CONTINUOUS FILTERING AND ITS SPECTRAL SEQUENCE

By

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0. A filtering f of a ring A is a integer valued function on A satisfying the following three conditions:

(0.1) $$f(x+y) \geq \min\{f(x), f(y)\}, \qquad x, y \in A,$$

(0.2) $$f(xy) \geq f(x)+f(y),$$

(0.3) $$f(0) = +\infty.$$

Thus, the notion of filtering can be regarded as a generalization of discrete valuation of a field. For purely algebraic interest, it seems to be natural to consider a continuous filtering as the generalization of continuous valuation.

In this note, we consider a real valued function F on A satisfying the above three conditions. We call F a *continuous filtering* of A, and the ring A is said to be a *continuously filtered ring.*

Sections 1 and 2 are devoted to describe analogous definitions notations and relations to those of J. Leray [1], and the main parts of this note are sections 3 and 4.

1. A ring A is called a *continuosly graded ring* if

$$A = \sum_{p \in R} A^{[p]} \qquad \text{(direct sum, } R \text{ is the set of reals)}$$

where $\{A^{[p]}\}$ are submodules of A and satisfy

$$A^{[p]} \cdot A^{[q]} \subset A^{[p+q]}.$$

A continuously filtered ring A is called a *continuosly filtered differential ring* if A has a differentiation (d, a) subjected to

$$d^2 = 0,$$
$$adx + dax = 0, \qquad x, y \in A,$$
$$d(xy) = dx \cdot y + ax \cdot dy, \qquad (a \text{ is an automorphism of } A),$$

and

$$F(ax) = F(x).$$

A differentiation (d, a) is called homogeneous of degree r $(r \in R)$ if

(d, a) is a differentiation of a continuously graded ring A and

$$dA^{[p]} \subset A^{[p+r]} \qquad \text{for any } p \in R.$$

If B is an ideal of a continuously filtered ring A, then A/B becomes a continuously filtered ring if we define

$$\bar{F}(\bar{x}) = \sup_{x \in \bar{x}} F(x) \qquad \text{for } \bar{x} \in A/B. \tag{1.1}$$

2. From now on, A means a continuously filtered differential ring. We set

$$A^p = \{x \mid x \in A,\ F(x) \geq p\} \qquad p \in R,$$

then this is a submodule of A, and

$$A^p \subset A^q \quad \text{if } p \geq q, \qquad \bigcup_p A^p = A,$$
$$A^p \cdot A^q \subset A^{p+q}.$$

Define, for $\varepsilon > 0$,

$$G_\varepsilon(A) = \sum_p A^p / A^{p+\varepsilon} \qquad \text{(direct sum)}$$

and define the multiplication by

$$(x^p \bmod A^{p+\varepsilon})(x^q \bmod A^{q+\varepsilon}) = x^p \cdot x^q \bmod A^{p+q+\varepsilon},$$

Then $G_\varepsilon(A)$ becomes a continuously graded ring, called the *ε-graded ring* of A. If we put

$$C = \text{kernel of } d, \qquad D = \text{image of } d,$$
$$C^p = A^p \cap C, \qquad D^p = A^p \cap D,$$
$$C_r^p = \{x \mid x \in A^p,\ dx \in A^{p+r}\}, \qquad dC_r^p = D_r^{p+r},$$

then we have

$$D_r^p \subset D_{r+\varepsilon}^p, \qquad \bigcup_{r \in R} D_r^p = D^p, \qquad D^p \subset C^p, \qquad C_{r+\varepsilon}^p \subset C_r^p, \tag{2.1}$$

$$C_{r-\varepsilon}^{p+\varepsilon} = C_r^p \cap A^{p+\varepsilon} \subset C_r^p, \tag{2.2}$$

$$D_{r-\varepsilon}^{p+\varepsilon} = D_r^p \cap A^{p+\varepsilon} \subset D_r^p, \tag{2.3}$$

$$C_r^p \cdot C_r^q \subset C_r^{p+q}, \tag{2.4}$$

$$C_r^p \cdot D_{r-\varepsilon}^q \subset C_{r-\varepsilon}^{p+q+\varepsilon} + D_{r-\varepsilon}^{p+q}, \qquad D_{r-\varepsilon}^q \cdot C_r^p \subset C_{r-\varepsilon}^{d+q+\varepsilon} + D_{r-\varepsilon}^{p+q}, \tag{2.5}$$

(2.4) implies that $\sum_{p \in R} C_r^p$ (direct sum of modules C_r^p) can be considered to be a continuously graded ring, while (2.5) means that

$$\sum_{p \in R} (^{p+\varepsilon}_{r-\varepsilon} + D_{r-\varepsilon}^p) \qquad \text{(direct sum)}$$

is an ideal of $\sum_p C_r^p$.

We define

$$H_{r,\varepsilon}(A)=\sum_p C_r^p/(C_{r-\varepsilon}^{p+\varepsilon}+D_{r-\varepsilon}^p)\,.$$

Then $H_{r,\varepsilon}(A)$ has a differentiation $(d_{r,\varepsilon}, a_{r,\varepsilon})$ of homogeneus of degree r by

$$d_{r,\varepsilon}h_r^{[p]}=dc_r^p \bmod (C_{r-\varepsilon}^{p+r+\varepsilon}+D_{r-\varepsilon}^{p+r})\,,$$

$$a_{r,\varepsilon}h_r^{[p]}=ac_\varepsilon^p \bmod (C_{r-\varepsilon}^{p+\varepsilon}+D_{r-\varepsilon}^p)\,,$$

where $h_r^{[p]}\in H_{r,\varepsilon}(A)$ is homogeneous of degree p and $c_r^a\in h_r^{[p]}$. Next, we define the cohomology ring of $H_{r,\varepsilon}(A)$, we use the notation $H(H_{r,\varepsilon}(A))$. A parallel argument to that of J. Leray [1] Chap. I, § 9 shows that

$$C(H_{r,\varepsilon}(A))=\text{kernel of } d_{r,\varepsilon}=\sum_p (C_{r+\varepsilon}^p+C_{r-\varepsilon}^{p+\varepsilon})/(C_{r-\varepsilon}^{p+\varepsilon}+D_{r-\varepsilon}^p)\,,$$

$$D(H_{r,\varepsilon}(A))=\text{image of } d_{r,\varepsilon}=\sum_p (C_{r-\varepsilon}^{p+\varepsilon}+D_{r-\varepsilon}^p+D_r^p)/(C_{r-\varepsilon}^{p+\varepsilon}+D_{r-\varepsilon}^p)$$

$$=\sum_p (C_{r-\varepsilon}^{p+\varepsilon}+D_r^p)/(C_{r-\varepsilon}^{p+\varepsilon}+D_{r-\varepsilon}^p)\,,$$

whence

$$H(H_{r,\varepsilon}(A))=\sum_p (C_{r-\varepsilon}^p+C_{r-\varepsilon}^{p+\varepsilon})/(C_{r-\varepsilon}^{p+\varepsilon}+D_r^p)$$

$$=\sum_p C_{r+\varepsilon}^p/\{C_{r+\varepsilon}^p\cap(C_{r-\varepsilon}^{p+\varepsilon}+D_r^p)\}$$

$$=\sum_p C_{r+\varepsilon}^p/(C_r^{p+\varepsilon}+D_r^p)=H_{r+\varepsilon,\varepsilon}(A)\,.$$

3. In this section, we proceed to define an inverse mapping system of $\{H_{r,\varepsilon}(A)\}_{R\ni\varepsilon>0}$ and consider the projective limit of this system. Since

$$C_{r-\sigma}^{p+\sigma}+D_{r-\sigma}^p\supset C_{r-\tau}^{p+\tau}+D_{r-\tau}^p$$

for $0<\sigma<\tau$, we can define a natural inverse mapping π_σ^τ:

$$\pi_\sigma^\tau:\quad H_{r,\tau}(A)\to H_{r,\sigma}(A)\,.$$

The projective limit of this system is denoted by

(3.1) $$p\text{-}\lim_\sigma H_{r,\sigma}(A)=H_r\,,$$

and we define a differentiation (d_r, a_r) by

$$d_rh_r=(\cdots, d_{r,\sigma}h_{r,\sigma},\cdots)\,,$$

$$a_rh_r=(\cdots, a_{r,\sigma}h_{r,\sigma},\cdots)$$

for $$h_r=(\cdots, h_{r,\sigma},\cdots)\in H_r\ (\pi_\sigma^\tau h_{r,\tau}=h_{r,\sigma})\,.$$

It is easy to see that the above definition of (d_r, a_r) has no incovenience. Also we can define naturally an inverse system of $\{H_{r+\sigma,\sigma}(A)\}$ and the projective limit

(3.2) $$p\text{-}\lim_\sigma H_{r+\sigma,\sigma}(A)\,,$$

because of

$$C^p_{r+\sigma}\supset C^p_{r+\tau}\,,\qquad C^{p+\sigma}_r+D^p_r\supset C^{p+\tau}_r+D^p_r\,.$$

For (3.1) and (3.2), the following relation is true:

$$H(H_r)=H(p\text{-}\lim_\sigma H_{r,\sigma}(A))=p\text{-}\lim_\sigma\,(H(H_{r,\sigma}(A)))$$
$$=p\text{-}\lim_\sigma H_{r+\sigma,\sigma}(A)\,.$$

For the proof, a straightforward computation shows that

$$C(H_r)=\text{kernel of } d_r=p\text{-}\lim_\sigma C(H_{r,\sigma}(A))$$
$$D(H_r)=\text{image of } d_r=p\text{-}\lim_\sigma D(H_{r,\sigma}(A))\,,$$

so that we get

$$H(H_r)=C(H_r)/D(H_r)\cong p\text{-}\lim_\sigma\,\{C(H_{r,\sigma}(A))/D(H_{r,\sigma}(A))\}$$
$$=p\text{-}\lim_\sigma H(H_{r,\sigma}(A))=p\text{-}\lim_\sigma H_{r+\sigma,\sigma}(A)$$

(π^τ_σ induce the natural inverse system of $C(H_{r,\sigma}(A))/D(H_{r,\sigma}(A))$).

4. We define another continuously graded ring

$$H_{\infty,\sigma}(A)=\sum_p C^p/(C^{p+\sigma})+D^p)\,.$$

Then we have

$$H_{\infty,\sigma}(A)=G_\sigma(H(A))\,, \tag{4.1}$$

where $H(A)$ is the cohomology ring of A with the filtering defined as (1.1). The proof is analogous to that for discrete filtration and is omitted.

Next we consider

$$I_{r,\sigma}=\sum_p\,(\bigcap_{n>0} C^p_{r+n\sigma})/(C^{p+\sigma}_{r-\sigma}+D^p_{r-\sigma})$$

and an ideal of $I_{r,\sigma}$

$$J_{r,\sigma}=\sum_p\,(C^{p+\sigma}+D^p)/(C^{p+\sigma}_{r-\sigma}+D^p_{r-\sigma})\,.$$

Then we have easily

$$I_{r,\sigma}/J_{r,\sigma}\cong I_{r+t,\sigma}/J_{r+t,\sigma}\qquad\text{for } t>\sigma,$$

therefore we identify all $I_{\sigma+t,\sigma}$, and denote

$$\lim_{r\to\infty} H_{r,\sigma}(A)\,.$$

An analogous relation to (4.1) holds

$$G_\sigma(H(A))\subset\lim_{r\to\infty} H_{r,\sigma}(A)\,.$$

Again, if we use the natural inverse system, then we get

$$p\text{-}\lim_\sigma\,(G_\sigma(H(A)))\subset p\text{-}\lim_\sigma\,(\lim_{r\to\infty} H_{r,\sigma}(A))\,.$$

5. In 3 and 4, we defined two limits of $H_{r,\sigma}(A)$, $p\text{-}\lim_{\sigma}$ and $\lim_{r\to\infty}$. These two operations are not commutative, because $p\text{-}\lim_{\sigma}(\lim_{r\to\infty} H_{r,\sigma}(A))$ can be always defined, while $\lim_{r\to\infty}(p\text{-}\lim_{\sigma} H_{r,\sigma}(A))$ cannot be defined so far as we use only the natural procedure.

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References

[1] J. Leray, L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue, Jour. de Math. pures et appl. VIII, 29 (1950), 1-139.

[2] H. Cartan and S. Eilenberg, Homological Algebra, Princeton, (1956).