ON $*$-MODULAR RIGHT IDEALS OF AN ALTERNATIVE RING

By

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It is well known that, in any alternative ring A, the Smiley radical SR(A) is contained in every modular maximal right ideal M. E. Kleinfeld has shown that every primitive alternative, non-associative ring is a Cayley-Dickson algebra.

Now we introduce the notion of $*$-modularity as follows: a right ideal I of an alternative ring A is called $*$-modular if there exist two elements a, $u \in A$ such that

$$(1) \qquad x+ax+(a, x, u) \in I$$

for all $x \in A$, where (a, x, u) denotes the associator $ax \cdot u - a \cdot xu$ of a, x, u, and in this case we call a a *left $*$-modulo unit of I.* Clearly, modularity implies $*$-modularity.

In this note, we shall show that the above results are also true if we replace modular ideals by $*$-modular ideals.

If a ring A is assumed to be alternative, then (a, b, c) becomes a skew-symmetric function of its three variables.

The Smiley radical SR(A) of an alternative ring A is defined as the totality of elements $z \in A$ for which each element of $(z)_r$ is right quasi-regular.

In the next lemma we develop an important property of $*$-modular right ideals.

Lemma 1. *Let I^* be a $*$-modular right ideal of an alternative ring A, and suppose that a left $*$-modulo unit a of I^* is right quasi-regular. Then $I^*=A$.*

Proof. Let b be a right quasi-inverse of a:

$$(2) \qquad a+b+ab=0\,.$$

Since a is a left $*$-modulo unit of I^* and since $(a, a, u)=0$, we have $a+a^2 \in I^*$ by putting $x=a$ in (1), while $(a+a^2)b-(a, b, u)=ab+a^2b-(a, a, u)-(a, b, u)=ab+a^2b-(a, a+b, u) = ab+a^2b+(a, ab, u) \in I^*$ by (2). Hence it follows that $(a, b, u) \in I^*$. On the other hand, if we put $x=b$ in (1), we have $b+ab+(a, b, u)\in I^*$, whence $b+ab\in I^*$. Thus $a\in I^*$, which implies together with (1) that every $x\in A$ is in I^*, that is, $I^*=A$.

As any modular maximal right ideal is a member of a set of $*$-modular maximal right ideals, the intersection of all the modular maximal right ideals is contained in the intersection of modular maximal right right ideals.

Now, we show a connection between the intersection of all the $*$-modular maximal right ideals and the radical SR(A) in an alternative ring A.

Theorem 1. *Let A be an alternative ring. Then the Smiley radical $SR(A)$ is contained in the intersection of all the $*$-modular maximal right ideals M^*:*

$$\mathrm{SR}(A)\subseteq \bigcap M^*.$$

Proof. Let $z\in A$ be an element not contained in the intersection $\bigcap M^*$. Then there exists a $*$-modular maximal right ideal M^* does not contain z. And we have $A=M^*+(z)_r$. Let a, u be elements such that $x+ax+(a, x, u)\in M^*$ for all $x\in A$, and let m^* and z' be elements of M^* and $(z)_r$ respectively such that $a=m^*+z'$. Then $x+z'x+(z', x, u)=x+(a-m^*)x+(a-m^*, x, u)=x+ax+(a, x, u)-m^*x+(m^*, x, u)\in M^*$ for all $x\in A$. Thus z' is also a left $*$-modulo unit of M^*. But, since $M^*\neq A$, z' is not right quasi-regular by Lemma 1, and so z' is not in SR(A). This proves our theorem.

Next we refer to the structure of $*$-primitive alternative ring.

An alternative ring is defined to be $*$-primitive in case it contains a $*$-modular maximal right ideal whose quotient is zero.

Lemma 2. *The quotient $(I^*:A)=\{x\in A;\ Ax\subseteq I^*\}$ of a $*$-modular right ideal I^* is an ideal of A.*

Proof. The $*$-modularity of I^* assures the existence of $a, u\in A$ with the property that $x+ax+(a, x, u)\in I^*$ for every $x\in A$. Since $ax\in I^*$ for $x\in(I^*:A)$, we have $x+(a, x, u)=x+a\cdot ux-au\cdot x\in I^*$. And further $ux+a\cdot ux+(a, ux, u)\in I^*$. Combining this with $ux\in I^*$, we obtain $a\cdot ux\in I^*$ and eventually $x\in I^*$. Hence, for any $y\in A$, $A\cdot xy$ and $A\cdot yx$ are both in I^*.

By the light of this lemma, it is clear that A is $*$-primitive if and only if A has a $*$-modular maximal right ideal which contains no nonzero two-sided ideals.

An alternative ring is called simple if it has no nonzero proper two-sided ideals and is not a nil ring.

The following lemma is due to E. Kleinfeld [1].

Lemma 3. *A simple alternative ring is either a Cayley-Dickson algebra or associative.*

Most results in primitive alternative rings which were stated in [2] are also true in our *-primitive case under a slight modification of the modularity.

We obtain the following:

Theorem 2. *Every *-primitive, alternative, non-associative ring A is a Cayley-Dickson algebra.*

Proof. We may prove, with the help of the proof in primitive case [2], that every *-modular maximal right ideal M^* of A is zero. And so, it is enough only to show that A is a simple alternative ring. For any left *-modulo unit a of M^*, $a+a^2=0$ and then $a^n=\pm a$ for every integer $n\geq 1$. It shows that a is not nilpotent, and hence A is simple by Lemma 3. Therefore, A is a Cayley-Dickson algebra.

Bibliography

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