ON SOME THEOREMS OF THE FOURIER TRANSFORM

By

Hiroshi ISHIKAWA and Sumiyuki KOIZUMI

1. **Introduction.** Some twenty years ago J. Marcinkiewicz [6] has stated a result about the interpolation of operations without proof. Recently A. Zygmund [9] has completed the theory with a certain application to the theory of Fourier series. This method can be applied to the theory of Fourier transform. This is the purpose of our paper.

2. The measure space which we consider is totally σ-finite in the sense of P. R. Halmos [3]. One of the authors [5] has extended a result of A. Zygmund [9] to the case of totally σ-finite measure in the following manner.

Theorem A. *Let X and Y be two measure spaces with totally σ-finite measures μ and ν, respectively. Let $h=\mathrm{T}f$ be a quasi-linear operation defined for all simple functions f in X, with h defined on Y. Suppose that* T *is simultaneously of weak types (a, a) and (b, b), where $1 \leqq a < b < \infty$. Suppose also that $\varphi(u)$ is a continuous increasing function for $u \geqq 0$ satisfying the condition:*

$$\varphi(0)=0, \tag{2.1}$$

$$\varphi(2u)=O(\varphi(u)), \tag{2.2}$$

$$\int_u^\infty \frac{\varphi(t)}{t^{b+1}}dt=O\left(\frac{\varphi(u)}{u^b}\right), \tag{2.3}$$

$$\int_1^u \frac{\varphi(t)}{t^{a+1}}dt=O\left(\frac{\varphi(u)}{u^a}\right), \tag{2.4}$$

for $u \to \infty$ and further

$$\varphi(2u)=O(\varphi(u)), \tag{2.5}$$

$$\int_u^1 \frac{\varphi(t)}{t^{b+1}}dt=O\left(\frac{\varphi(u)}{u^b}\right), \tag{2.6}$$

$$\int_0^u \frac{\varphi(t)}{t^{a+1}}dt=O\left(\frac{\varphi(u)}{u^a}\right), \tag{2.7}$$

for $u\to 0$. *Then* $h=\mathrm{T}f$ *belongs to the* L_ν^φ *and we have*

$$\int_Y \varphi(|h|)d\nu \leqq A_\varphi \int_X \varphi(|f|)d\mu, \tag{2.8}$$

where A_φ *is a constant independent of* f. *In particular, the operation* T *can be uniquely extended to the whole space* L_μ^φ *preserving the relation* (2.8).

Another theorem which we need is due to A. P. Calderón and A. Zygmund [1]. This reads as follows.

Theorem B. *In the measure space of the same types as in Theorem A, let a quasi-linear operation* $h=\mathrm{T}f$, *which is defined for all simple functions* f, *be weak type* (1, 1). *Then we have for any subset* Y_0 *of* Y *with finite* ν-*measure*,

$$\int_{Y_0} |h|^{1-\varepsilon}d\nu \leqq \frac{A}{\varepsilon}\nu(Y_0)^\varepsilon\Big(\int_X |f|d\mu\Big)^{1-\varepsilon}, \tag{2.9}$$

where $0<\varepsilon<1$ *and* A *is an absolute constant. Further the operation* T *can be uniquely extended to the whole space preserving the relation* (2.9).

3. We shall give a new proof of a theorem due to G. H. Hardy and J. E. Littlewood [4]. Let us suppose that X and Y are one dimensional Euclidean spaces where $d\mu(x)$ is an ordinary Lebesgue measure on X and $d\nu(y)=\frac{dy}{y^2}(y\neq 0)$ on Y. We consider a linear operation

$$h=\mathrm{T}f=y\hat{f}=\frac{y}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{iyt}\,dt, \tag{3.1}$$

where f is simple. Then we have

Lemma 1. *The operation* T *defined by* (3.1) *is of strong type* (2, 2).

This is equivalent to the Plancherel theorem.

Lemma 2. *The operation* T *defined by* (3.1) *is of weak type* (1, 1).

Proof of Lemma 2. We define two sets

$$E_r=\{y\,\big|\,|y\hat{f}|>r\} \tag{3.2}$$

and

$$O_r=\Big\{y\Big|\ |y|>\frac{\sqrt{2\pi}}{\|f\|_1}r\Big\}, \tag{3.3}$$

where r is any positive number.

It is sufficient to prove that

(3.4) $$\nu(E_r) \leqq \frac{M}{r}\int_{-\infty}^{\infty} |f|\,dx = \frac{M}{r}\,||f||_1.$$

By the definition of $\hat{f}$, we have

$$|\hat{f}| \leqq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |f(t)|\,dt = \frac{||f||_1}{\sqrt{2\pi}},$$

from which it follows easily $E_r \subset O_r$. Therefore we have

$$\nu(E_r) < \nu(O_r) = 2\int_{\sqrt{2\pi}\,r/||f||_1} \frac{dy}{y^2} = \frac{\sqrt{2/\pi}\,||f||_1}{r}.$$

Thus we have proved Lemma 2.

By Theorem A, Lemmas 1 and 2 we get

Theorem 1. *Let f be a measurable function such that $\varphi(|f|)$ is μ-integrable. Then the Fourier transform $\hat{f}$ of f can be defined and we have*

(3.5) $$\int_{-\infty}^{\infty} \varphi(|y\hat{f}|)\frac{dy}{y^2} \leqq A_\varphi \int_{-\infty}^{\infty} \varphi(|f|)\,dx,$$

where $\varphi(u)$ is a function of Theorem A with $a=1$ and $b=2$.

In particular, we have

Theorem 2. *If f belongs to L^p $(1<p\leqq 2)$, then we have*

(3.6) $$\left(\int_{-\infty}^{\infty} |\hat{f}|\,|y|^{p-2}\,dy\right)^{\frac{1}{p}} \leqq A_p\left(\int_{-\infty}^{\infty} |f|^p\,dx\right).$$

If $\varphi(u)=u^r\psi(u)$, where $1<r\leqq 2$ and $\psi(u)$ is a positive slowly varying function as $u\to 0$ and $u\to\infty$ in the sense of J. Karamata and A. Zygmund [9], then $\varphi(u)$ satisfies the property of Theorem A. Taking $\psi(u)=\log^+ u$, we get the following

Theorem 3. *Let f be a function such that*

(3.7) $$\int_{-\infty}^{\infty} |f|^p \log^+ |f|\,dx < \infty \quad (1<p\leqq 2),$$

then the Fourier transform $\hat{f}$ of f can be defined and we have

(3.8) $$\int_{-\infty}^{\infty} |\hat{f}|^p (\log^+ |y\hat{f}|)\,|y|^{p-2}\,dy \leqq A_p \int_{-\infty}^{\infty} |f|^p \log^+ |f|\,dx.$$

4. We shall now interchange the measure functions μ and ν on X and Y respectively, that is, we take $d\mu=\frac{dx}{x^2}(x\neq 0)$ and $d\nu$ as an ordinary Lebesgne measure on X and Y respectively. Then we have

Theorem 4. *If* $|f|^{p'}|x|^{p'-2}$ *belongs to* L, *then the Fourier transform* $\hat{f}$ *of* f *exists and we have*

$$(4.1)\qquad \left(\int_{-\infty}^{\infty}|\hat{f}|^{p'}dy\right)^{\frac{1}{p'}}\leqq A_{p'}\left(\int_{-\infty}^{\infty}|f|^{p'}|x|^{p'-2}dx\right)^{\frac{1}{p'}},$$

where $p'>2$.

Proof of Theorem 4. We have

$$\left(\int_{-\infty}^{\infty}|\hat{f}|^{p'}dy\right)^{\frac{1}{p'}}=\sup_{\|g\|_p\leqq 1}\int_{-\infty}^{\infty}\hat{f}\cdot g\,dy,$$

where p' is a conjugate index of p.

Let $\hat{g}(t,n)$ be a trancated Fourier transform of $g(t)$, that is,

$$\hat{g}(t,n)=\frac{1}{\sqrt{2\pi}}\int_{-n}^{n}g(u)e^{-itu}\,du.$$

Since f is simple, we have

$$\int_{-n}^{n}\hat{f}\cdot g\,dy=\int_{-\infty}^{\infty}f(t)\hat{g}(t,n)\,dt$$

$$=\int_{-\infty}^{\infty}fx^{\frac{2}{p}-1}x\hat{g}\cdot x^{-\frac{2}{p}}\,dx.$$

By the Hölder inequality and Theorem 2,

$$\left|\int_{-n}^{n}\hat{f}\cdot g\,dy\right|\leqq\left(\int_{-\infty}^{\infty}|f||x|^{\frac{2}{p}-1}dx\right)^{\frac{1}{p'}}\left(\int_{-n}^{n}|x||\hat{g}|^{p}\frac{dx}{x^{2}}\right)^{\frac{1}{p}}$$

$$\leqq A_{p'}\left(\int_{-\infty}^{\infty}|f|^{p'}|x|^{p'-2}dx\right)\left(\int_{-n}^{n}|g|^{p}dx\right)^{\frac{1}{p}}$$

$$\leqq A_{p'}\left(\int_{-\infty}^{\infty}|f|^{p'}|x|^{p'-2}dx\right)^{\frac{1}{p'}}.$$

Hence we have

$$\left(\int_{-\infty}^{\infty}|\hat{f}|^{p'}dy\right)^{\frac{1}{p'}}\leqq A_{p'}\left(\int_{-\infty}^{\infty}|f|^{p'}|x|^{p'-2}dx\right)^{\frac{1}{p'}}.$$

This completes the proof.

Here we remark that the constant $A_{p'}$ of (4.1) is equal to A_p of (3.6).

Combing Theorem A, Theorem 1 and 4, we get the following

Theorem 5. *Let* f *be a measurable function such* $\varphi(|f|)$ *is* μ*-integrable. Then the Fourier transform* $\hat{f}$ *of* f *is defined and we have*

(4.2) $$\int_{-\infty}^{\infty}\varphi(|\hat{f}|)\,dy \leqq A_{\varphi}\int_{-\infty}^{\infty}\varphi(|xf|)\frac{dx}{x^2},$$

where $\varphi(u)$ is a function of Theorem A with $a=2$ and $b=p'$, $(2<p'<\infty)$.

5. Finally we shall treat about the class L. Let us consider the measure spaces defined in § 3. We begin with the following

Lemma 3. *Let f belong to L, then we have for any sub-interval with finite ν-measure*

(5.1) $$\int_I |y\hat{f}|^{1-\varepsilon}\frac{dy}{y^2} \leqq \frac{A}{\varepsilon}\nu^{\varepsilon}(I)\Big(\int_{-\infty}^{\infty}|f|\,dx\Big)^{1-\varepsilon},$$

where $0<\varepsilon<1$.

Proof of Lemma 3. This is an immediate consequence of Lemma 2 and Theorem 3.

Let us consider the interval

(5.2) $$J_n=\Big\{y\,\Big|\,\frac{1}{2^n}<|y|<\frac{1}{2^{n-1}}\Big\},\quad (n=1,2,\cdots),$$

Then its ν-measure becomes

(5.3) $$\nu(J_n)=2\int_{1/2^n}^{1/2^{n-1}}\frac{dy}{y^2}=2^n.$$

If we take J_n as I in Lemma 3, we get

(5.4) $$\int_{J_n} y\,|y\hat{f}|^{1-\varepsilon}\frac{dy}{y^2} \leqq \frac{A}{\varepsilon}2^{n\varepsilon}\,||f||_1^{1-\varepsilon}.$$

Therefore we have

(5.5) $$\int_{|y|<1}|y|^{\delta-\varepsilon-1}|\hat{f}|^{1-\varepsilon}\,dy=\sum_{n=1}^{\infty}2^{-n\delta_1}\int_{J_n}|y\hat{f}|^{1-\varepsilon}|y|^{-2}\,dy$$
$$\leqq \frac{A}{\varepsilon}\sum_{n-1}^{\infty}\frac{1}{2^{-n(\delta_1-\varepsilon)}}\,||f||_1^{1-\varepsilon}$$
$$\leqq \frac{A}{\varepsilon(\delta_1-\varepsilon)}\,||f||_1^{1-\varepsilon}\quad (\delta_1>\varepsilon).$$

On the other hand if we write

(5.6) $$I_n=\{y\,|\,2^{n-1}<|y|<2^n\},$$

then its ν-measure is 2^{-n}. Thus by similar argument

(5.7) $$\int_{|y|>1}|y|^{\delta-\varepsilon-1}|\hat{f}|^{1-\varepsilon}\,dy \leqq \frac{A}{\varepsilon(\varepsilon-\delta_2)}\,||f||_1^{1-\varepsilon}\quad (\delta_2<\varepsilon).$$

From (5.6) and (5.7) we get the following result.

Theorem 6. *Let f belong to L, then we have*

$$\int_{-\infty}^{\infty} |\hat{f}|^{1-\varepsilon} |y|^{\delta(y)-\varepsilon-1} dy \leqq \frac{A}{\varepsilon\cdot\delta_0}\left(\int_{-\infty}^{\infty}|f|\,dx\right)^{1-\varepsilon}, \tag{5.8}$$

where $\delta(y)$ is such that

$$\delta(y)=\begin{cases}\delta_1 & if \quad |y|<1\\ \delta_2 & if \quad |y|<1,\end{cases} \tag{5.9}$$

δ_1 and δ_2 being constants greater and less than ε respectively, and

$$\min(|\delta_1-\varepsilon|, |\delta_2-\varepsilon|)=\delta_0.$$

We remark that in the above theorem $\delta(y)$ can be replaced by any bounded continuous function such that $\delta(y)>\varepsilon$ if $|y|<1$, $\delta(y)<\varepsilon$ if $|y|<1$, and

$$\inf_{\substack{|y|<\frac{1}{2}\\ |y|<2}} |\delta(y)-\varepsilon|=\eta>0.$$

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DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY