta}\right\}, \qquad \text{as} \quad t\to 0, \tag{1.7}$$

then the derived Fourier series of $f(x)$ is summable H_2 to s at a point x, that is,

$$\sum_{n=1}^{m} |\tau_n(x)-s|^2 = o(m). \tag{1.8}$$

This theorem is the analogue of F. T. Wang's theorem for Fourier series ([8]). As the analogue of another theorem due to G. H. Hardy and J. E. Littlewood ([3], cf. [5]), we have proved [4]

Theorem 3. *If $f(t)$ is a continuous function of bounded variation such that*

$$\int_0^t |dg(u)| = o(t),$$

then

$$\sum_{n=1}^{m} |\tau_n(x)-s|^2 = o(m\log m).$$

We have also the following theorem.

Theorem 4. *If $f(t)$ is a continuous function of bounded variation such that*

$$\int_0^{\delta} \frac{|dg(u)|^2}{u\,du} < \infty, \tag{1.9}$$

then (1.8) *holds.*

The integral in (1.9) is taken in the Hellinger sense, that is, it is defined as the limit of

$$\sum \frac{|g(u_i)-g(u_{i-1})|^2}{u_i(u_i-u_{i-1})} .$$

Theorem 5. *If $f(t)$ is a continuous function of bounded variation such that*

$$\frac{1}{v-u}\int_u^v |dg(t)| \to 0 , \qquad (v>u,\ v\to 0) ,$$

then (1.8) *holds.*

It is known that $f(t)$ is differentiable at $t=x$ when and only when

$$(f(u)-f(v))/(u-v) \to 0$$

as $u \uparrow x$, $v \downarrow x$. Then we get

Theorem 6. *If $f(t)$ is a continuous function of bounded variation and it is monotone in a neighbourhood of $t=x$ and is differentiable at $t=x$, then the Fourier series of $f(t)$ is strongly summable H_2 at $t=x$.*

We have the following generalizations of the PRASAD-SINGH theorem concerning the series (1.4).

Theorem 7. *If $f(t)$ is a continuous function of bounded variation which is differentiable at $t=x$ and if for some $\alpha>1$*

$$H(t)=\int_0^t |dh(u)| = o\left\{t\Big/\left(\log\frac{1}{t}\right)^{\alpha}\right\} , \quad as \quad t\to 0 ,$$

where

$$h(u)=h_x(u)=f(x+u)+f(x-u)-2f(x) ,$$

then

$$\sum_{n=1}^{m} |\bar{\tau}_n(x)-H_m(x)|^k = o(m)$$

for any $k>0$, where $\bar{\tau}_n(x)$ is the n-th partial sum of the series (1.4) *and*

$$H_m(x) = -\frac{1}{4\pi}\int_{1/m}^{\pi} h_x(t)\operatorname{cosec}^2\frac{t}{2}\,dt .$$

Theorem 8. *If $f(t)$ is a continuous function of bounded variation which is differentiable at $t=x$ and if for some $\beta>1/2$*

$$H(t)=\int_0^t |dh(u)| = o\left\{t\Big/\left(\log\frac{1}{t}\right)^{\beta}\right\} , \quad as \quad t\to 0 ,$$

then

$$\sum_{n=1}^{m} |\bar{\tau}_n(x)-H_m(x)|^2 = o(m) .$$

Proof of sketch of Theorem 1 and Theorem 2 is given in [4] and [6]. We give here their complete proof in §2 and §3. Theorem 3 is stated in [4], but the proof is not given there, so that we prove it in §4. In the paragraphs §5–§8, we prove the remaining theorems.

2. Proof of Theorem 1.[1]

For the proof of Theorem 1, we need the following lemma.

Lemma. *If $f(x)$ be a continuous function of bounded variation and a_n, b_n are its Fourier coefficients, then*

$$\sum_{n=1}^{m}(|na_n|^k+|nb_n|^k)=o(m),$$

where $k\geqq 1$.

For, by the WIENER theorem (cf. ZYGMUND [9], p. 221), we have

$$\sum_{n=1}^{m} n\sqrt{a_n^2+b_n^2}=o(m).$$

Hence

$$\sum_{n=1}^{m} n|a_n|=o(m), \qquad \sum_{n=1}^{m} n|b_n|=o(m).$$

Without loss of generality, we may assume that $n|a_n|\leqq 1$, $n|b_n|\leqq 1$, since the function is of bounded variation. Thus we get

$$\sum_{n=1}^{m}|na_n|^k\leqq\sum_{n=1}^{m}|na_n|=o(m)$$

and

$$\sum_{n=1}^{m}|nb_n|^k\leqq\sum_{n=1}^{m}|nb_n|=o(m),$$

which complete the proof of the lemma.

We shall now prove Theorem 1. We have

$$\begin{aligned}(2.1)\qquad \tau_n(x)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)\left\{\frac{d}{dx}\frac{\sin(n+1/2)(x-u)}{\sin(x-u)/2}\right\}du\\&=-\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)\left\{\frac{d}{du}\frac{\sin(n+1/2)(x-u)}{\sin(x-u)/2}\right\}du\\&=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x+t)-f(x-t)\}\frac{d}{dt}\frac{\sin(n+1/2)t}{\sin t/2}dt.\end{aligned}$$

By the integration by parts,

$$\tau_n(x)=\frac{1}{2\pi}\int_{0}^{\pi}\frac{\sin(n+1/2)t}{\sin t/2}d\{f(x+t)-f(x-t)\}.$$

Hence we have

1) Cf. HARDY-LITTLEWOOD [2].

$$(2.2)\qquad \tau_n(x)-s=\frac{1}{2\pi}\int_0^\pi \frac{\sin(n+1/2)t}{\sin t/2}\,d\{f(x+t)-f(x-t)-2ts\}$$
$$=\frac{1}{2\pi}\int_0^\pi \frac{\sin(n+1/2)t}{\sin t/2}\,dg(t)$$
$$=\frac{1}{2\pi}\int_0^\pi \frac{\sin nt}{\tan t/2}\,dg(t)+\frac{1}{2\pi}\int_0^\pi \cos nt\,dg(t)\,.$$

For any $\varepsilon>0$, there is a δ such that

$$\int_0^t |dg(u)|<\varepsilon t\Big/\Big(\log\frac{1}{t}\Big)^\alpha, \qquad \text{for } 0<t<\delta\,.$$

Let us put

$$g(u)=g_1(u)+g_2(u)\,,$$

where

$$g_1(u)=g(u) \qquad \text{in } (0,\ \delta/2),$$
$$=0 \qquad \text{in } (\delta,\ \pi)$$

and $g_1(u)$ is linear in $(\delta/2, \delta)$ and is continuous in $(0, \pi)$. Hence $g_2(u)$ is also a continuous function of bounded variation which vanishes in the interval $(0, \delta/2)$. So we have

$$\tau_n(x)-s=\frac{1}{2\pi}\int_0^\pi \frac{\sin nt}{\tan t/2}\,dg_1(t)$$
$$+\frac{1}{2\pi}\int_0^\pi \frac{\sin nt}{\tan t/2}\,dg_2(t)+\frac{1}{2\pi}\int_0^\pi \cos nt\,dg(t)$$
$$=P_n+Q_n+R_n\,.$$

Since Q_n and R_n are n times of the n-th Fourier coefficients of continuous functions of bounded variation, respectively, we have, by Lemma,

$$\sum_{n=1}^m |Q_n|^k=o(m) \quad \text{and} \quad \sum_{n=1}^m |R_n|^k=o(m)\,.$$

However

$$P_n=\frac{1}{2\pi}\int_0^{\delta/2}\frac{\sin nt}{\tan t/2}\,dg(t)+\frac{C}{2\pi}\int_{\delta/2}^{\delta}\frac{\sin nt}{\tan t/2}\,dt$$
$$=\frac{1}{2\pi}\int_0^{\delta/2}\frac{\sin nt}{\tan t/2}\,dg(t)+o(1)$$
$$=S_n+o(1), \qquad (\text{as } n\to\infty)\,.$$

Hence, it is sufficient to show that

$$T_m^k = \sum_{n=1}^{m} |S_n|^k = o(m) .$$

For this purpose we set

$$c_n = |S_n|^{k-1} \cdot \operatorname{sgn} S_n ,$$

$$A_m(t) = \sum_{n=1}^{m} c_n \sin nt ,$$

$$\Gamma_m = \sum_{n=1}^{m} |c_n| .$$

Then

$$|A_m(t)| \leqq \begin{cases} \Gamma_m \\ mt\Gamma_m . \end{cases}$$

Using these formulas, we have

$$\begin{aligned} 2\pi T_m^k &= 2\pi \sum_{n=1}^{m} |S_n|^{k-1} \cdot S_n (\operatorname{sgn} S_n) \\ &= 2\pi \sum_{n=1}^{m} c_n S_n = \sum_{n=1}^{m} c_n \int_0^{\delta/2} \frac{\sin nt}{\tan t/2} dg(t) \\ &= \int_0^{\delta/2} A_m(t) \cot t/2 \, dg(t) = \int_0^{1/m} + \int_{1/m}^{\delta/2} = I_1 + I_2 , \end{aligned}$$

say, where

$$|I_1| \leqq m\Gamma_m \int_0^{1/m} |dg(t)| \leqq \varepsilon\Gamma_m$$

and

$$\begin{aligned} |I_2| &\leqq \left| \int_{1/m}^{\delta/2} A_m(t) \cot t/2 \, dg(t) \right| \\ &\leqq P_m \int_{1/m}^{\delta/2} \cot t/2 \, |dg(t)| \\ &\leqq \Gamma_m \left\{ \left[\cot \frac{t}{2} G(t) \right]_{1/m}^{\delta/2} + \frac{1}{2} \int_{1/m}^{\delta/2} \operatorname{cosec}^2 t/2 \, G(t) \, dt \right\} , \end{aligned}$$

where

$$G(t) = \int_0^t |dg(u)| .$$

Hence we get[1]

$$|I_2| \leqq \varepsilon\Gamma_m + \varepsilon\Gamma_m \int_{1/m}^{\delta/2} \frac{dt}{t(\log 1/t)^{\alpha}} < A\varepsilon\Gamma_m .$$

1) In the following, we denote by A an absolute constant, which may be different in each occurences.

Collecting above results, we get

$$T_m^k \leqq A\varepsilon \Gamma_m .$$

However, by the Hölder inequality, we see that, for $k>1$,

$$\Gamma_m = \sum_{n=1}^{m} |S_n|^{k-1} \leqq \left(\sum_{n=1}^{m} |S_n|^k\right)^{1/k'} \cdot m^{1/k}$$
$$= T_m^{k/k'} m^{1/k} , \qquad (1/k+1/k'=1) .$$

Hence we have

$$T_m^k \leqq A\varepsilon m^{1/k} T_m^{k/k'} ,$$

that is,
$$T_m^k/m \leqq A^k \varepsilon^k .$$

Thus we have

$$\limsup_{m\to\infty} \frac{T_m^k}{m} \leqq A^k \varepsilon^k .$$

Since ε is arbitrary, we get the required result.

3. **Proof of Theorem 2.** We can replace O in (1.7) by o, and then for any $0<\varepsilon<1$, there is a δ such that

$$G(t) = \int_0^t |dg(u)| < \varepsilon t \Big/ \left(\log \frac{1}{t}\right)^{\beta} , \qquad (0 < t < \delta) .$$

Let us put

$$g(u) = g_1(u) + g_2(u) ,$$

where

$$g_1(u) = g(u) \qquad \text{in } (0, \delta/2) ,$$
$$= 0 \qquad \text{in } (\delta, \pi)$$

and $g_1(u)$ is linear in $(\delta/2, \delta)$ and is continuous in $(0, \pi)$. Hence $g_2(u)$ is also a continuous function of bounded variation which vanishes in the interval $(0, \delta/2)$.

By the argument in §2, we have

$$\tau_n(x) - s = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin nt}{\tan t/2} dg_1(t) + \frac{1}{2\pi}\int_0^{\pi} \frac{\sin nt}{\tan t/2} dg_2(t) + \frac{1}{2\pi}\int_0^{\pi} \cos nt\, dg(t)$$
$$= P_n + Q_n + R_n .$$

It is sufficient to show that

$$\sum_{n=1}^{m} |P_n|^2 = o(m) ,$$

since, as in §2,

$$\sum_{n=1}^{m} |Q_n|^2 = o(m) \quad \text{and} \quad \sum_{n=1}^{m} |R_n|^2 = o(m) .$$

Since

$$P_n = \frac{1}{2\pi}\int_0^{\delta/2} \frac{\sin nt}{\tan t/2}\, dg(t) + \frac{C}{2\pi}\int_{\delta/2}^{\delta} \frac{\sin nt}{\tan t/2}\, dt$$
$$= \frac{1}{2\pi}\int_0^{\delta/2} \frac{\sin nt}{\tan t/2}\, dg(t) + o(1) ,$$

we have

$$\sum_{n=1}^{m} |P_n|^2 = \frac{1}{4\pi^2}\int_0^{\delta/2}\int_0^{\delta/2} \frac{dg(u)}{\tan u/2}\cdot\frac{dg(v)}{\tan v/2} \sum_{n=1}^{m} \sin nu \cdot \sin nv + o(m)$$
$$= \frac{1}{4\pi^2}\left\{\int_0^{1/m}\int_0^{1/m} + \int_0^{1/m}\int_{1/m}^{\delta/2} + \int_{1/m}^{\delta/2}\int_0^{1/m} + \int_{1/m}^{\delta/2}\int_{1/m}^{\delta/2}\right\} + o(m)$$
$$= I_1 + I_2 + I_3 + I_4 + o(m) ,$$

where

$$|I_1| \leqq A\int_o^{1/m} |dg(u)| \int_o^{1/m} |dg(v)| \left(\sum_{n=1}^{m} n^2\right) \leqq A\varepsilon m ,$$

$$|I_2| \leqq A\int_o^{1/m} |dg(u)| \int_{1/m}^{\delta/2} \frac{|dg(v)|}{v}\left(\sum_{n=1}^{m} n\right) < \frac{\varepsilon m}{(\log m)^{\beta}} \int_{1/m}^{\delta/2} \frac{|dg(v)|}{v}$$
$$\leqq \frac{A\varepsilon m}{(\log m)^{\beta}} \left\{\left[\frac{G(v)}{v}\right]_{1/m}^{\delta/2} + \int_{1/m}^{\delta/2} \frac{G(v)}{v^2}\, dv\right\}$$
$$\leqq \frac{A\varepsilon m}{(\log m)^{\beta}} \left\{O(1) + O(\log m)^{1-\beta}\right\} \leqq A\varepsilon m$$

and

$$|I_3| \leqq A\varepsilon m .$$

We next consider the remaining part I_4: We have

$$I_4 = \int_{1/m}^{\delta/2}\int_{1/m}^{\delta/2} \frac{dg(u)}{\tan u/2}\,\frac{dg(v)}{\tan v/2}\,\frac{\sin(m+1/2)(u-v)}{2\sin(u-v)/2}$$
$$+ \int_{1/m}^{\delta/2}\int_{1/m}^{\delta/2} \frac{dg(u)}{\tan u/2}\,\frac{dg(v)}{\tan v/2}\,\frac{\sin(m+1/2)(u+v)}{2\sin(u+v)/2} = J + L ,$$

where

$$|L| \leqq A\int_{1/m}^{\delta/2} \frac{|dg(u)|}{u}\int_{1/m}^{\delta/2} \frac{|dg(v)|}{v(u+v)} \leqq A(\log m)^{1-\beta}\int_{1/m}^{\delta/2} \frac{|dg(v)|}{v^2}$$

$$= A(\log m)^{1-\beta}\left\{\left[\frac{G(v)}{v^2}\right]_{1/m}^{\delta/2}+\int_{1/m}^{\delta/2}\frac{G(v)}{v^3}\,dv\right\}$$

$$\leqq A(\log m)^{1-\beta}\left\{\frac{\varepsilon m}{(\log m)^{\beta}}+\varepsilon\int_{1/m}^{\delta/2}\frac{dv}{v^2(\log 1/v)^{\beta}}\right\}$$

$$\leqq A\varepsilon m(\log m)^{1-2\beta}+A\varepsilon(\log m)^{1-\beta}\left\{\int_{1/m}^{(1/m)^{\Delta}}+\int_{(1/m)^{\Delta}}^{\delta/2}\right\}\frac{dv}{v^2(\log 1/v)^{\beta}},$$

$$(0<\Delta<1),$$

$$\leqq A\varepsilon m(\log m)^{1-2\beta}+A\varepsilon(\log m)^{1-2\beta}\int_{1/m}^{(1/m)^{\Delta}}\frac{dv}{v^2}+A\varepsilon(\log m)^{1-\beta}\int_{(1/m)^{\Delta}}^{\delta/2}\frac{dv}{v^2}$$

$$\leqq A\varepsilon m(\log m)^{1-2\beta}+A\varepsilon m(\log m)^{1-2\beta}+A\varepsilon m^{\Delta}(\log m)^{1-\beta}\leqq A\varepsilon m\,.$$

It is then sufficint to show that $|J| \leqq A\varepsilon m$. We have

$$2J=\int_{1/m}^{\delta/2}\int_{1/m}^{\delta/2}\frac{dg(u)}{\tan u/2}\,\frac{dg(v)}{\tan v/2}\,\frac{\sin(m+1/2)(u-v)}{\sin(u-v)/2}$$

$$=\int_{1/m}^{\delta/2}\frac{dg(u)}{\tan u/2}\int_{u}^{\delta/2}\frac{dg(v)}{\tan v/2}\,\frac{\sin(m+1/2)(u-v)}{\sin(u-v)/2}$$

$$+\int_{1/m}^{\delta/2}\frac{dg(u)}{\tan u/2}\int_{1/m}^{u}\frac{dg(v)}{\tan v/2}\,\frac{\sin(m+1/2)(u-v)}{\sin(u-v)/2}=J_1+J_2\,,$$

say. Let us first estimate J_1. We write

$$J_1=\int_{1/m}^{\delta/2}\frac{dg(u)}{\tan u/2}\left\{\int_{u}^{2u}+\int_{2u}^{\delta/2}\right\}\frac{dg(v)}{v}\,\frac{\sin(m+1/2)(u-v)}{\sin(u-v)/2}=J_{11}+J_{12}\,.$$

Then

$$|J_{12}|\leqq A\int_{1/m}^{\delta/2}\frac{|dg(u)|}{u}\int_{2u}^{\delta/2}\frac{|dg(v)|}{v}\,,$$

where the inner integral becomes, by integration by parts,

$$\int_{2u}^{\delta/2}\frac{|dg(v)|}{v^2}=\left[\frac{G(v)}{v^2}\right]_{2u}^{\delta/2}+2\int_{2u}^{\delta/2}\frac{G(v)}{v^3}\,dv$$

$$\leqq\frac{A\varepsilon}{u(\log 1/u)^{\beta}}+\frac{A}{\delta(\log 1/\delta)^{\beta}}\leqq\frac{A\varepsilon}{u(\log 1/u)^{\beta}}\,.$$

Hence

$$|J_{12}| \leqq A\varepsilon \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2 (\log 1/u)^\beta}$$
$$= A\varepsilon \left[\frac{G(u)}{u^2 (\log 1/u)^\beta} \right]_{1/m}^{\delta/2} + A\varepsilon \int_{1/m}^{\delta/2} \frac{G(u)}{u^3 (\log 1/u)^\beta} du$$
$$\leqq A\varepsilon^2 \frac{m}{(\log m)^{2\beta}} + A\varepsilon^2 m \int_{1/m}^{\delta/2} \frac{du}{u (\log 1/u)^{2\beta}} \leqq A\varepsilon^2 m .$$

Concerning J_{11}, we have

$$J_{11} = \int_{1/m}^{\delta/2} \frac{dg(u)}{u} \left\{ \int_u^{u+1/m} + \int_{u+1/m}^{2u} \right\} \frac{dg(v)}{v} \frac{\sin (m+1/2)(u-v)}{\sin (u-v)/2}$$
$$= J_{111} + J_{112} ,$$

where

$$|J_{111}| \leqq Am \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u} \int_u^{u+1/m} \frac{|dg(v)|}{v}$$
$$= Am \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u} \left\{ \left[\frac{G(v)}{v} \right]_u^{u+1/m} + \int_u^{u+1/m} \frac{G(v)}{v} dv \right\}$$
$$\leqq A\varepsilon m \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u} \left\{ \frac{1}{\{\log 1/(u+1/m)\}^\beta} + \right.$$
$$\left. + \frac{1}{\{\log 1/(u+1/m)\}^\beta} \int_u^{u+1/m} \frac{dv}{v} \right\}$$
$$\leqq A\varepsilon m \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u \{\log 1/(u+1/m)\}^\beta}$$
$$\leqq A\varepsilon m \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u (\log 1/u)^\beta}$$
$$\leqq A\varepsilon m \left\{ \varepsilon \left[\frac{1}{(\log 1/u)^{2\beta}} \right]_{1/m}^{\delta/2} + \varepsilon \int_{1/m}^{\delta/2} \frac{du}{u (\log 1/u)^{2\beta}} \right\} \leqq A\varepsilon^2 m$$

and

$$|J_{112}| \leqq \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u} \int_{u+1/m}^{2u} \frac{|dg(v)|}{v(v-u)} .$$

Since $\dfrac{1}{v(v-u)} = \dfrac{1}{u} \left(\dfrac{1}{v-u} - \dfrac{1}{v} \right)$, we obtain

$$|J_{112}| \leqq \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2} \int_{u+1/m}^{2u} \frac{|dg(v)|}{v} + \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2} \int_{u+1/m}^{2u} \frac{|dg(v)|}{v-u}$$
$$= J'_{112} + J''_{112} ,$$

where

$$|J'_{112}| \leqq |J''_{112}| \leqq A\varepsilon \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2} \left\{ \left[\frac{v}{v-u} \frac{1}{(\log 1/v)^\beta} \right]_{u+1/m}^{2u} + \int_{u+1/m}^{2u} \frac{v}{(\log 1/v)^\beta} \frac{1}{(v-u)^2} dv \right\}$$

$$\leqq A\varepsilon \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u} \frac{mu}{(\log 1/u)^\beta} \leqq A\varepsilon^2 m .$$

Hence we have proved that

$$|J_1| \leqq A\varepsilon^2 m .$$

Finally we consider the remaining part

$$J_2 = \int_{1/m}^{\delta/2} \frac{dg(u)}{\tan u/2} \int_{1/m}^{u} \frac{dg(v)}{\tan v/2} \frac{\sin (m+1/2)(u-v)}{\sin (u-v)/2} .$$

Since $\dfrac{1}{v(u-v)} = \dfrac{1}{u}\left(\dfrac{1}{v} + \dfrac{1}{u-v}\right)$, we have

$$|J_2| \leqq A \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u} \int_{1/m}^{u} |dg(v)| \frac{1}{v(u-v)} |\sin (m+1/2)(u-v)|$$

$$= A \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2} \frac{|dg(v)|}{u-v} |\sin (m+1/2)(u-v)|$$

$$+ A \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2} \int_{1/m}^{u} \frac{|dg(v)|}{v} |\sin (m+1/2)(u-v)| = J_{21} + J_{22} ,$$

say, where

$$J_{21} \leqq Am \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2} \int_{1/m}^{u} |dg(v)| \leqq A\varepsilon m \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u(\log 1/u)^\beta}$$

$$\leqq A\varepsilon m ,$$

(cf. J_{111}), and

$$J_{22} \leqq A \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2} \int_{1/m}^{u} \frac{|dg(v)|}{v}$$

$$\leqq A\varepsilon \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^3 (\log 1/u)^\beta} + A\varepsilon (\log m)^{1-\beta} \int_{1/m}^{\delta/2} \frac{|dg(u)|}{u^2}$$

$$\leqq A\varepsilon^2 m + A\varepsilon^2 (\log m)^{1-\beta} \left\{ \frac{m}{(\log m)^\beta} + \int_{1/m}^{(1/m)^\Delta} \frac{du}{u^2 (\log 1/u)^\beta} + \int_{(1/m)^\Delta}^{\delta/2} \frac{du}{u^2 (\log 1/u)^\beta} \right\} , \quad (0 < \Delta < 1) ,$$

$$\leqq A\varepsilon^2 m + A\varepsilon^2 (\log m)^{1-\beta} \left\{ \frac{m}{(\log m)^\beta} + \frac{m}{(\log m)^\beta} + m^\Delta \right\}$$

$$\leqq A\varepsilon^2 m .$$

Thus we have

$$|J_2| \leqq A\varepsilon^2 m .$$

Collecting above estimations, we get

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} |P_n|^2 \leqq A\varepsilon ,$$

which completes the proof of Theorem 2.

4. Proof of Theorem 3[1].

We see that the assumption of Theorem 3

$$G(t) = \int_0^t |dg(u)| = o(t) \tag{4.1}$$

implies that

$$\int_{1/n}^{\pi} \frac{|dg(t)|}{t} = o(\log n) \tag{4.2}$$

and

$$\int_{1/n}^{\pi} \frac{|dg(t)|}{t^2} = o(n) . \tag{4.3}$$

By integration by parts, we have

$$\begin{aligned}\int_{1/n}^{\pi} \frac{|dg(t)|}{t^2} &= \left[G(t)\frac{1}{t^2}\right]_{1/n}^{\pi} - \int_{1/n}^{\pi} \frac{G(t)}{t^3}\,dt \\ &= o\left\{\left[1/t\right]_{1/n}^{\pi}\right\} + o\left\{\int_{1/n}^{\pi} \frac{dt}{t^2}\right\} = o(n) ,\end{aligned}$$

which shows (4.2). (4.3) is similarly proved.

For the proof of Theorem 3, we have

$$\begin{aligned}\sum_{n=1}^{m} |\tau_n(x)-s|^2 &= \frac{1}{4\pi^2}\sum_{n=1}^{m}\int_0^{\pi} \frac{\sin(n+1/2)t}{\sin t/2}\,dg(t)\int_0^{\pi} \frac{\sin(n+1/2)u}{\sin u/2}\,dg(u) \\ &= \frac{1}{4\pi^2}\sum_{n=1}^{m}\int_0^{\pi} \frac{\sin nt}{\tan t/2}\,dg(t)\int_0^{\pi} \frac{\sin nu}{\tan u/2}\,dg(u) \\ &\quad + \frac{1}{2\pi}\sum_{n=1}^{m} n(b_n\cos nx - a_n \sin nx)\int_0^{\pi} \frac{\sin nt}{\tan t/2}\,dg(t) \\ &\quad + \frac{1}{4}\sum_{n=1}^{m} n^2(b_n\cos nx - a_n\sin nx)^2 + o(1) \\ &= J+K+L+o(1) ,\end{aligned}$$

1) Cf. T. KAWATA [5].

where, by Lemma,

$$|L| \leqq \sum_{n=1}^{m} n^2 (a_n^2 + b_n^2) = o(m)$$

and

$$\begin{aligned} |K| &\leqq \sum_{n=1}^{m} n (|a_n| + |b_n|) \left\{ \int_0^{1/n} \frac{nt}{t} |dg(t)| + \int_{1/n}^{\pi} \frac{|dg(t)|}{t} \right\} \\ &= o\left\{ \sum_{n=1}^{m} n (|a_n| + |b_n|) \log n \right\} = o\left\{ \log m \sum_{n=1}^{m} n (|a_n| + |b_n|) \right\} \\ &= o(m \log m) . \end{aligned}$$

Hence it is sufficient for the proof to show that

$$4\pi^2 J = \sum_{n=1}^{m} \int_0^{\pi} \frac{\sin nt}{\tan t/2} dg(t) \int_0^{\pi} \frac{\sin nu}{\tan u/2} dg(u) = o(m \log m) .$$

For this purpose, we have

$$\begin{aligned} 4\pi^2 J &= \int_0^{\pi} \frac{dg(t)}{\tan t/2} \int_0^{\pi} \frac{dg(u)}{\tan u/2} \sum_{n=1}^{m} \sin nt \sin nu \\ &= \int_0^{1/m}\int_0^{1/m} + \int_0^{1/m}\int_{1/m}^{\pi} + \int_{1/m}^{\pi}\int_0^{1/m} + \int_{1/m}^{\pi}\int_{1/m}^{\pi} = J_1 + J_2 + J_3 + J_4 , \end{aligned}$$

where

$$|J_1| \leqq \int_0^{1/m} |dg(t)| \int_0^{1/m} |dg(u)| \left(\sum_{n=1}^{m} n^2 \right) = o(m^3/m^2) = o(m) ,$$

$$|J_2| \leqq \int_0^{1/m} |dg(t)| \int_{1/m}^{\pi} \frac{|dg(u)|}{u} \left(\sum_{n=1}^{m} n \right) = o(m^2 \log m/m) = o(m \log m) ,$$

$$|J_3| \leqq \int_{1/m}^{\pi} \frac{|dg(t)|}{t} \int_0^{1/m} |dg(u)| \left(\sum_{n=1}^{m} n \right) = o(m \log m)$$

and

$$\begin{aligned} J_4 &= \int_{1/m}^{\pi} \frac{dg(t)}{\tan t/2} \int_{1/m}^{\pi} \frac{dg(u)}{\tan u/2} \sum_{n=1}^{m} \left[\cos n(u-t) - \cos n(u+t) \right] \\ &= \int_{1/m}^{\pi/2} \frac{dg(t)}{\tan t/2} \int_{1/m}^{\pi} \frac{dg(u)}{\tan u/2} \frac{\sin m(u-t)}{\sin (u-t)/2} \\ &\quad + \int_{\pi/2}^{\pi} \frac{dg(t)}{\tan t/2} \int_{1/m}^{\pi} \frac{dg(u)}{\tan u/2} \frac{\sin m(u-t)}{\sin (u-t)/2} \\ &\quad - \int_{1/m}^{\pi} \frac{dg(t)}{\tan t/2} \int_{1/m}^{\pi} \frac{dg(u)}{\tan u/2} \frac{\sin m(u+t)}{\sin (u+t)/2} = J_{41} + J_{42} + J_{43} . \end{aligned}$$

We consider first J_{41}. We have

$$J_{41} = \int_{1/m}^{\pi/2} \frac{dg(t)}{\tan t/2} \left\{ \int_{|u-t|<1/2m} + \int_{|u-t|\geqq 1/2m} \right\} \frac{1}{\tan u/2} \frac{\sin m(u-t)}{\sin (u-t)/2} dg(u)$$
$$= J'_{41} + J'_{42} ,$$

where

$$|J'_{41}| \leqq Am \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \int_{|u-t|<1/2m} \frac{|dg(u)|}{u}$$
$$= Am \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \int_{t-1/2m}^{t+1/2m} \frac{|dg(u)|}{u}$$
$$= Am \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \left\{ \frac{G(t+1/2m)}{t+1/2m} - \frac{G(t-1/2m)}{t-1/2m} \right\}$$
$$+ Am \int_{1/m}^{\pi} \frac{|dg(t)|}{t} \int_{t-1/2m}^{t+1/2m} \frac{G(u)}{u^2} du$$
$$= o(m \log m) + O\left\{ m \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \log \frac{t+1/2m}{t-1/2m} \right\}$$
$$= o(m \log m) + O\left\{ m \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \right\} = o(m \log m) ,$$

since

$$1 \leqq \frac{t+1/2m}{t-1/2m} \leqq 3 \qquad \text{for} \quad 1/m \leqq t \leqq \pi/2 ,$$

and

$$|J'_{41}| \leqq A \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \int_{t+1/2m}^{\pi} \frac{|dg(u)|}{u} + A \int_{3/2m}^{\pi/2} \frac{|dg(t)|}{t} \int_{1/m}^{t-1/2m} \frac{|dg(u)|}{u(t-u)}$$
$$= A \{ J_{411} + J_{412} \} .$$

Integrating by parts, we get

$$J_{411} = \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \left[\frac{G(u)}{u(u-t)} \right]_{t+1/2m}^{\pi} + \int_{1/m}^{\pi} \frac{|dg(t)|}{t} \int_{t+1/2m}^{\pi} \frac{G(u)(2u-t)}{u^2 (u-t)^2} du$$
$$= O\left\{ \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t(\pi-t)} \right\} + o\left\{ m \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \right\} + O\left\{ \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \int_{t+1/2m}^{\pi} \frac{du}{(u-t)^2} \right\}$$
$$= O\left\{ \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \right\} + o(m \log m) + O\left\{ m \int_{1/m}^{\pi/2} \frac{|dg(t)|}{t} \right\}$$
$$= o(m \log m)$$

and we have similarly

$$J_{412} = \int_{3/2m}^{\pi/2} \frac{|dg(t)|}{t}\left[\frac{G(u)}{u(t-u)}\right]_{1/m}^{t-1/2m} + \int_{3/2m}^{\pi/2} \frac{|dg(t)|}{t}\int_{1/m}^{t-1/2m} G(u)\frac{t-2u}{u^2(t-u)^2}\,du$$

$$= O\left\{m\int_{3/2m}^{\pi/2}\frac{|dg(t)|}{t}\right\} + o\left\{\int_{32/m}^{\pi/2}\frac{|dg(t)|}{t(t-1/m)}\right\} + O\left\{\int_{3/2m}^{\pi/2}\frac{|dg(t)|}{t}\int_{1/m}^{t-1/2m}\frac{du}{u(t-u)}\right\}$$

$$= o(m\log m) + O\left(m\int_{3/2m}^{\pi/2}\frac{|dg(t)|}{t}\right) + O\left\{\int_{3/2m}^{\pi/2}\frac{|dg(t)|}{t}\left[\log\frac{u}{t-u}\right]_{1/m}^{t-1/2m}\right\}$$

$$= o(m\log m) + o(m\log m) + O\left\{\log m\int_{3/2m}^{\pi/2}\frac{|dg(t)|}{t}\right\} = o(m\log m)\,.$$

Hence we get

$$J_{41} = o(m\log m)\,.$$

The circumstances for the remaining parts J_{42} and J_{43} are very simple. We have namely,

$$|J_{42}| \leqq A\int_{\pi/2}^{\pi}\frac{|dg(t)|}{t}\int_{1/m}^{\pi}\frac{|dg(u)|}{u} = o(\log m)\,,$$

and

$$|J_{43}| \leqq A\int_{1/m}^{\pi}\frac{|dg(t)|}{t}\int_{1/m}^{\pi}\frac{|dg(u)|}{u(u+t)} = o\left(\log m\int_{1/m}^{\pi}\frac{|dg(u)|}{u^2}\right)$$

$$= o(m\log m)\,.$$

Thus we get Theorem 3.

5. Proof of Theorem 4. Since $f(t)$ is a continuous function of bounded variation, it is sufficient to prove (1.8), replaced $\tau_n(x)$ by

$$\tau_n^*(x) = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{d}{dx}\,\frac{\sin n(x-u)}{2\tan(x-u)/2}\right)f(u)\,du\,.$$

We have

$$\tau_n^*(x) - s = \frac{1}{\pi}\int_0^{\pi}\frac{\sin nt}{2\tan t/2}\,dg(t)\,.$$

We shall prove that

$$(5.1)\qquad \sum_{\nu=1}^{\infty}(\tau_\nu^*(x)-s)^2 r^n = o(1/(1-r)) \quad (r\uparrow 1)\,,$$

from which we get easily[1]

1) It follows from the inequality (valid for any $p_n \geqq 0$)

$$\sum_0^m p_n \leqq (1-1/m)^{-m}\sum_0^m p_n(1-1/m)^n \leqq 4\sum_0^\infty p_n(1-1/m)^n,\ m\geqq 2\,.$$

Cf. O. SZÁSZ [10].

$$\sum_{n=1}^{m} |\tau_n^*(x)-s|^2 = o(m) .$$

The left side of (5.1) is

$$(5.2)\quad \sum_{n=1}^{\infty}(\tau_n^*(x)-s)^2 r^n = \frac{1}{\pi^2}\sum_{n=1}^{\infty} r^n \int_0^{\pi}\int_0^{\pi} \frac{\sin nt \sin nu}{2\tan t/2 \cdot 2\tan u/2}\, dg(u)\, dg(t)$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \frac{dg(u)}{2\tan u/2}\,\frac{dg(t)}{2\tan t/2}\left(\sum_{n=1}^{\infty} r^n \sin nt \sin nu\right)$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} P(u,t,r)\, dg(u)\, dg(t) .$$

Then

$$P(u,t,r) = \frac{1}{2\tan u/2 \cdot 2\tan t/2}\sum_{n=1}^{\infty} r^n \sin nt \sin nu$$

$$= \frac{4r(1-r^2)}{\pi^2}\,\frac{\cos t/2 \cos u/2}{[1-2r\cos(u+t)+r^2][1-2r\cos(u-t)+r^2]}$$

$$> 0 \qquad (0<t<\pi,\ 0<u<\pi,\ 0<r<1) .$$

The right side integral of (5.2) is majorated by[1]

$$(5.3)\quad \int_0^{\pi}\int_0^{\pi} P(u,t,r)\,\frac{|dg(t)|^2}{dt}\,du$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\frac{1}{2\tan t/2}\,\frac{|dg(t)|^2}{dt}\int_0^{\pi}\frac{du}{2\tan u/2}\left(\sum_{n=1}^{\infty} r^n \sin nt \sin nu\right)$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\frac{1}{2\tan t/2}\,\frac{|dg(t)|^2}{dt}\cdot\frac{\pi}{2}\sum_{n=1}^{\infty} r^n \sin nt$$

$$= \frac{1}{2\pi} F'\left\{\int_0^{\pi}\frac{1}{(1-re^{it})\,2\tan t/2}\,\frac{|dg(t)|^2}{dt}\right\} .$$

The integral may be written as

$$(5.4)\quad \int_0^{1-r} + \int_{1-r}^{\pi} = O\left(\frac{1}{1-r}\int_0^{\pi}\frac{|dg(t)|^2}{t\,dt}\right) .$$

---

1) For, the integral is the limit of

$$\sum\sum P(u_i, t_j, r)\,|g(u_i)-g(u_{i-1})|\,|g(t_j)-g(t_{j-1})|$$

$$= \sum\sum \sqrt{P(u_i,t_j,r)}\,\frac{|g(u_i)-g(u_{i-1})|}{\sqrt{u_i-u_{i-1}}}\,\sqrt{t_j-t_{j-1}}$$

$$\cdot \sqrt{P(u_i,t_j,r)}\,\frac{|g(t_j)-g(t_{j-1})|}{\sqrt{t_j-t_{j-1}}}\,\sqrt{u_i-u_{i-1}}$$

$$\leqq \sum\sum P(u_i,t_j,r)\,\frac{|g(t_j)-g(t_{j-1})|^2}{t_j\, t_{j-1}}\,(u_i-u_{i-1}) ,$$

which tends to the integral (5.3).

Since the relation (1.8) depends only on the local property of $f(t)$ at $t=x$, we can suppose, from the begining, that $f(t)=0$ for $|x-t|>\delta$, and then it may be supposed that the integral of (5.4) in the right side is sufficiently small, by the condition (1.9). Thus we have proved (5.1).

6. Proof of Theorem 5. From (5.2), it is sufficient to prove that

$$(6.1)\qquad \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \frac{(1-r)^2\,|\,dg(u)\,|\,|\,dg(v)\,|}{[(1-r)^2+(u-v)^2][(1-r)^2+(u+v)^2]} = o(1) \quad (r\uparrow 1)\,.$$

By the symmetry, it suffices to consider the integral

$$\int_0^{\pi/2} du \int_u^{\pi/2} dv\,.$$

Let D_k be the set

$$0\leqq u\leqq \pi\,,\qquad 2^{k-1}(1-r)\leqq v-u\leqq 2^k(1-r)\,,$$

contained in the triangle $0\leqq u\leqq\pi/2$, $u\leqq v\leqq\pi/2$.

The integral (6.1) on D_k is less than

$$\iint_{D_k} \frac{|\,dg(u)\,|\,|\,dg(v)\,|}{2^{2(k-1)}[(1-r)^2+u^2]} \leqq \frac{1}{2^{2(k-1)}}\int_0^{\pi/2} \frac{|\,dg(u)\,|}{[(1-r)^2+u^2]} \int_{u+2^{k-1}(1-r)}^{u+2^k(1-r)} |\,dg(v)\,|$$

$$\leqq \frac{1-r}{2^{k-2}}\, o\left\{\int_0^{\pi/2} \frac{|\,dg(u)\,|}{(1-r)^2+u^2}\right\} = \frac{1-r}{2^{k-1}}\, o\left\{\int_0^{1-r} + \int_{1-r}^{\pi}\right\} = I_k + J_k\,,$$

where

$$I_k = \frac{1}{2^{k-1}}\, o\left\{\frac{1}{1-r}\int_0^{1-r} |\,dg(u)\,|\right\}$$

and

$$J_k = \frac{1}{2^{k-1}}\, o\left\{\frac{1}{1-r}\int_0^{1-r} |\,dg(u)\,|\right\} + \frac{1}{2^{k-1}}\, o\left\{(1-r)\int_{1-r}^{\pi} \frac{1}{u^2}\int_0^u |\,dg(u)\,|\right\}$$

o being uniform in k. Thus, summing up above estimations, we get (6.1). Thus the theorem is proved.

7. Proof of Theorem 7. As usual if we put

$$\overline{D}_n^*(t) = (1-\cos nt)/2\tan t/2\,,$$

then

$$(7.1)\qquad \bar{\tau}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\,\frac{d}{dx}\,\overline{D}_n^*(u-x)\,du$$

$$-\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\,\frac{d}{du}\left[-\frac{1}{2}\sin n(u-x)\right]du$$

$$= I(n)+J(n)\,.$$

We easily see that

$$J(n) = -\frac{1}{2}(na_n \cos nx + nb_n \sin nx) .$$

Hence we have, by the lemma,

$$\sum_{n=1}^{m} |J(n)|^k \leqq A \sum_{n=1}^{m} (|na_n|^k + |nb_n|^k) = o(m) . \tag{7.2}$$

For the first part of (7.1), we have, for $n \leqq m$,

$$\begin{aligned} I(n) &= -\frac{1}{\pi}\int_0^{\pi} \frac{d}{dt}\overline{D}_n^*(t)\left\{f(x+t)+f(x-t)\right\} dt \\ &= -\frac{1}{\pi}\int_0^{\pi} \overline{D}_n^*(t)\, dh(t) \\ &= -\frac{1}{\pi}\left\{\int_0^{1/m} + \int_{1/m}^{\pi}\right\} = I_1(n)+I_2(n) . \end{aligned}$$

By the assumption, we have

$$I_1(n) = O\left\{\int_0^{1/m} (nt)/t\,|dh(t)|\right\} = O\left\{n\int_0^{1/m} |\,dh(t)|\right\} = o(1)$$

and

$$\begin{aligned} I_2(n) &= -\frac{1}{2\pi}\int_{1/m}^{\pi} \cot\frac{t}{2}\, dh(t) + \frac{1}{2\pi}\int_{1/m}^{\pi} \cot\frac{t}{2}\cos nt\, dh(t) \\ &= -\frac{1}{2\pi}\left[\cot\frac{t}{2}\, h(t)\right]_{1/m}^{\pi} - \frac{1}{2\pi}\int_{1/m}^{\pi} \frac{1}{2}\operatorname{cosec}^2\frac{t}{2}\, h(t)\, dt \\ &\qquad + \frac{1}{2\pi}\int_{1/m}^{\pi} \cot\frac{t}{2}\cos nt\, dh(t) = G_m + H_m + L_{mn} \end{aligned}$$

where

$$G_m = -\frac{1}{2\pi}\cot\frac{1}{2m}\, h\left(\frac{1}{m}\right) = o(1) ,$$

for $h(t)/t \to 0$ with t, as $f'(x)$ exists.

Collecting above results, we get

$$\bar{\tau}_n(x) - H_m = L_{mn} + J + o(1) .$$

By virtue of (7.2), it is sufficient to show that

$$\sum_{n=1}^{m} |\,L_{mn}|^k = o(m) .$$

For any $\varepsilon > 0$, there is a δ such that

$$\int_0^t |dh(u)| < \varepsilon t \Big/ \left(\log \frac{1}{t}\right), \qquad (0<t<\delta).$$

Let us put

$$h(u) = h_1(u)+h_2(u),$$

where

$$\begin{aligned} h_1(u) &= h(u) && \text{in } (0, \delta/2), \\ &= 0 && \text{in } (\delta, \pi) \end{aligned}$$

and $h_1(u)$ is linear in $(\delta/2, \delta)$ and is continuous in $(0, \pi)$. Hence $h_2(u)$ is also a continuous function of bounded variation which vanishes in $(0, \delta/2)$. So we have

$$\begin{aligned} L_{mn}(x) &= \frac{1}{2\pi}\int_{1/m}^{\pi} \cot\frac{t}{2}\cos nt\, dh(t) \\ &= \frac{1}{2\pi}\int_{1/m}^{\pi} \cot\frac{t}{2}\cos nt\, dh_1(t) + \frac{1}{2\pi}\int_{1/m}^{\pi} \cot\frac{t}{2}\cos nt\, dh_2(t) \\ &= \bar{P}_n+\bar{Q}_n . \end{aligned}$$

Since $\bar{Q}_n$ is n times of the n-th Fourier coefficient of a continuous function of bounded variation, we have, by Lemma,

$$\sum_{n=1}^{m} |\bar{Q}_n|^k = o(m).$$

However

$$\begin{aligned} \bar{P}_n &= \frac{1}{2\pi}\int_{1/m}^{\delta/2} \cot\frac{t}{2}\cos nt\, dh(t)+o(1) \\ &= \bar{S}_n+o(1). \end{aligned}$$

Hence it suffices to show that

$$\bar{T}_m = \sum_{n=1}^{m} |\bar{S}_n|^k = o(m).$$

Similarly as in the proof of Theorem 1, we put

$$\bar{c}_n = |\bar{S}_n|^{k-1}\operatorname{sgn}\bar{S}_n, \qquad \bar{A}_m(t) = \sum_{n=1}^{m} \bar{c}_n \cos nt.$$

and

$$\bar{T}_m = \sum_{n=1}^{m} |\bar{c}_n| .$$

Then we have

$$\bar{T}_m^k = \sum_{n=1}^{m} \bar{c}_n \bar{S}_n = \sum_{n=1}^{m} \bar{c}_n \cdot \frac{1}{2\pi} \int_{1/m}^{\delta/2} \cot \frac{t}{2} \cos nt \, dh(t)$$
$$= \frac{1}{2\pi} \int_{1/m}^{\delta/2} \left(\sum_{n=1}^{m} \bar{c}_n \cos nt \right) \cot \frac{t}{2} \, dh(t)$$
$$= \frac{1}{2\pi} \int_{1/m}^{\delta/2} \bar{A}_m(t) \cot \frac{t}{2} \, dh(t) .$$

Hence

$$\bar{T}_m^k \leqq \bar{\Gamma}_m \int_{1/m}^{\delta} \cot \frac{t}{2} \, |dh(t)| = o(\bar{\Gamma}_m) + o\left\{ \bar{\Gamma}_m \int_{1/m}^{\delta} \frac{dt}{t (\log 1/t)^{\alpha}} \right\}$$
$$= o(\bar{\Gamma}_m) + o(\bar{T}_m^{k/k'} m^{1/k}) .$$

Thus we get

$$\bar{T}_m^k = o(m) ,$$

which is the required.

8. **Proof of Theorem 8.** As in the proof of Theorem 7, it is sufficient to show that

$$\sum_{n=1}^{m} |\bar{S}_n|^2 = \sum_{n=1}^{m} \left| \frac{1}{2\pi} \int_{1/m}^{\delta/2} \frac{\cos nt}{\tan t/2} \, dh(t) \right|^2 = o(m) .$$

For this purpose we have

$$\left| \int_{1/m}^{\delta/2} \frac{\cos nt}{\tan t/2} \, dh(t) \right|^2 = \int_{1/m}^{\delta/2} \frac{\cos nt}{\tan t/2} \, dh(t) \int_{1/m}^{\delta/2} \frac{\cos nu}{\tan u/2} \, dh(u)$$
$$= \frac{1}{2} \int_{1/m}^{\delta/2} \int_{1/m}^{\delta/2} \frac{dh(t)}{\tan t/2} \frac{dh(u)}{\tan u/2} \left\{ \cos n(u-t) + \cos n(u+t) \right\}.$$

Hence we get

$$\sum_{n=1}^{m} |\bar{S}_n|^2 = \frac{1}{2(2\pi)^2} \int_{1/m}^{\delta/2} \int_{1/m}^{\delta/2} \frac{dh(t)}{\tan t/2} \frac{dh(u)}{\tan u/2} \left\{ \frac{\sin (m+1/2)(u-t)}{2 \sin (u-t)/2} \right.$$
$$\left. + \frac{\sin (m+1/2)(u+t)}{2 \sin (u+t)/2} \right\} + o(m).$$

Thus the required result shall be obtained by the same argument used in the proof of Theorem 2.

References

[1] H. C. CHOW: Criteria for the strong summiablity of the derived Fourier series and its conjugate series, Journ. London Math. Soc. 31 (1956), 57–64.

[2] G. H. HARDY and J. E. LITTLEWOOD: On the strong summability of Fourier series, Proc. London Math. Soc. 26 (1926–7), 273–286.

[3] G. H. HARDY and J. E. LITTLEWOOD: The strong summability of Fourier series, Fund. Math. 25 (1935), 162–189.

[4] S. IZUMI and M. KINUKAWA: On the strong summability of the derived Fourier series II, Proc. Japan Acad. 31 (1955), 107–110.

[5] T. KAWATA: A proof of a theorem of HARDY-LITTLEWOOD, Proc. Imp. Acad. Japan, 15 (1939), 243–246.

[6] M. KINUKAWA: On the strong summability of the derived Fourier series, Proc. Japan Acad. 30 (1954), 801–804.

[7] B. N. PRASAD and U. N. SINGH: On the strong summability of the derived Fourier series and its conjugate series, Math. Zeits. 56 (1952), 280–288.

[8] F. T. WANG: Note on H_2 summability of Fourier series, Jour. London Math. Soc. 19 (1944), 208–209.

[9] A. ZYGUMUND: Trigonometrical series.

[10] O. SZÁSZ: On strong summability of Fourier series, Trans. American Math. Soc. 48 (1940), 117–125.

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