ON THE RATIOS OF THE NORMS DEFINED BY MODULARS

By

Tetsuya SHIMOGAKI

§1. Let R be a *modulared semi-ordered linear space* and $m(x)$ $(x \in R)$ be a *modular*[1)] on R. Since $0 \leq m(\xi x)$ is a non-trivial convex function of real number $\xi \geq 0$ for every $0 \neq x \in R$, we can define two kinds of norms by the modular m as follows:

$$\|x\| = \inf_{\xi>0} \frac{1+m(\xi x)}{\xi}, \quad |||x||| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|} \quad (x \in R). \tag{1.1}$$

The former of them is said to be the *first norm* by m and the latter to be the *second* (or *modular*) *norm* by m.

Let $\overline{R}^m$ be the *modular conjugate space* of R and $\overline{m}$ be the *conjugate modular*[2)] of m. Then we can also define the norms on $\overline{R}^m$ by $\overline{m}$ as above. It is well-known [4; §40] that *if R is semi-regular*[3)] *the first norm by the conjugate modular $\overline{m}$ is the conjugate one of the second norm by m and the second norm by $\overline{m}$ is the conjugate one of the first norm by m.* Since $\|\cdot\|$ and $|||\cdot|||$ are semi-continuous, they are reflexive [3]. We have always $|||x||| \leq \|x\| \leq 2|||x|||$ for all $x \in R$, that is, $1 \leq \frac{\|x\|}{|||x|||} \leq 2$ for all $0 \neq x \in R$.

When the ratios of these two norms are equal to a constant number, i.e. $\frac{\|x\|}{|||x|||} = \gamma$ holds for each $0 \neq x \in R$, S. Yamamuro [8] and I. Amemiya [1] succeeded in showing that the modular m is of L^p-type essentially, i.e. $m(\xi x) = \xi^p m(x)$ for all $x \in R$ and $\xi \geq 0$, where $1 \leq p$.

In the earlier paper [7] the author investigated the case that the

1) For the definition of a modular see [4]. The notations and terminologies used here are the same as in [4 or 7].

2) $\overline{R}^m$ is the totality of all linear functionals $\overline{a}$ on R such that $\inf_{\lambda \in \Lambda} |\overline{a}(x_\lambda)| = 0$ for every $x_\lambda \downarrow_{\lambda \in \Lambda} 0$ and $\sup_{m(x) \leq 1} |\overline{a}(x)| < +\infty$. The conjugate modular $\overline{m}$ of m on $\overline{R}^m$ is defined as

$$\overline{m}(\overline{a}) = \sup_{x \in R} \{\overline{a}(x) - m(x)\} \quad (\overline{a} \in \overline{R}^m).$$

3) R is said to be *semi-regular*, if $\overline{a}(x) = 0$ for all $\overline{a} \in \overline{R}^m$ implies $x = 0$.

ratios satisfy the condition:

$$\inf_{0 \neq x \in R} \frac{\|x\|}{|||x|||} > 1, \tag{1.2}$$

and proved that it is equivalent to uniform finiteness of both m and $\overline{m}$, provided that R is non-atomic.

In an Orlicz space $L_{\Phi}^{*}(G)$[4], which is one of the concrete examples of modulared semi-ordered linear spaces, the similar results concerning the ratios were found independently by D. V. Salekhov in [6] under more restricted circumstances.

In this paper we shall consider the following conditions on the ratios of the norms by a modular m:

$$\frac{\|x\|}{|||x|||} < 2 \quad \text{for all } 0 \neq x \in R; \tag{1.3}$$

or

$$\sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||} < 2, \tag{1.4}$$

and study their relations to the properties of the modular m. We shall show in § 2 that *if the condition* (1.3) *is satisfied, then either* $m(\xi x) < \xi^2 m(x)$ (*for all* $\xi > 1$ *and* $x \in R$ *with* $m(x) \geq 1$), *or* $m(\xi x) > \xi^2 m(x)$ (*for all* $\xi > 1$ *and* $x \in R$ *with* $+\infty > m(x) \geq 1$) *holds, provided that* R *is non-atomic.* And as for (1.4) we shall show in § 3 that (1.4) *implies that either* $m(\xi x) \leq \xi^p m(x)$ *for all* $\xi \geq 1$ *and* $x \in R$ *with* $m(x) \geq 1$ *or* $m(\xi x) \geq \xi^{p'} m(x)$ *for all* $\xi \geq 1$ *and* $x \in R$ *with* $m(x) \geq 1$ *holds, where* p, p' *are real numbers with* $1 \leq p < 2 < p' \leq +\infty$, *provided that* R *is non-atomic.*

The difference between the conditions (1.2) and (1.4) exists in the point of their topological properties, that is, the former of them remains valid for any modular m' equivalent[5] to the original one except a finite dimensional space, but the later dose not hold in general. Thus we can not obtain the explicit conditions equivalent to (1.4) with respect to the modular m in general case. For a modular of unique spectra, however, we shall estimate $\sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||}$ and $\inf_{0 \neq x \in R} \frac{\|x\|}{|||x|||}$ exactly in § 4 by applying the results obtained in §§ 2 and 3.

Throughout this paper we denote by R a *modulared semi-ordered linear space* and by m a *modular* on R. For any $p \in R$ we denote by $[p]$

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4) For the definition of Orlicz space $L_{\Phi}^{*}(G)$ see [2] or [9].

5) Two modulars m and m' on R are called *equivalent*, if their norms are equivalent to each other.

a *projection operator defined by* $p:[p]x=\bigcup_{n=1}^{\infty}(n|p|\frown x)$ for all $0\leq x\in R$. R is called to be *non-atomic*, if any $0\neq a\in R$ is decomposed into $a=b+c$ such that $|b|\frown|c|=0$, $b\neq 0$ and $c\neq 0$. Since $m(x+y)=m(x)+m(y)$ for any $x, y\in R$ with $|x|\frown|y|=0$, $a\in R$ with $m(a)<+\infty$ can be decomposed into $a=[p]a+(1-[p])a$ for some $p\in R$ such that $m([p]a)=m((1-[p])a)$, if R is non-atomic. Here we note that $m(\xi x)$ is a *continuous function of* $\xi\in[0,\eta]$ *for each* $x\in R$, if $m(\eta x)<+\infty$, because $m(\xi x)$ is a positive convex function of $\xi\geq 0$ for each $x\in R$.

§2. We put for every $x\in R$ with $m(x)<+\infty$

(2.1) $$\pi_{+}(x)=\inf_{\varepsilon>0}\frac{1}{\varepsilon}\{m((1+\varepsilon)x)-m(x)\}$$

and

(2.2) $$\pi_{-}(x)=\sup_{\varepsilon>0}\frac{1}{\varepsilon}\{m(x)-m((1-\varepsilon)x)\},$$

and for $x\in R$ with $m(x)=+\infty$ we put

(2.3) $$\pi_{+}(x)=\pi_{-}(x)=+\infty .$$

Then it follows from the definitions that $0\leq\pi_{-}(x)\leq\pi_{+}(x)$ for all $x\in R$ and *both* $\pi_{+}(\xi x)$ *and* $\pi_{-}(\xi x)$ *are non-decreasing functions of* $\xi\geq 0$ *for every* $x\in R$ *and are orthogonally additive, that is,* $\pi_{\pm}(x+y)=\pi_{\pm}(x)+\pi_{\pm}(y)$ *if* $x\perp y$[6], $x, y\in R$. Furthermore $\pi_{+}(\xi x)$ is a *right-hand continuous function of* $\xi\geq 0$ *for every* $x\in R$, since $m(\xi x)$ is a convex function of $\xi\geq 0$. In fact, we have for each $\xi_0\geq 0$ $\lim_{\xi\downarrow\xi_0}\pi_{+}(\xi x)=\inf_{\xi>\xi_0}\pi_{+}(\xi x)=\inf_{\xi>\xi_0}\left[\inf_{\varepsilon>0}\frac{1}{\varepsilon}\{m((1+\varepsilon)\xi x)-m(\xi x)\}\right]=\inf_{\varepsilon>0}\left[\inf_{\xi>\xi_0}\frac{1}{\varepsilon}\{m((1+\varepsilon)\xi x)-m(\xi x)\}\right]=\inf_{\varepsilon>0}\frac{1}{\varepsilon}\{m((1+\varepsilon)\xi_0 x)-m(\xi_0 x)\}=\pi_{+}(\xi_0 x)$, if $m(\alpha\xi_0 x)<+\infty$ for some $\alpha>1$. If $m(\alpha\xi_0 x)=+\infty$ for all $\alpha>1$ or $m(\xi_0 x)=+\infty$, we have $\pi_{+}(\xi_0 x)=+\infty$ and $\lim_{\xi\downarrow\xi_0}\pi_{+}(\xi x)=\pi_{+}(\xi_0 x)$. Similarly $\pi_{-}(\xi x)$ is a *left-hand continuous function of* ξ *for every* $x\in R$, that is, $\lim_{\xi\uparrow\xi_0}\pi_{-}(\xi x)=\pi_{-}(\xi_0 x)$ for each $\xi_0\geq 0$.

We put also

$$S=\{x: x\in R,\ |||x|||=1\},$$

$$S_m=\{x: x\in R,\ m(x)=1\}\quad\text{and}\quad S_c=\{x: x\in S,\ m(x)<1\}.$$

From the definition of the second norm it is clear that $S=S_m\smile S_c$, $S_m\frown S_c=\phi$ and $m(\xi x)=+\infty$ for all $x\in S_c$ and $\xi>1$.

6) Two elements x, y are called *mutually orthogonal*, if $|x|\cap|y|=0$ and then we write $x\perp y$. For a subset A of R, $A^{\perp}$ denotes the set of all $x\in R$ with $x\perp y$ for all $y\in A$.

Lemma 1. *We have* $\|x\|=2$ *for* $x\in S$ *if and only if* $x\in S_m$ *and* $\pi_-(x)\leq 2\leq \pi_+(x)$.

Proof. If $\|x\|=2$ for $x\in S$, then we have by the formula (1.1) $2\xi\leq 1+m(\xi x)$ for every $\xi>0$. This implies $m(x)=1$ and $\frac{1-m(\xi x)}{1-\xi}\leq 2 \leq \frac{m(\eta x)-1}{\eta-1}$ for every $0<\xi<1<\eta$. It follows therefore $\pi_-(x)\leq 2\leq\pi_+(x)$. Conversely $\pi_-(x)\leq 2\leq \pi_+(x)$ implies $\frac{m(x)-m(\xi x)}{1-\xi}\leqq 2\leqq \frac{m(\eta x)-m(x)}{\eta-1}$ for every $0<\xi<1<\eta$, which yields $2\xi\leqq 1+m(\xi x)$ for all $\xi>0$ in virtue of $m(x)=1$. Hence we have $\|x\|=2$. Q.E.D.

Lemma 2. *We have* $\pi_-(x)>2$ *for* $x\in S_m$, *if and only if* $\|x\|<2$ *and* $\|x\|=\inf_{\frac{1}{2}\leq\xi<1}\frac{1+m(\xi x)}{\xi}$.

Proof. If $\pi_-(x)>2$ for $x\in S_m$, then we have for some $0<\xi<1$ $2<\frac{1-m(\xi x)}{1-\xi}$, which implies $2>\frac{1+m(\xi x)}{\xi}$ and $\|x\|<2$. Since $\pi_+(x)\geq\pi_-(x)>2$, we obtain $\frac{m(\eta x)-1}{\eta-1}\geq \pi_-(x)>2$ for all $\eta>1$. It follows from above that $\frac{1+m(\eta x)}{\eta}>2$ for $\eta>1$ and $\|x\|=\inf_{\frac{1}{2}<\xi<1}\frac{1+m(\xi x)}{\xi}$. Conversely let $\|x\|<2$, $\|x\|=\inf_{\frac{1}{2}<\xi<1}\frac{1+m(\xi x)}{\xi}$ and $x\in S_m$, then there exists ξ_0 $(0<\xi_0<1)$ such that $2>\frac{1+m(\xi_0 x)}{\xi_0}$. This implies $2<\frac{1-m(\xi_0 x)}{1-\xi_0}$ and $\pi_-(x)>2$. Q.E.D.

Lemma 3. *We have* $\pi_+(x)<2$ *for* $x\in S$ *if and only if* $\|x\|<2$ *and* $\|x\|=\inf_{1<\xi}\frac{1+m(\xi x)}{\xi}$.

Proof. If $x\in S$ and $\pi_+(x)<2$, then we have $m(\xi_0 x)<+\infty$ for some $1<\xi_0$ by the definition of $\pi_+(x)$. Thus we have $x\in S_m$. The remainder of the proof can be obtained by the similar way as above. Q.E.D.

Now we put

$$S_*=\left\{x: x\in S,\ \|x\|=\inf_{\frac{1}{2}\leq\xi<1}\frac{1+m(\xi x)}{\xi}\right\}$$

and

$$S^*=\left\{x: x\in S,\ \|x\|=\inf_{1<\xi}\frac{1+m(\xi x)}{\xi}\right\}.$$

It is clear by Lemmata 1–3 that $S_*\cup S^*=S$, $S_c\subset S_*$ and that $x\in S_*\cap S^*$ implies $\|x\|=2$.

The following lemma plays essential rôle in this paper.

Lemma 4. *If there exist mutually orthogonal elements* $x, y \in R$ *with* $x \in S_*$ *and* $y \in S^*$, *then there exists an element* $z \in S$ *such that* $\|z\| = 2$.

Proof. If $\|x\| = 2$ (or $\|y\| = 2$) holds, then the above assertion is clearly true. Hence we suppose $\|x\| < 2$ and $\|y\| < 2$. We put for every positive number α with $0 \leq \alpha \leq 1$

(2.4) $$\varphi(\alpha) = \sup_{\|| \alpha x + \beta y \|| = 1} \beta .$$

Since $y \in S^*$ implies $m(y) = 1$, it is easily seen that $\varphi(\alpha)$ is a continuous function of α $(0 \leq \alpha \leq 1)$ and if α runs decreasingly from 1 to 0, $\varphi(\alpha)$ does increasingly from ξ_0 to 1, where $\xi_0 = \sup_{m(\xi y) = 1 - m(x)} \xi$.

Now we put $z_\alpha = \alpha x + \varphi(\alpha) y$. It is clear that $z_\alpha \in S_m$ and

(2.5) $$\pi_{\pm}(z_\alpha) = \pi_{\pm}(\alpha x) + \pi_{\pm}(\varphi(\alpha) y) \quad \text{for all } 0 \leq \alpha \leq 1.$$

By Lemma 3, $\|y\| < 2$ and $y \in S^*$ imply

(2.6) $$\pi_-(z_0) \leq \pi_+(z_0) = \pi_+(y) < 2 .$$

And if $x \in S_m$, then we have by Lemma 2

(2.7) $$\pi_-(z_1) = \pi_-(x) + \pi_-(\xi_0 y) \geq \pi_-(x) > 2 .$$

On the other hand, $x \in S_c$ implies $\pi_+(x) = \pi_+(z_1) = +\infty$, because $m(\xi x) = +\infty$ for all $\xi > 1$. Thus we have $\|z_1\| = 2$ by Lemma 1, if $\pi_-(z_1) \leq 2$. Therefore, we may also suppose that both (2.6) and (2.7) hold good.

In virtue of (2.6) we can put $\alpha_0 = \sup_{\pi_+(z_\alpha) < 2, 0 \leq \alpha \leq 1} \alpha$. For such $\alpha_0 \geq 0$ there exists a sequence of positive numbers such that $0 \leq \alpha_n \uparrow_{n=1}^{\infty} \alpha_0$ and $\pi_-(z_{\alpha_n}) \leq \pi_+(z_{\alpha_n}) < 2$. We have by (2.5) and by the fact that $1 \geq \alpha' \geq \alpha \geq 0$ implies $0 \leq \varphi(\alpha') \leq \varphi(\alpha) \leq 1$

$$\pi_-(z_{\alpha_n}) = \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_n) y) \geq \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_0) y) .$$

Since $\pi_-(\xi x)$ is left-hand continuous, we obtain

$$2 \geq \overline{\lim_{n\to\infty}} \pi_-(z_{\alpha_n}) \geq \lim_{n\to\infty} \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_0) y) = \pi_-(\alpha_0 x) + \pi_-(\varphi(\alpha_0) y) ,$$

which implies $\pi_-(z_{\alpha_0}) \leq 2$. On the other hand, since $\pi_-(z_{\alpha_0}) \leq 2$ implies $\alpha_0 < 1$, we can find a sequence $\{\alpha'_n\}_{n=1}^{\infty}$ such that $1 \geq \alpha'_n \downarrow_{n=1}^{\infty} \alpha_0$ and $\pi_+(z_{\alpha'_n}) \geq 2$. We have also by (2.5)

$$2 \leq \pi_+(z_{\alpha'_n}) = \pi_+(\alpha'_n x) + \pi_+(\varphi(\alpha'_n) y) \leq \pi_+(\alpha'_n x) + \pi_+(\varphi(\alpha_0) y) .$$

Since $\pi_+(\xi x)$ is right-hand continuous, we obtain

$$2 \leq \underline{\lim_{n\to\infty}} \pi_+(z_{\alpha'_n}) \leq \lim_{n\to\infty} \pi_+(\alpha'_n x) + \pi_+(\varphi(\alpha_0) y) = \pi_+(\alpha_0 x) + \pi_+(\varphi(\alpha_0) y) ,$$

which yields $\pi_+(z_{\alpha_0}) \geq 2$. Therefore we have $\pi_-(z_{\alpha_0}) \leq 2 \leq \pi_+(z_{\alpha_0})$ and a

fortiori $\|z_{\alpha_0}\|=2$ by Lemma 1. Q.E.D.

We denote by S_{xy} the totality of all elements $z\in S$ such that $z=\alpha x+\beta y$ for some $0\leq\alpha, \beta$.

Lemma 5. *If there exist mutually orthogonal elements $x, y\in S$ and $z\in S_{x,y}$ such that $x, y\in S^*$ and $z\in S_*$, then there exists $z_0\in S_{x,y}$ such that $\|z_0\|=2$.*

Proof. Here we may also assume without loss of generality that $\pi_+(x)<2$, $\pi_+(y)<2$ and $\pi_-(z)>2$. We also define $\varphi(\alpha)$ by the formula (2.4) and put $z_\alpha=\alpha x+\varphi(\alpha)y$ for $0\leq\alpha\leq1$. It follows from $x, y\in S^*$ that z_α belongs to S_m for each $0\leq\alpha\leq1$. Since there exists some $1>\alpha'>0$ such that $\pi_-(z)=\pi_-(z_{\alpha'})>2$ we can put $\alpha_0=\inf_{\pi_-(z_\alpha)\geq2}\alpha$. Then we can find a sequence of positive numbers $\{\alpha_n\}_{n=1}^\infty$ such that $1\geq\alpha_n\downarrow_{n=1}^\infty\alpha_0$ and $\pi_-(z_{\alpha_n})\geq2$ $(n=1, 2, \cdots)$. Since

$$2\leq\pi_+(z_{\alpha_n})=\pi_+(\alpha_n x)+\pi_+(\varphi(\alpha_n)y)\leq\pi_+(\alpha_n x)+\pi_+(\varphi(\alpha_0)y)$$

implies

$$2\leq\underline{\lim}_{n\to\infty}\{\pi_+(\alpha_n x)+\pi_+(\varphi(\alpha_0)y)\}=\pi_+(\alpha_0 x)+\pi_+((\alpha_0)y)=\pi_+(z_{\alpha_0}),$$

we have $2\leq\pi_+(z_{\alpha_0})$. On the other hand, $\pi_+(z_{\alpha_0})\geq2$ implies $\alpha_0>0$ and hence we can find also a sequence of positive numbers $\{\alpha'_n\}_{n=1}^\infty$ such that $0\leq\alpha'_n\uparrow_{n=1}^\infty\alpha_0$ and $\pi_-(z_{\alpha'_n})<2$ $(n=1, 2, \cdots)$. Since

$$2>\pi_-(z_{\alpha'_n})=\pi_-(\alpha'_n x)+\pi_-(\varphi(\alpha'_n)y)\geq\pi_-(\alpha'_n x)+\pi_-(\varphi(\alpha_0)y)$$

implies

$$2\geq\overline{\lim}_{n\to\infty}\{\pi_-(\alpha'_n x)+\pi_-(\varphi(\alpha_0)y)\}=\pi_-(a_0 x)+\pi_-(\varphi(\alpha_0)y),$$

we have $2\geq\pi_-(z_{\alpha_0})$. Therefore we obtain $\pi_-(z_{\alpha_0})\leq2\leq\pi_+(z_{\alpha_0})$, which implies $\|z_{\alpha_0}\|=2$. Q.E.D.

Here we note that if there exist mutually orthogonal elements $x, y\in S$ and $z\in S_{x,y}$ such that $z\in S^*$ then $m(x+y)>1$. Hence applying the similar method as in the proof of Lemma 5, we have

Lemma 6. *If there exist mutually orthogonal elements $x, y\in S$ and $z\in S_{x,y}$ such that $x, y\in S_*$ and $z\in S^*$, then there exists $z_0\in S_{x,y}$ with $\|z_0\|=2$.*

Collecting the results of the above Lemmata, we have

Theorem 2.1. *In order that the condition* (1.3) *is satisfied, that is, $\frac{\|x\|}{|||x|||}<2$ for all $0\neq x\in R$, it is necessary and sufficient that either*

(2.8) $$\pi_+(x)<2 \quad \textit{for all } x\in S$$

or

$$(2.9') \qquad \pi_-(x) > 2 \quad \textit{for all } x \in S_m$$

holds.

Proof. Necessity. When R is one-dimensional, the assertion comes directly from Lemmata 2 and 3. Thus we may assume that the dimension of R is greater than two. Now let $R = N_1 \oplus N_2$, where N_i $(i=1, 2)$ are normal manifolds and $N_1^{\perp} = N_2$. For an element $x_0 \in N_1 \frown S$ the condition (1.3) implies either $x_0 \in S_*$ or $x_0 \in S^*$.

First let $x_0 \in S_*$. Then Lemma 4 and the condition (1.3) imply $N_2 \frown S \subseteq S_*$, which implies also $N_1 \frown S \subseteq S_*$ by Lemma 4. Therefore we obtain $S \subset S_*$ by Lemma 6. Thus we can see that (2.9′) holds good in virtue of Lemma 2.

Secondly let $x_0 \in S^*$, then we have by the same manner $N_2 \frown S = S^*$ and $N_1 \frown S = S^*$. This implies that (2.8) holds good in virtue of Lemmata 3 and 5. Q.E.D.

Sufficiency. Since $x \in S_c$ implies $1 + m(x) < 2$, we have $\|x\| < 2$ for all $x \in S_c$. Thus we can see that Lemmata 2 and 3 assure that (2.8) (or (2.9′)) implies the condition (1.3). Q.E.D.

A modular m on R is said to be *finite* if $m(x) < +\infty$ for all $x \in R$. Since we have $S = S_m$, in case R is finite, we have immediately from Theorem 2.1

Corollary 1. *Let a modular m be finite. In order that the condition* (1.3) *holds, it is necessary and sufficient that either the condition* (2.8) *or*

$$(2.9) \qquad \pi_-(x) > 2 \quad \text{for all } x \in S$$

holds.

In order that we discuss the condition (1.3) more precisely in case of non-atomic R, we need to prove

Lemma 7. *If R non-atomic and $S = S^*$, then the modular m is finite.*

Proof. If there exists $0 \leq x \in R$ with $m(x) = +\infty$, we put $\alpha_0 = \inf_{m(\xi x) = +\infty} \xi$ and $x_0 = \alpha_0 x$. It follows $m(\eta x_0) = +\infty$ for all $\eta > 1$ and $m(\xi x_0) < +\infty$ for all $0 \leq \xi < 1$. When $m(x_0) < +\infty$ holds, we can find an element $p \in R$ such that $m(\eta [p] x_0) = +\infty$ for all $\eta > 1$ and $m([p] x_0) \leq 1$, since R has no atomic elements. For such $[p] x_0$ we have $[p] x_0 \in S_c$, which is inconsistent with $S^* = S$. Now let $m(x_0) = +\infty$ hold. Since R is non-atomic, we can find $p \in R$ with $m\left(\frac{2}{3}[p] x_0\right) \leq \frac{1}{4}$ and $\lim_{\xi \uparrow 1} m(\xi [p_0] x_0) = +\infty$. Now we put

$\alpha=\dfrac{1}{\left|\left|\left|\frac{2}{3}[p]x_0\right|\right|\right|}$ and $y=\alpha\frac{2}{3}[p]x_0$. Then we have $1<\alpha<\frac{3}{2}$ and $y\in S_m$, hence

$$\pi_-(y)=\pi_-\left(\alpha\frac{2}{3}[p]\pi_0\right)\geqq\frac{m\left(\alpha\frac{2}{3}[p]x_0\right)-m\left(\frac{2}{3}[p]x_0\right)}{\frac{\alpha-1}{\alpha}}\geq\frac{1-\frac{1}{4}}{\frac{1}{3}}>2\,.$$

This contradicts the assumption: $S=S^*$ by Lemmata 1 and 2. Therefore we have proved that $m(x)<+\infty$ for all $x\in R$. Q.E.D.

Thoreme 2.2. *Let R be non-atomic and the condition* (1.3) *be satisfied, then the modular m satisfies one of the following conditions*:

(2.10) $\quad m(\xi x)<\xi^2m(x)$ *for all* $\xi>1$ *and* $x\in R$ *with* $m(x)\geq 1$;

(2.11) $\quad m(\xi x)>\xi^2m(x)$ *for all* $\xi>1$ *and with* $+\infty>m(x)\geq 1$.

Proof. In virtue of the foregoing theorem we know that one of the conditions (2.8) or (2.9′) is true. First we suppose that (2.8) holds. Then Lemma 7 together with Lemma 3 implies that m is finite. If $m(x)=N+\frac{m}{n}$ (where N, m and n are natural numbers with $m\leq n$), we can decompose orthogonally x into $x=\sum_{i=1}^{N-1}x_i+\sum_{j=1}^{n+m}y_j$ such that $m(x_i)=1$ ($i=1, 2,\cdots, N-1$) and $m(y_j)=\frac{1}{n}$ ($j=1, 2,\cdots, n+m$). The number of j satisfying $\pi_+(y_j)\geq 2m(y_j)$ is less than n, because if there exist $j_1, j_2,\cdots, j_n$ with $\pi_+(y_{j_k})\geq 2m(y_{j_k})$ ($k=1, 2,\cdots, n$), we have $\sum_{k=1}^{n}y_{j_k}\in S_m$ and $\pi_+\left(\sum_{k=1}^{n}y_{j_k}\right)=\sum_{k=1}^{n}\pi_+(y_{j_k})\geq\sum_{k=1}^{n}2m(y_{j_k})=2m\left(\sum_{k=1}^{n}y_{j_k}\right)=2$, which is inconsistent with (2.8). Hence we can find $\{j_p\}$ ($1\leq p\leq m$) such that $\pi_+(y_{j_p})<2m(y_{j_p})$ ($p=1, 2,\cdots, m$). Putting $y_0=\sum_{p=1}^{m}y_{j_p}$, we obtain $m\left(\sum_{j=1}^{m+n}y_j-y_0\right)=1$ and $\pi_+\left(\sum_{j=1}^{m+n}y_j-y_0\right)<2$. Therefore we have

$$\pi_+(x)=\sum_{i=1}^{N-1}\pi_+(x_i)+\pi_+\left(\sum_{j=1}^{m+n}y_j-y_0\right)+\pi_+(y_0)$$
$$<2(N-1)+2+2m(y_0)=2\left(N+\frac{m}{n}\right)=2m(x)\,,$$

hence $\pi_+(x)<2m(x)$. In general, if $1<m(x)$, we can find $\{\alpha_n\}_{n=1}^{\infty}$ with $\alpha_n\downarrow_{n=1}^{\infty}1$ and $m(\alpha_nx)$ is a rational number for each $n\geq 1$. It follows

$$\pi_+(x)=\lim_{n\to\infty}\pi_+(\alpha_nx)\leq\lim_{n\to\infty}2m(\alpha_nx)=2m(x)\,,$$

hence

(2.12) $$\pi_+(x) \leq 2m(x) \quad \text{for all } x \in R \text{ with } 1 \leq m(x).$$

Now we put

(2.13) $$m'_x(\xi) = \lim_{\xi \to 0} \frac{m((\xi+\varepsilon)x) - m(\xi x)}{\varepsilon}$$

for each $x \in R$ and $\xi \geq 0$. It is clear that $\xi \cdot m'_x(\xi) = \pi_+(\xi x)$ for all $\xi > 0$ and $x \in R$. (2.12) implies

(2.14) $$\frac{m'_x(\xi)}{m(\xi x)} \leq \frac{2}{\xi} \quad (\xi > 1)$$

for every $x \in R$ with $m(x) \geq 1$. Integrating both sides of (2.14) from 1 to $\eta > 1$ with respect to ξ, we have

(2.15) $$\log \frac{m(\eta x)}{m(x)} \leq 2 \log \xi \quad (\eta > 1).$$

In formula (2.15), however, the equal sign does not hold in any case. Indeed, since as is shown above, the set of all ξ satisfying $\xi m'_x(\xi) = \pi_+(\xi x) < 2m(\xi x)$ is dense in $[1, +\infty)$ and $m(\xi x)$ is a continuous function of ξ, there exists an interval $(\xi_0, \eta_0) \subseteq (1, \eta)$ such that

$$\xi m'_x(\xi) = \pi_+(\xi x) < 2m(\xi x)$$

holds for all $\xi \in (\xi_0, \eta_0)$. Therefore we have

$$m(\eta x) < \eta^2 m(x)$$

for all $\eta > 1$ and $x \in R$ with $m(x) \geq 1$.

By the quite same manner we can prove that the condition (2.11) is satisfied, if we assume that the condition (2.9′) is true. Q.E.D.

§ 3. Here we consider the case that the norms defined by a modular m satisfy (1.4), that is, $\sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||} < 2$. From the results proved in § 2 we have

Theorem 3.1. *If the condition* (1.4) *is satisfied, then either*

(3.1) $$\sup_{x \in S} \pi_+(x) < 2$$

or

(3.2) $$\inf_{x \in S_m} \pi_-(x) > 2$$

holds.

Proof. In virtue of Theorem 2.1, we can see that either (2.8) or (2.9′) holds. First let (2.8) be true and set $\gamma = \sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||}$. Then for each $x \in S$ and $\varepsilon > 0$, there exists $\xi > 1$ such that $\frac{1+m(\xi x)}{\xi} < \gamma + \varepsilon$ by Lemma 3. From this

$$\pi_+(x) \leq \frac{m(\xi x)-1}{\xi - 1} < \gamma + \varepsilon$$

follows, if $\gamma + \varepsilon < 2$. Hence we have $\gamma \geq \pi_+(x)$ for all $x \in S$.

On the other hand, we can prove by the same way[7] that (2.9′) together with (1.4) implies (3.2). Q.E.D.

Remark 1. The converse of Theorem 3.1 does not remain true in general. It is easily verified that there exists a modular which does not fulfil (1.4) but satisfies (3.1) (or (3.2)).

Remark 2. As is seen in the proof of Theorem 3.1, it is clear that $\sup_{x \in S} \pi_+(x) \leq \gamma$ or $\inf_{x \in S_m} \pi_-(x) \geq \frac{\gamma}{\gamma - 1}$ holds respectively, where $\gamma = \sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||} < 2$.

As for non-atomic R, corresponding to Theorem 2.2, we have

Theorem 3.2. *Let R be non-atomic and the condition* (1.4) *be satisfied, then either*

(3.3) $\quad m(\xi x) \leq \xi^p m(x)$ *for all* $\xi \geq 1$ *and* $x \in R$ *with* $m(x) \geq 1$;

(3.4) $\quad m(\xi x) \geq \xi^{p'} m(x)$ *for all* $\xi \geq 1$ *and* $x \in R$ *with* $m(x) \geq 1$,

where p and p' are real numbers with $1 \leq p < 2 < p' \leq +\infty$[8].

Proof. In virtue of the preceding theorem we need only to verify implications: (3.1)$\rightarrow$(3.3) and (3.2)$\rightarrow$(3.4). And these implications can be ascertained by the same manner as in the proof of Theorem 2.2. Here we may choose p, p' as $p = \gamma$ and $p' = \frac{\gamma}{\gamma - 1}$ respectively, where $\gamma = \sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||}$. Q.E.D.

Remark 3. As is easily verified by calculating $\|x\|$ of $x \in S$, the condition (3.3) is the sufficient one for (1.4) at the same time. On the other hand, (3.4) is not such a one in general.

§4. At last we deal with a modular of unique spectra [4; §54] and estimate exactly $\sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||}$ and $\inf_{0 \neq x \in R} \frac{\|x\|}{|||x|||}$ as applications of the

7) We note that $2 - \gamma\xi \geq \frac{\gamma}{\gamma - 1}(1 - \xi)$ holds if $1 < \gamma < 2$ and $1 < \gamma\xi$.

8) When $p' = +\infty$, we put $\xi^\infty = \infty$ if $\xi > 1$.

preceding theorems and those of [7]. An element $0 \leq s \in R$ is said to be *simple*, if $m(s) < +\infty$ and $m([p]s) = 0$ implies $[p]s = 0$. And a modular m is said to be *of unique spectra* if $m(\xi s) = \int_{[s]} \xi^{\rho(\mathfrak{p})} m(d\mathfrak{p} s)$[9] for all $\xi \geq 0$ and simple elements $s \in R$. Function spaces $L^{p(t)}$ [5] (where $p(t)$ is a measurable function with $p(t) \geq 1$ $(0 \leq t \leq 1)$): the totality of all measurable functions $\varphi(t)$ such that

$$(4.1) \qquad \int_0^1 |\alpha\varphi(t)|^{p(t)} dt < +\infty \qquad \text{for some } \alpha > 0,$$

and sequence spaces l^{p_ν} (where $p_\nu \geq 1$ $(\nu \geq 1)$): the totality of all sequences $\mathfrak{x} = (\xi_\nu)_{(\nu \geq 1)}$ such that

$$(4.2) \qquad \sum_{\nu=1}^{\infty} |\alpha\xi_\nu|^{p_\nu} < +\infty \qquad \text{for some } \alpha > 0.$$

are the examples of modulared spaces whose modulars are of unique spectra, where the modulars are defined as $m(\varphi) = \int_0^1 |\varphi(t)|^{p(t)} dt$ and $m(\mathfrak{x}) = \sum_{\nu=1}^{\infty} |\xi_\nu|^{p_\nu}$ respectively. When m is of unique spectra, we denote by ρ_u, ρ_l the *upper exponent* of m: $\rho_u = \sup_{\mathfrak{p} \in \mathfrak{E}} \rho(\mathfrak{p})$ and the *lower exponent* of m: $\rho_l = \inf_{\mathfrak{p} \in \mathfrak{E}} \rho(\mathfrak{p})$ respectively. There exist normal manifolds N_1, N_2 such that $R = N_1 \oplus N_2$ and $\rho(\mathfrak{p})$ is finite for all $\mathfrak{p} \in U_{[N_1]}$ and $\rho(\mathfrak{p}) = +\infty$ for all $\mathfrak{p} \in U_{[N_2]}$, that is, m is singlar in N_2. For any $0 \neq x \in N_2$ we have $\|x\| = |||x|||$ and $S = S_*$. Therefore we obtain

Theorem 4.1. *If a modular m is of unique spectra, then we have* $\left(\textit{with the conventions } \frac{1}{\infty} = 0 \textit{ and } \infty^0 = 1\right)$

$$\sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||} \begin{cases} = 2, & \textit{if } \rho_l \leq 2 \leq \rho_u, \\ = \rho_u^{\frac{1}{\rho_u}} q_u^{\frac{1}{q_u}}, & \textit{if } \rho_u < 2, \\ = \rho_l^{\frac{1}{\rho_l}} q_l^{\frac{1}{q_l}}, & \textit{if } \rho_l > 2, \end{cases}$$

where q_u and q_l are real numbers with $\frac{1}{\rho_u} + \frac{1}{q_u} = 1$ and $\frac{1}{\rho_l} + \frac{1}{q_l} = 1$.

Proof. As $\frac{\|x\|}{|||x|||} = 1$ for all $0 \neq x \in N_2$, we may consider only the

9) $\mathfrak{p}$ is a point of *representation space* $\mathfrak{E}$ of R, i.e. the maximal ideal of normal manifolds of R. For N, we denote by $U_{[N]}$ the totality of all $\mathfrak{p} \in \mathfrak{E}$ with $N \in \mathfrak{p}$. $\rho(\mathfrak{p})$ is a continuous function on $\mathfrak{E}$ with $\rho(\mathfrak{p}) \geq 1$.

ratios of the norms in N_1. When $\gamma \leq \rho(\mathfrak{p}) \leq \gamma'$ for all $\mathfrak{p} \in U_{[p]} \subseteq U_{[N_1]}$, then we have for all $x \in R$ $\xi^{\gamma} m([p]x) \leq m(\xi[p]x) \leq \xi^{\gamma'} m([p]x)$ $(\xi \geq 1)$ and $\eta^{\gamma} m([p]x) \geq m(\eta[p]x) \geq \eta^{\gamma'} m([p]x)$ $(0 \leq \eta \leq 1)$. From this and Lemma 4 we have $\sup_{0 \neq x \in R} \frac{\|x\|}{|||x|||} = 2$, if $\rho_l \leq 2 \leq \rho_u$. Since $\sup_{x \in S} \pi_+(x) < 2$ if and only if $\rho_u < 2$ and $\inf_{x \in S} \pi_-(x) > 2$ if and only if $\rho_l > 2$ for m, we have by Lemmata 2 and 3 that $\|x\| \leq \rho_u^{\frac{1}{\rho_u}} q_u^{\frac{1}{q_u}} (x \in S)$ and $\|x\| \leq \rho_l^{\frac{1}{\rho_l}} q_l^{\frac{1}{q_l}}$ respectively according to (3.1) and (3.2). Therefore we complete the proof. Q.E.D.

Similarly we can conclude by Theorem 3.1 in [7]

Theorem 4.2. *If a modular m is of unique spectra, then we have*

$$\inf_{0 \neq x \in R} \frac{\|x\|}{|||x|||} = \operatorname{Min} \left\{ \rho_l^{\frac{1}{\rho_l}} q_l^{\frac{1}{q_l}},\ \rho_u^{\frac{1}{\rho_u}} q_u^{\frac{1}{q_u}} \right\}.$$

Department of Mathematics,
Hokkaido University, Sapporo.

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