ON CONJUGATELY SIMILAR TRANSFORMATIONS

By

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Introduction. H. Nakano in this book [1] defined the *modulared semi-ordered linear space* $R(m)$, that is, R is a universally continuous[1] semi-ordered linear space where a functional $m(a)$ $(a \in R)$ is defined such as the following seven properties are satisfied:

$$(M)\quad \begin{array}{ll} 1) & 0 \leqq m(a) \leqq +\infty \text{ for all } a \in R; \\ 2) & \text{if } m(\xi a)=0 \text{ for all } \xi \geqq 0, \text{ then } a=0; \\ 3) & \text{for any } a \in R \text{ there exists } \alpha>0 \text{ such that } m(\alpha a)<+\infty; \\ 4) & \text{for any } a \in R,\ m(\xi a) \text{ is a convex function of } \xi; \\ 5) & |a| \leqq |b| \text{ implies } m(a) \leqq m(b); \\ 6) & a \frown b=0 \text{ implies } m(a+b)=m(a)+m(b); \\ 7) & 0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda} a^{2)} \text{ implies } \sup_{\lambda \in \Lambda} m(a_\lambda)=m(a). \end{array}$$

This functional $m(a)$ $(a \in R)$ is called a *modular* on R. The well-known space $L_p([0,1])$ $(p \geqq 1)$ is one of examples of the modulared semi-ordered linear space, putting $m_p(a)=\int_0^1 \frac{1}{p}|a(t)|^p\, dt$ $(p \geqq 1)$.

Let R be a universally continuous semi-ordered linear space and $\bar{R}$ be the *conjugate space* of R, that is, the space of all universally continuous[3] linear functionals on R. Especially when R is a modulared semi-ordered linear space by modular $m(a)$ $(a \in R)$, a functional $\bar{a} \in \bar{R}$ is said to be *modular bounded* if $\sup_{m(a)\leqq 1} |(a, \bar{a})| < +\infty$. The space of all modular bounded functionals $\bar{R}^m$ is a universally continuous semi-ordered linear space. When we put for $\bar{a} \in \bar{R}$

$$(1)\qquad \bar{m}(\bar{a})=\sup_{a \in R}\{(a, \bar{a})-m(a)\} \quad (\bar{a} \in \bar{R}),$$

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1) A semi-ordered linear space R is said to be universally continuous if for any system $a_\lambda \geqq 0$ $(\lambda \in \Lambda)$ there exists an element $\bigcap_{\lambda \in \Lambda} a_\lambda$ in R ([1], p. 17).

2) For any $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda_3 \in \Lambda$ such that $a_{\lambda_1} \smile a_{\lambda_2} \leqq a_{\lambda_3}$ and $\bigcup_{\lambda \in \Lambda} a_\lambda = a$.

3) A linear functional $\bar{a}$, $(a, \bar{a})$ $(a \in R)$, is said to be universally continuous, if for any $a_\lambda \downarrow_{\lambda \in \Lambda} 0$ we have $\inf_{\lambda \in \Lambda} |(a_\lambda, \bar{a})|=0$ ([1], p. 81).

$\bar{a}$ is modular bounded if and only if we can find $\alpha>0$ such that $\overline{m}(\alpha\bar{a})<+\infty$ ([1], p. 169). And $\overline{m}(\bar{a})$ ($\bar{a}\in\bar{R}^m$) is a modular on $\bar{R}^m$. Namely $\bar{R}^m(\overline{m})$ is also a modulared semi-ordered linear space. This $\bar{R}^m(\overline{m})$ is called a *modular conjugate space* of $R(m)$ and $\overline{m}(\bar{a})$ ($\bar{a}\in\bar{R}^m$) is called a conjugate modular of $m(a)$ ($a\in R$). If we put $R(m)=L_p([0,1])$ (m_p) $(p>1)$, then we have $\bar{R}=\bar{R}^m=L_q([0,1])$ and $\overline{m}_p=m_q$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$.

In [1] the concept of a *conjugately similar transformation* on R was introduced as one method to construct a modular on the universally continuous semi-ordered linear space R and it was tried to represent the modular as an integration of a conjugately similar transformation.

A conjugately similar transformation T is a mapping from R^+, the positive cone of R, to $\bar{R}^+$ satisfying the following two condition:

(2)
1) if $a\geqq b\geqq 0$, then $Ta\geqq Tb\geqq 0$;
2) for any $a\in R^+$ and a normal manifold[4] N in R we have $(Ta)[N]$[5] $=T([N]a)$, where $[N]$ is the projection operator[6] of N.

If R is reflexive $(R=\bar{\bar{R}})$ and there exists a conjugately similar transformation T on R such that T is an onto- and one to one-mapping, R is called a *conjugately similar space* by T. For instance $L_p([0,1])$ $(p>1)$ is a conjugately similar space by a conjugately similar transformation $T|a|=|a|^{\frac{p}{q}}\in L_q([0,1])$ $(a\in L_p([0,1]))$.

Main results in [1] concerning the relation between modulars and conjugately similar transformations are the following two: I and II.

I. If we put $m_T(a)=\int_0^1(|a|, T\xi|a|)d\xi$ $(a\in R)$ for any conjugately similar transformation T, then we have a finite modular[7] m_T on R.

II. If R is a conjugately similar space by T, then
1) m_T and $\overline{m_T}$ are normal[8] and monotone complete[9];

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4) N is a linear manifold of R and if $N\ni|a|\geqq|b|$ then $b\in N$ and for any $a_\lambda\in N$ $(\lambda\in\Lambda)$ $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda} a$ we have $a\in N$ ([1], §4).

5) For $\bar{a}\in\bar{R}$ the notation $\bar{a}[N]$ means $(a,\bar{a}[N])=([N]a,\bar{a})$ $(a\in R)$.

6) For any $a\in R$ we have an unique decomposition of a such that $a=a_1+a_2$, $a_1\in N$, $a_2\in N^\perp=\{x;\ |x|\frown|y|=0$ for all $y\in N\}$. We put $[N]a=a_1$ ([1], §4).

7) The modular m is finite if and only if $m(a)<+\infty$ for all $a\in R$ ([1], p. 196).

8) The modular m is normal if and only if m is finite and for any $a\in R$ $m(\xi a)$ is a strictly convex function of ξ ([1], p. 263).

9) The modular m is monotone complete if and only if for any $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}$, $\sup_{\nu\in\Lambda} m(a_\lambda)<+\infty$ there exists $\bigcup_{\lambda\in\Lambda} a_\lambda$ in R ([1], p. 157).

2) T^{-1} is also a conjugately similar transformation;

3) $\bar{R}^{m_T}=\bar{R}$ and $\overline{m_T}=m_{T^{-1}}$;

4) $m_T(|a|)+m_{T^{-1}}(T|a|)=(|a|, T|a|)$ $(a\in R)$.

Conversely for the modulared semi-ordered linear space $R(m)$, if m and $\bar{m}$ are normal and monotone complete, then there exists a conjugately similar transformation T such that R is a conjugately similar space by T and $m=m_T$ and $\bar{m}=m_{T^{-1}}$.

The purpose of this paper is a generalization of the above results to the most general case. We shall discuss in §1 a generalized conjugately similiar transformation and a representation of the modular as the integration of a generalized conjugately similar transformation. In §2 we shall generalize the concept of an inverse transformation of a conjugately similar transformation and study the relation between a conjugate modular and a generalized inverse transformation of a conjugately similar transformation. In §3 we shall state the classification of several known types of modulars in other words, that is, according to the types of conjugately similar transfermations. In §4 we shall treat new types of modulars and their conjugate types.

Throughout this paper we shall use notations and terminologies according to H. Nakano's book [1].

Before entering into the details I wish to express my gratitude to Professor Nakano for his kind encouragement and advice.

§1. Modulars and Conjugately similar transformations. Let R be a universally continuous semi-ordered linear space and $\bar{R}$ be its conjugate space.

Definition. A mapping T from a subset M of R^+ into $\bar{R}^+$ is called a *conjugately similar transformation* on R and M is called the domain of T, if the following conditions are satisfied

(C)

1) if $M\ni a\geqq b\geqq 0$, then $b\in M$;
2) for any $a, b\in M$ we have $a\smile b\in M$;
3) for any $a\in R^+$ there exists $\alpha>0$ such that $\alpha a\in M$;
4) if $M\ni\xi a>0$ for all $\xi>0$, then there exists $\xi_0>0$ such that $T\xi_0 a>0$;
5) $M\ni a\geqq b\geqq 0$ implies $Ta\geqq Tb\geqq 0$;
6) for any $a\in M$ and normal manifold N in R we have $T([N]a)=(Ta)[N]$;

7) if $M \ni a_\lambda$ $(\lambda \in \Lambda)$, $a_\lambda \uparrow_{\lambda \in \Lambda} a$ and $\sup_{\lambda \in \Lambda} (a_\lambda, Ta_\lambda) < +\infty$, then we have $\alpha a \in M$ for all $0 \leqq \alpha < 1$.

Evidently the above definition is a generalization of (2). We shall use the notation (T, M) to show a conjugately similar transformation having a domain M.

From the definition we can see easily the following lemma.

Lemma 1. i) *If $(a, Ta)=0$ for some $a \in M$, then we have $Ta=0$. Especially $T0=0$.*

ii) *For any $a, b \in M$ such that $a \frown b=0$ we obtain $a+b \in M$ and $T(a+b)=Ta+Tb$.*

iii) *For $a, b \in M$ we have $T(a \asymp b) \doteq Ta \asymp Tb$.*

For instance, if $a, b \in M$ and $a \frown b=0$, then from 2) of (C) $a+b \in M$ and from 6) of (C)

$$\begin{aligned}(x, T(a+b)) &= ([a+b]x, T(a+b)) = ([a]x, T(a+b)) + ([b]x, T(a+b)) \\ &= (x, T[a](a+b)) + (x, T[b](a+b)) = (x, Ta+Tb)\end{aligned}$$

for all $x \in R$. Therefore $T(a+b)=Ta+Tb$.

Theorem 1. *For a conjugately transformation T and its domain M*

$$m_T(a) = \int_0^1 (|a|, T\xi|a|)\, d\xi \quad (a \in R)$$

is a modular on R, where we put $(|a|, T\xi|a|)=+\infty$ for $\xi|a| \notin M$.

Proof. 1) of (M) is evident. Lemma 1 and 4) of (C) imply 2) and 6) of (M). 3) of (C) implies 3) of (M). 5) of (C) implies 4) and 5) of (M). We prove only 7) of (M).

At first, for any $[p_\lambda]\uparrow_{\lambda \in \Lambda}[a]$ we see $\sup_{\lambda \in \Lambda} m_T([p_\lambda]a) = m_T(a)$. Because from 3) of (C) there exists ξ_0, $0 < \xi_0 \leqq +\infty$ such that

$$\begin{cases} \xi_1 |a| \in M & \text{for all } 0 \leqq \xi_1 < \xi_0, \\ \xi_2 |a| \notin M & \text{for all } \xi_2 > \xi_0. \end{cases}$$

For ξ_1, $0 \leqq \xi_1 < \xi_0$ we have $([p_\lambda]|a|, T\xi_1[p_\lambda]|a|) = ([p_\lambda]|a|, T\xi_1|a|)$ from 6) of (C). Therefore $\sup_{\lambda \in \Lambda}([p_\lambda]|a|, T\xi_1[p_\lambda]|a|) = (|a|, T\xi_1|a|)$ for all $0 \leqq \xi_1 < \xi_0$. If for some ξ_2, $+\infty > \xi_2 > \xi_0$ we have $\sup_{\lambda \in \Lambda}([p_\lambda]|a|, T\xi_2[p_\lambda]|a|) < +\infty$, then from 7) of (C) we obtain $\alpha\xi_2|a| \in M$ for all α, $0 \leqq \alpha < 1$. This contradicts the property of ξ_0. Therefore we have $\sup_{\lambda \in \Lambda}([p_\lambda]|a|, T\xi_2[p_\lambda]|a|) = +\infty$ $= (|a|, T\xi_2|a|)$ for all ξ_2, $+\infty > \xi_2 > \xi_0$. Hence we have

$$\sup_{\lambda \in \Lambda}([p_\lambda]|a|, T\xi[p_\lambda]|a|) = (|a|, T\xi|a|) \text{ for all } \xi,\ \xi \neq \xi_0.$$

Therefore

$$\sup_{\lambda\in\Lambda} m_T([p_\lambda]a)=\sup_{\lambda\in\Lambda}\int_0^1([p_\lambda]|a|,\ T\xi[p_\lambda]|a|)d\xi=\int_0^1\sup_{\lambda\in\Lambda}([p_\lambda]|a|,\ T\xi[p_\lambda]|a|)d\xi$$
$$=\int_0^1(|a|,\ T\xi|a|)d\xi=m_T(a).$$

Next for any $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}a$ and $0\leqq\alpha<1$ we put $p_\lambda=(a_\lambda-\alpha a)^+$, then we have $[p_\lambda]\uparrow_{\lambda\in\Lambda}[a]$ and $a\geqq a_\lambda\geqq[p_\lambda]a_\lambda\geqq\alpha[p_\lambda]a$. Therefore $m_T(a)\geqq m_T(a_\lambda)\geqq m_T(\alpha[p_\lambda]a)$ and this implies $m_T(a)\geqq\sup_{\lambda\in\Lambda}m_T(a_\lambda)\geqq\sup_{\lambda\in\Lambda}m_T(\alpha[p_\lambda]a)=m_T(\alpha a)$ for all $0\leqq\alpha<1$. On the other hand $\sup_{0\leqq\alpha<1}m_T(\alpha a)=\sup_{0\leqq\alpha<1}\int_0^\alpha(a,\ T\xi a)d\xi=$ $=\int_0^1(a,\ T\xi a)\,d\xi$. Therefore we have $\sup_{\lambda\in\Lambda}m_T(a_\lambda)=m_T(a)$ Q.E.D.

For (T,M) satisfying 1)$\sim$7) of (C), we put

$$(3)\quad\begin{aligned}&M_+=\{a;\ \alpha a\in M\text{ for some }\alpha>1\text{ depending on }a\},\\&M_-=\{a;\ \alpha a\in M\text{ for all }0\leqq\alpha<1\text{ and }\bigcup_{0\leqq\alpha<1}T\alpha a\text{ exists}\},\\&T_+a=\bigcap_{\alpha>1}T\alpha a\text{ for }a\in M_+,\\&T_-a=\bigcup_{0\leqq\alpha<1}T\alpha a\text{ for }a\in M_-.\end{aligned}$$

Evidently we have $M_+\subset M\subset M_-$ and $T_-a\leqq Ta\leqq T_+a$ $(a\in M_+)$.

We can see easily $(T_+,\ M_+)$ and $(T_-,\ M_-)$ have the following properties stronger than (C).

$(C_\pm)$

1) if $M_\pm\ni a\geqq b\geqq 0$, then $b\in M_\pm$,
2) for any $a, b\in M_\pm$ we have $a\smile b\ni M_\pm$;
3) for any $a\in R^+$ there exists $\alpha>0$ such that $\alpha a\in M_\pm$;
4) if $M_\pm\ni\xi a>0$ for all $\xi>0$, then there exists $\xi_0>0$ such that $T_\pm\xi_0 a>0$;
5) $M_\pm\ni a\geqq b\geqq 0$ implies $T_\pm a\geqq T_\pm b\geqq 0$;
6) for any $a\in M_\pm$ and normal manifold N in R we have $T_\pm([N]a)=(T_\pm a)[N]$;
7) (+) if $M_+\ni a_\lambda(\lambda\in\Lambda)$, $a_\lambda\uparrow_{\lambda\in\Lambda}a$ and $\sup_{\lambda\in\Lambda}(a_\lambda,\ T_+a_\lambda)<+\infty$, then we have $\alpha a\in M_+$ for all $0\leqq\alpha<1$;
   (−) if $M_-\ni a_\lambda$ $(\lambda\ni\Lambda)$, $a_\lambda\uparrow_{\lambda\in\Lambda}a$ and $\sup_{\lambda\in\Lambda}(a_\lambda,\ T_-a_\lambda)<+\infty$, then we have $a\in M_-$;
8) (+) for any $a\in M_+$ there exists $\alpha>1$ such that $\alpha a\in M_+$ and we have $T_+a=\bigcap_{\alpha>1}T_+\alpha a$;
   (−) for any $a\in M_-$ we have $T_-a=\bigcup_{0\leqq\alpha<1}T_-\alpha a$.

From the above $(C_\pm)$ we can prove

8′) (+) for $a_\lambda \in M$ $(\lambda \in \Lambda)$ and $a_\lambda \downarrow_{\lambda\in\Lambda} a \geqq 0$ we have $T_+[a]a_\lambda \downarrow_{\lambda\in\Lambda} T_+a$;
(−) for $a_\lambda \in M$ $(\lambda \in \Lambda)$ and $a_\lambda \uparrow_{\lambda\in\Lambda} a \in M_-$ we have $T_-a_\lambda \uparrow_{\lambda\in\Lambda} T_-a$.

(+): Without loss of generally we can put $a_\lambda \leqq a_0 \in M_+$ $(\lambda \in \Lambda)$. If we put $p_\lambda = ([a]a_\lambda - \alpha a)$ for $\alpha > 1$ such as $\alpha a \in M_+$, then we have $[p_\lambda] \downarrow_{\lambda\in\Lambda} 0$. Since $[a]a_\lambda = [p_\lambda]a_\lambda + (1-[p_\lambda])a_\lambda \leqq [p_\lambda]a_0 + (1-[p_\lambda])\alpha a$, we have from the above 5) and 6) $T_+[a]a_\lambda \leqq T_+[p_\lambda]a_\lambda + T_+(1-[p_\lambda])\alpha a \leqq [p_\lambda]T_+a_0 + T_+\alpha a$, therefore $\bigcap_{\lambda\in\Lambda} T_+[a]a_\lambda \leqq \bigcap_{\lambda\in\Lambda} [p_\lambda]T_+a_0 + T_+\alpha a$. Hence $T_+a \leqq \bigcap_{\lambda\in\Lambda} T_+[a]a_\lambda \leqq \bigcap_{\lambda\in\Lambda} T_+\alpha a = T_+a$.

(−): For an arbitrary α such as $0 \leqq \alpha < 1$ we put $p_\lambda = (a_\lambda - \alpha a)^+$, then we have $[p_\lambda] \uparrow_{\lambda\in\Lambda} [a]$. Since $a_\lambda = [p_\lambda]a_\lambda + (1-[p_\lambda])a_\lambda \geqq [p_\lambda]\alpha a$, we have from the above 5) and 6) $T_-a_\lambda \geqq T_-[p_\lambda]\alpha a = [p_\lambda]T_-\alpha a$, therefore $T_-a \geqq \bigcup_{\lambda\in\Lambda} T_-a_\lambda \geqq \bigcup_{\lambda\in\Lambda} [p_\lambda]T_-\alpha a = T_-\alpha a$. Hence $T_-a \geqq \bigcup_{\lambda\in\Lambda} T_-a_\lambda \geqq \bigcup_{0\leqq\alpha<1} T_-\alpha a = T_-a$.

Lemma 2. *(T_+, M_+) and (T_-, M_-) are also conjugately similar transformations on R and we have $m_T = m_{T_+} = m_{T_-}$.*

It is clear from $(C_\pm)$ that (T_+, M_+) and (T_-, M_-) are conjugately similar transformations on R. From the definition of T_+ and T_- we have for all $\xi \geqq 0$ $(|a|, T_+\xi|a|) = \inf_{\xi<\eta} (|a|, T\eta|a|)$ and $(|a|, T_-\xi|a|) = \sup_{0\leqq\eta<\xi} (|a|, T\xi|a|)$. As $(|a|, T\xi|a|)$ is a monotone function of $\xi \geqq 0$, $(|a|, T\xi|a|)$ is equal to $(|a|, T_\pm\xi|a|)$ for almost everywheae $\xi \geqq 0$. Therefore $m_T(a) = m_{T_+}(a) = m_{T_-}(a)$ $(a \in R)$.

If we put for a modular m on R

$$(4)\qquad \begin{aligned} D_+ m(a) &= \begin{cases} \inf\limits_{\xi>1} \dfrac{m(\xi a) - m(a)}{\xi - 1} & \text{for } m(a) < +\infty \\ +\infty & \text{for } m(a) = +\infty, \end{cases} \\ D_- m(a) &= \begin{cases} \sup\limits_{0\leqq\xi<1} \dfrac{m(a) - m(\xi a)}{1 - \xi} & \text{for } m(a) < +\infty \\ +\infty & \text{for } m(a) = +\infty, \end{cases} \end{aligned}$$

then we can see easily

$$(5)\qquad \begin{aligned} &M_\pm = \{|a|;\ D_\pm m_T(a) < +\infty\}, \\ &(a, T_\pm a) = D_\pm m_T(a) \quad \text{for } a \in M_\pm. \end{aligned}$$

Next, in the following (Theorem 2 and 3) we shall prove the converse of Theorem 1, that is, for any modulared semi-ordered linear space $R(m)$ there exists a conjugately similar transformation (T, M) such that $m = m_T$. Theorem 2 is fundamental. However we shall assume some knowledge about the spectral theory of a semi-ordered linear space ([1], Chap. II.).

At first we state the properties of $D_\pm m(a)$ $(a \in R)$ ([4], §1).

Let $D_{+\xi}m(\xi a)$ $(D_{-\xi}m(\xi a))$ be the derivative at ξ of a function $m(\xi a)$ from the right (left) side. We see easily

$$(6)\quad \begin{aligned} &\xi D_{\pm\xi}m(\xi a)=D_{\pm}m(\xi a)\quad (\xi\geqq 0,\ a\in R),\\ &m(a)=\int_0^1 D_{\pm\xi}m(\xi a)\,d\xi=\int_0^1 \frac{D_{\pm}m(\xi a)}{\xi}\,d\xi\quad (a\in R). \end{aligned}$$

From the convexity of $m(\xi a)$ $D_{+\xi}m(\xi a)$ $(D_{-\xi}m(\xi a))$ is a right (left) continuous increasing function and $D_{-\xi}m(\xi a)\leqq D_{+\xi}m(\xi a)$.

The property that characterizes the functionals $D_{\pm}m(a)$ $(a\in R)$ is the following.

Lemma 3. 1) $0\leqq D_{\pm}m(a)\leqq +\infty$ *for all* $a\in R$;

2) *if* $D_{\pm}m(\xi a)=0$ *for all* $\xi\geqq 0$, *then* $a=0$;

3) *for any* $a\in R$ *there exists* $\alpha>0$ *such that* $D_{\pm}m(\alpha a)<+\infty$;

$(D_{\pm})$ 4) $(+)$ $\dfrac{D_{+}m(\xi a)}{\xi}$ *is a right continuous increasing function of* $\xi>0$;

$(-)$ $\dfrac{D_{-}m(\xi a)}{\xi}$ *is a left continuaus increasing function of* $\xi>0$;

5) $|a|\leqq|b|$ *implies* $D_{\pm}m(a)\leqq D_{\pm}m(b)$;

6) $a\frown b=0$ *implies* $D_{\pm}m(a+b)=D_{\pm}m(a)+D_{\pm}m(b)$;

7) $(+)$ *if* $a_0\geqq a_\lambda\downarrow_{\lambda\in\Lambda}a\geqq 0$ and $D_{+}m(a_0)<+\infty$, *then* $\inf_{\lambda\in\Lambda}D_{+}m(a_\lambda)=D_{+}m(a)$;

$(-)$ *if* $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}a$, *then* $\sup_{\lambda\in\Lambda}D_{-}m(a_\lambda)=D_{-}m(a)$.

1)$\sim$4) are all evident and (M) 6) implies 6).

The proof of 5): We may put $0\leqq a\leqq b$, $b=[a]b$ and $D_{+}m(b)<+\infty$, from the spectral theory we can find $b_n\geqq a$ $(n=1,2\cdots)$ such that $b_n\uparrow_{n=1}^{\infty}b$ and $b_n=\sum_{\nu=1}^{\kappa_n}\xi_{\nu,n}[p_{\nu,n}]a$, $\xi_{\nu,n}\geqq 1$ $(n=1,2\cdots;\ \nu=1,2\cdots\kappa_n)$, $\sum_{\nu=1}^{\kappa_n}[p_{\nu,n}]=[a]$. For any $\varepsilon>0$ $\dfrac{m((1+\varepsilon)b_n)-m(b_n)}{\varepsilon}\geqq D_{+}m(b_n)=\sum_{\nu=1}^{\kappa_n}D_{+}m(\xi_{\nu,n}[p_{\nu,n}]a)\geqq\sum_{\nu=1}^{\kappa_n}D_{+}m([p_{\nu,n}]a)$ $=D_{+}m(a)$, on the other hand from (M) 7) $\lim_{n\to\infty}\dfrac{m((1+\varepsilon)b_n)-m(b_n)}{\varepsilon}$ $=\dfrac{m((1+\varepsilon)b)-m(b)}{\varepsilon}$, therefore $\dfrac{m((1+\varepsilon)b)-m(b)}{\varepsilon}\geqq D_{+}m(a)$, hence $D_{+}m(b)$ $\geqq D_{+}m(a)$. Similary we have $D_{-}m(b)\geqq D_{-}m(a)$.

The proof of 7) $(+)$: If $a=0$, then we have for all $\xi>1$ $0\leqq\inf_{\lambda\in\Lambda}D_{+}m(a_\lambda)$

$\leqq \inf_{\lambda\in\Lambda} \frac{m(\xi a_\lambda)-m(a_\lambda)}{\xi-1} \leqq \inf_{\lambda\in\Lambda} \frac{m(\xi a_\lambda)}{\xi-1}=0$, because we can prove $\inf_{\lambda\in\Lambda} m(\xi a_\lambda)=0$ from (*M*) 7) and the assumption $m(\xi a_0)<+\infty$ ([1], p. 155). For a general case, putting $p_\lambda=([a]a_\lambda-\alpha a)^+$ for $\alpha>1$, we have $a_\lambda=[p_\lambda]a_\lambda+(1-[p_\lambda])a_\lambda+(1-[a])a_\lambda \leqq [p_\lambda]a_0+(1-[p_\lambda])\alpha a+(1-[a])a_\lambda$. Therefore from 5) and 6) $D_+m(a_\lambda)\leqq D_+m([p_\lambda]a_0)+D_+m((1-[p_\lambda])\alpha a)+D_+m((1-[a])a_\lambda) \leqq D_+m([p_\lambda]a_0)+D_+m(\alpha a)+D_+m((1-[a])a_\lambda)$. As $\inf_{\lambda\in\Lambda}[p_\lambda]a_0=0$ and $\inf_{\lambda\in\Lambda}(1-[a])a_\lambda=0$, we have $\inf_{\lambda\in\Lambda} D_+m([p_\lambda]a_0)=0$ and $\inf_{\lambda\in\Lambda} D_+m((1-[a])a_\lambda)=0$. Therefore $D_+m(a) \leqq \inf_{\lambda\in\Lambda} D_+m(a)\leqq D_+m(\alpha a)$ for all $\alpha>1$. From 4) (+) we have $D_+m(a)=\inf_{\lambda\in\Lambda} D_+m(a_\lambda)$.

The proof of 7) (−): With the similar technique for the case of $D_-m(a)<+\infty$ we can obtain $\sup_{\lambda\in\Lambda} D_-m(a_\lambda)=D_-m(a)$. At first if $a_\lambda=[p_\lambda]a$ and $[p_\lambda]\uparrow_{\lambda\in\Lambda}[a]$, then from 6) and 7) of (*M*) we have $\sup_{\lambda\in\Lambda} \frac{m([p_\lambda]a)-m(\xi[p_\lambda]a)}{1-\xi} = \frac{m(a)-m(\xi a)}{1-\xi}$ for $0\leqq\xi<1$ and therefore we have $\sup_{\lambda\in\Lambda} D_-m([p_\lambda]a)=\sup_{\lambda\in\Lambda} \sup_{0\leqq\xi<1} \frac{m([p_\lambda]a)-m(\xi[p_\lambda]a)}{1-\xi} = \sup_{0\leqq\xi<1}\sup_{\lambda\in\Lambda} \frac{m([p_\lambda]a)-m(\xi[p_\lambda]a)}{1-\xi} = \sup_{0\leqq\xi<1} \frac{m(a)-m(\xi a)}{1-\xi}=D_-m(a)$. Next, putting $p_\lambda=(a_\lambda-\alpha a)^+$ for $0\leqq\alpha<1$, we have $[p_\lambda]\uparrow_{\lambda\in\Lambda}[a]$ and $a_\lambda=[p_\lambda]a_\lambda+(1-[p_\lambda])a_\lambda\geqq[p_\lambda]\alpha a$, therefore from the above $\sup_{\lambda\in\Lambda} D_-m(a_\lambda)\geqq \sup_{\lambda\in\Lambda} D_-m([p_\lambda]\alpha a)=D_-m(\alpha a)$, Hence $D_-m(a)\geqq \sup_{\lambda\in\Lambda} D_-m(a_\lambda) \geqq \sup_{0\leqq\alpha<1} D_-m(\alpha a)=D_-m(a)$. If $D_-m(a)=+\infty$ and $\sup_{\lambda\in\Lambda} D_-m(a_\lambda)=\gamma<+\infty$, then we have a contradiction. Because: There exists some ξ_0, $0<\xi_0\leqq 1$ such that $D_-m(\xi a)=+\infty$ for $\xi>\xi_0$ and $D_-m(\xi a)<+\infty$ for $0\leqq\xi<\xi_0$. And $D_-m(\xi_0 a) = \sup_{0\leqq\xi<\xi_0} D_-m(\xi a) = \sup_{0\leqq\xi<\xi_0} \sup_{\lambda\in\Lambda} D_-m(\xi a_\lambda) = \sup_{\lambda\in\Lambda} \sup_{0\leqq\xi<\xi_0} D_-m(\xi a) = \sup_{\lambda\in\Lambda} D_-m(\xi_0 a_\lambda)\leqq \sup_{\lambda\in\Lambda} D_-m(a_\lambda)<+\infty$. Therefore $m(\xi_0 a)<+\infty$ and $m(a_\lambda)\leqq(1-\xi_0) D_-m(a_\lambda)+m(\xi_0 a_\lambda) \leqq (1-\xi_0)\gamma+m(\xi_0 a)<+\infty$. Hence we have $m(a)=\sup_{\lambda\in\Lambda} m(a_\lambda)\leqq(1-\xi_0)\gamma+m(\xi_0 a)<+\infty$. This implies $D_-m(\xi a)<+\infty$ for all $1>\xi\geqq\xi_0$. This contradicts the property of ξ_0.

Remark. For a functional $f(a)$ $(a\in R)$ there exists a modular m on R such that $f(a)=D_-m(a)$ $(a\in R)$ if and only if f satisfies (D_-).

Theorem 2. *Let $a \in R^{+}$ and $\mu([p])$ be a universally additive*[10] *finite measure on the Boolean ring of projectors satisfying*

(7) $$D_{-}m([p]a) \leqq \mu([p]) \leqq D_{+}m([p]a) \text{ for all } [p],$$

then we can find uniquely $\bar{a} \in \bar{R}^{m}$ such that $\bar{a}[a] = \bar{a}$ and $\mu([p]) = ([p]a, \bar{a})$ for all $[p]$.

Proof. The expression (7) implies $\mu([p]) = 0$ for $[p][a] = 0$. Further from the universal additivity of μ we can find a projector $[a_1] \leqq [a]$ such that $\mu([p]) = 0$ for $[p][a_1] = 0$ and $\mu([p]) > 0$ for $0 \neq [p] \leqq [a_1]$. For $[a_1]$ we have also $m([p]a) = 0$ for $[p][a_1] = 0$, because the first inequality implies $m([p]a) \leqq \mu([p])$.

The condition (7) means $\dfrac{m(\xi[p]a) - m([p]a)}{\xi - 1} \geqq \mu([p])$ for all $\xi > 1$ and $\dfrac{m([p]a) - m(\xi[p]a)}{1 - \xi} \leqq \mu([p])$ for all $\xi < 1$. Therefore we have $\mu([p]) - m([p]a) + m(\xi[p]a) \geqq |\xi| \mu([p])$ for all ξ and $[p]$. Hence we obtain

(8) $$1 - \frac{m([p]a)}{\mu([p])} + \frac{m(\xi[p]a)}{\mu([p])} \geqq |\xi| \text{ for all } \xi \text{ and } 0 \neq [p] \leqq [a_1]$$

We consider the derivative of $m([p]x)$ by $\mu([p])$, that is, for any $x \in R$ and maximal ideal $\mathfrak{p}$ consisting of projectors such that $\mathfrak{p} \ni [a_1]$ we can define

(9) $$\rho(x, \mathfrak{p}) = \lim_{[p] \to \mathfrak{p}} \frac{m([p]x)}{\mu([p])} \quad (x \in R,\ \mathfrak{p} \ni [a_1]).$$

As the inverse expression of this we have

(10) $$m([p]x) = \int_{[p]} \rho(x, \mathfrak{p})\, \mu(d\mathfrak{p}) \text{ for all } [p] \leqq [a_1].$$

Tending $[p]$ to $\mathfrak{p}$ in (8), we obtain for any maximal ideal $\mathfrak{p} \ni [a_1]$

(11) $$1 - \rho(a, \mathfrak{p}) + \rho(\xi a, \mathfrak{p}) \geqq |\xi|.$$

Further we can prove for any $x \in R$ and maximal ideal $\mathfrak{p} \ni [a_1]$

(12) $$1 - \rho(a, \mathfrak{p}) + \rho(x, \mathfrak{p}) \geqq \left|\left(\frac{x}{a}, \mathfrak{p}\right)\right|,$$

where $\left(\dfrac{x}{a}, \mathfrak{p}\right)$ means a relative spectrum of x by a at $\mathfrak{p}$ ([1]), p. 34).

Because, if $\left|\left(\dfrac{x}{a}, \mathfrak{p}_0\right)\right| > 0$ for some $\mathfrak{p}_0 \ni [a_1]$, then for any ξ, $0 \leqq \xi < \left|\left(\dfrac{x}{a}, \mathfrak{p}_0\right)\right|$,

10) If $[p][q] = 0$, then $\mu([p] \smile [q]) = \mu([p]) + \mu([q])$. And if $[p_\lambda] \uparrow_{\lambda \in \Lambda} [p]$, then $\sup_{\lambda \in \Lambda} \mu([p_\lambda]) = \mu([p])$.

there exists $[p_0]$ such that $\mathfrak{p}_0 \ni [p_0] \leqq [a_1]$ and $[p]|x| \geqq \xi[p]a$ for all $[p] \leqq [p_0]$. Therefore we have $\rho(x, \mathfrak{p}_0) \geqq \rho(\xi a, \mathfrak{p}_0)$, hence from (11) $1-\rho(a, \mathfrak{p}_0)+\rho(x, \mathfrak{p}_0) \geqq 1-\rho(a, \mathfrak{p}_0)+\rho(\xi a, \mathfrak{p}_0) \geqq |\xi|$, therefore $1-\rho(a, \mathfrak{p}_0)+\rho(x, \mathfrak{p}_0) \geqq \left|\left(\frac{x}{a}, \mathfrak{p}_0\right)\right|$.

Next the expression (12) shows that $\left(\frac{x}{a}, \mathfrak{p}\right)$ is integrable by μ on $[a_1]$. Because, for $x \in R$ we can find $\alpha>0$ such that $m(\alpha x)<+\infty$, from (12) we have $\frac{1}{\alpha}\{1-\rho(a, \mathfrak{p})+\rho(\alpha x, \mathfrak{p})\} \geqq \left|\left(\frac{x}{a}, \mathfrak{p}\right)\right|$, and considering (10), the left side is integrable by μ on $[a_1]$. Therefore

$$\int_{[a_1]}\left|\left(\frac{x}{a}, \mathfrak{p}\right)\right| \mu(d\mathfrak{p}) \leqq \frac{1}{\alpha}\{\mu([a_1])-m([a_1]a)+m(\alpha[a_1]x)\}<+\infty .$$

If we put $L(x)=\int_{[a_1]}\left(\frac{x}{a}, \mathfrak{p}\right) \mu(d\mathfrak{p})$ $(x \in R)$, evidently L is a positive linear functional and we have

$$\text{(13)} \quad \begin{aligned} & L([p]a)=\mu([p][a_1])=\mu([p]) \text{ for all } [p], \text{ and} \\ & |L([p]x)| \leqq \frac{1}{\alpha}\{\mu([p][a_1])-m([p][a_1]a)+m(\alpha[p][a_1]x)\}. \end{aligned}$$

L is universally continuous. Because for any $[p_\lambda]\downarrow_{\lambda \in \Lambda} 0$ we have $0 \leqq \inf_{\lambda \in \Lambda} |L([p_\lambda]x)| \leqq \frac{1}{\alpha} \inf_{\lambda \in \Lambda}\{\mu([p_\lambda][a_1])-m([p_\lambda][a_1]a)+m(\alpha[p_\lambda][a_1]x)\}=0$ from the universal additivity of μ and the modular condition 7) of (M). Therefore $L=\bar{a} \in \bar{R}$. We see easily $\bar{a}[a]=\bar{a} \geqq 0$ and $([p]a, \bar{a})=\mu([p])$ for all $[p]$ and for any $x \in R$ $(x, \bar{a}) \leqq \mu([a_1])-m([a_1]a)+m([a_1]x) \leqq \mu([a])+m(a)+m(x)$, considering $\mu([a_1])=\mu([a])$ and $m([a_1]a)\doteq m(a)$. Hence from the inequality $(x, \bar{a}) \leqq (a, \bar{a})-m(a)+m(x)$ we have $\bar{m}(\bar{a})\doteq(a, \bar{a})-m(a)<+\infty$ and $\bar{a} \in \bar{R}^m$.

Such $\bar{a}$ is unique, because if $([p]a, \bar{a})=([p]a, \bar{b})$ for all $[p]$, then from the spectral theory we have $([a]x, \bar{a})=([a]x, \bar{b})$ for all $x \in R$, that is, $\bar{a}=\bar{a}[a]=\bar{b}[a]=\bar{b}$. Q.E.D.

Lemma 4. *For any $a \in R^+$ and $\bar{a} \in \bar{R}^+$ we have $D_- m([p]a) \leqq ([p]a, \bar{a}) \leqq D_+ m([p]a)$ for all $[p]$ if and only if $\bar{a}[a] \in \bar{R}^m$ and $m(a)+\bar{m}(\bar{a}[a])=(a, \bar{a})$.*

Because, let be $D_- m([p]a) \leqq ([p]a, \bar{a}) \leqq D_+ m([p]a)$ for every $[p]$, then we have proved in the previous theorem $\bar{m}(\bar{a}[a])=(a, \bar{a})-m(a)$ and $\bar{a}[a] \in \bar{R}^m$. Conversely, let be $m(a)+\bar{m}(\bar{a}[a])=(a, \bar{a})$, then we have $m([p]a)+\bar{m}(\bar{a}[a][p])=([p]a, \bar{a})$ for all $[p]$ ([1], p. 178). On the other hand for $\xi>1$ we have $m(\xi[p]a)+\bar{m}(\bar{a}[a][p]) \geqq \xi([p]a, \bar{a})$ from (1). Therefore

$\dfrac{m(\xi[p]a)-m([p]a)}{\xi-1} \geqq ([p]a, \bar{a}) \, (\xi>1)$, hence $D_+m([p]a) \geqq ([p]a, \bar{a})$. Similary we have $D_-m([p]a) \leqq ([p]a, \bar{a})$.

We put

$$(14) \qquad \begin{aligned} M_+ &= \{|a|;\ D_+m(a) < +\infty\}, \\ M_- &= \{|a|;\ D_-m(a) < +\infty\}. \end{aligned}$$

Then $(D_\pm)$ 7) show that for $a \in M_\pm$ $D_\pm m([p]a)$ are the universally additive finite measure on the Boolean ring of all $[p]$. Therefore on account of theorem 2 we find a mapping $T_+(T_-)$ from the domain $M_+(M_-)$ into $\bar{R}^+$ such that

$$(15) \qquad \begin{aligned} &([p]a,\ T_+a) + D_+m([p]a) \text{ for all } [p] \text{ and } (T_+a)[a] = T_+a, \\ &([p]a,\ T_-a) + D_-m([p]a) \text{ for all } [p] \text{ and } (T_-a)[a] = T_-a. \end{aligned}$$

(T_+, M_+) satisfies (C_+) and (T_-, M_-) satisfies (C_-).

Because, $(D_\pm)$ 5) imply $(C_\pm)$ 1). $(C_\pm)$ 2) are proved by $(D_\pm)$ 5) and 6): For $a, b \in M_\pm$ $D_\pm m(a \smile b) = D_\pm m([(a-b)^+]a + [(b-a)^+]b) = D_\pm m([(a-b)^+]a) + D_\pm m([(b-a)^+]b) \leqq D_\pm m(a) + D_\pm m(b) < +\infty$. $(D_\pm)$ 3) imply $(C_\pm)$ 3) and $(D_\pm)$ 2) imply $(C_\pm)$ 4). The proof of $(C_\pm)$ 6): Since $D_\pm m([p][N]a) = D_\pm m([[N]p]a)$ for all $[p]$, we have $([p][N]a,\ T_\pm([N]a)) = ([[N]p]a,\ T_\pm a) = ([p]a,\ (T_\pm a)[N])$ for all $[p]$, therefore $T_\pm([N]a) = (T_\pm a)[N]$. (C_-) 7) is evident from (D_-) 7). And (C_+) 7) is proved easily, because, if $M_+ \ni a_\lambda \uparrow_{\lambda \in \Lambda}$ and $\sup_{\lambda \in \Lambda} (a_\lambda,\ T_+a_\lambda) < +\infty$, then we have $\sup_{\lambda \in \Lambda} (a_\lambda,\ T_-a_\lambda) \leqq \sup_{\lambda \in \Lambda} (a_\lambda,\ T_+a_\lambda) < +\infty$, therefore from (C_-) 7) we have $a \in M_-$, hence $\alpha a \in M_+$ for all $0 \leqq \alpha < 1$. $(C_\pm)$ 8) are implied from $(D_\pm)$ 4). The proof of (C_+) 5): If $M_+ \ni a \geqq b \geqq 0$, then there exist $b_n (n=1,2\cdots)$ such that $a \geqq b_n \downarrow_{n=1}^\infty b$ and $b_n = \sum_{\nu=1}^{\kappa_n} \xi_{\nu,n}[p_{\nu,n}]a$, where $0 \leqq \xi_{\nu,n} \leqq 1$ $(\nu=1,2\cdots,\ \kappa_n;\ n=1,2\cdots)$ and $\sum_{\nu=1}^{\kappa_n} [p_{\nu,n}] = [a]$ $(n=1,2\cdots)$. From (C_+) 6) we obtain $T_+b_n = \sum_{\nu=1}^{\kappa_n} T_+(\xi_{\nu,n}[p_{\nu,n}]a)$ and $T_+a = \sum_{\nu=1}^{\kappa_n} T_+([p_{\nu,n}]a)$ and from (D_+) 4) we have $T_+(\xi_{\nu,n}[p_{\nu,n}]a) \leqq T_+([p_{\nu,n}]a)$ $(n=1,2\cdots;\ \nu=1,2,\cdots,\kappa_n)$. Therefore $T_+b_n \leqq T_+a$ $(n=1,2,\cdots)$, hence $([p]b_n,\ T_+b_n) \leqq ([p]b_n,\ T_+a)$ for all $[p]$. On the other hand from (D_+) 7) we have $\inf_{n \geqq 1} ([p]b_n,\ T_+b_n) = ([p]b,\ T_+b)$, therefore $([p]b,\ T_+b) \leqq \inf_{n \geqq 1} ([p]b_n,\ T_+a) = ([p]b,\ T_+a)$ for all $[p]$. Hence $T_+b = (T_+b)[b] \leqq (T_+a)[b] \leqq T_+a$. In the same way we can prove (C_-) 5).

From the above we have obtained

Theorem 3. *For an arbitrary modulared semi-ordered linear space $R(m)$ (T_+, M_+) and (T_-, M_-) are conjugately similar transformations on*

R and $m=m_{T_+}=m_{T_-}$ *and we have* $m(a)+\overline{m}(T_{\pm}a)=(a, T_{\pm}a)$ *for* $a\in M_{\pm}$.

We shall consider the conjugately similar transformation on the modular function space ([1], appendix I) which is a concrete representation of a modulared semi-ordered linear space.

Let $\Omega(\mathfrak{B}, \mu)$ be a measure space, that is, Ω be a abstract space, $\mathfrak{B}$ be a totally additive class of subsets in Ω and $\mu(B)$ $(B\in\mathfrak{B})$ be a finite measure on $\mathfrak{B}$. Let $\Phi(\xi, \omega)$ be a function on $[0, +\infty)\times\Omega$ satisfying following conditions

1) When ξ is fixed, $\Phi(\xi, \omega)$ is a $\mathfrak{B}$-measurable function;
2) When ω is fixed, $\Phi(\xi, \omega)$ is a convex non-decreasing left continuous function of $\xi\geqq 0$;
3) $\Phi(0, \omega)=0$ for every $\omega\in\Omega$;
4) $\lim_{\xi\to+\infty}\Phi(\xi, \omega)=+\infty$ for every $\omega\in\Omega$;
5) for any $\omega\in\Omega$ there exists $\alpha_\omega>0$ such that $\Phi(\alpha_\omega, \omega)<+\infty$.

We shall denote by R_Φ the class of all measurable functions $a(\omega)$ $(\omega\in\Omega)$ such that for some $\alpha>0$ we have $\int_\Omega \Phi(\alpha\,|\,a(\omega)\,|, \omega)\,d\mu<+\infty$. Putting $m_\Phi(a)=\int_\Omega \Phi(|\,a(\omega)\,|, \omega)\,d\mu$ for all $a\in R_\Phi$, $R_\Phi(m_\Phi)$ is a modulared semi-ordered linear space.

Let $\overline{\Phi}(\xi, \omega)$ be a complementary convex function of $\Phi(\xi, \omega)$ in the sense of Young for every fixed $\omega\in\Omega$. $\overline{\Phi}(\xi, \omega)$ satisfies the same condisions as $\Phi(\xi, \omega)$. Therefore for $\overline{\Phi}(\xi, \omega)$ we obtain a modulared semi-ordered linear space $R_{\overline{\Phi}}(m_{\overline{\Phi}})$ as we have obtained $R_\Phi(m_\Phi)$ for $\Phi(\xi, \omega)$.

For $\overline{a}\in R_{\overline{\Phi}}$, putting $(a, \overline{a})=\int_\Omega a(\omega)\overline{a}(\omega)d\mu$ $(a\in R_\Phi)$, we obtain a universally continuous linear functional on R_Φ and we can prove $R_{\overline{\Phi}}(m_{\overline{\Phi}})$ is the modular conjugate space of $R_\Phi(m_\Phi)$.

Since $\Phi(\xi, \omega)$ is a convex function of $\xi\geqq 0$, we denote by $\varphi(\xi, \omega)$ the left derivative at $\xi\geqq 0$ of $\Phi(\xi, \omega)$. Then $\varphi(\xi, \omega)$ satisfies also the same conditions as $\Phi(\xi, \omega)$ except the convexity about $\xi\geqq 0$ and 3). We see easily $D_-m_\Phi(a)=\int_\Omega |\,a(\omega)\,|\,\varphi(|\,a(\omega)\,|, \omega)\,d\mu$ for all $a\in R_\Phi$. Therefore we have $M_-=\{a;\ a\geqq 0 \text{ and } \int_\Omega a(\omega)\varphi(a(\omega), \omega)\,d\mu<+\infty\}$. By Young's inequality we have for $a\in M_-$ $\Phi(a(\omega), \omega)+\overline{\Phi}(\varphi(a(\omega), \omega), \omega)=a(\omega)\varphi(a(\omega), \omega)$, hence $\int_\Omega \Phi(a(\omega), \omega)\,d\mu$

$+\int_\Omega \bar{\Phi}(\varphi(a(\omega),\omega),\omega)\,d\mu=\int_\Omega a(\omega)\varphi(a(\omega),\omega)\,d\mu<+\infty$. Therefore $\varphi(a(\omega),\omega)\in R_{\bar{\Phi}}$. Furthermore, since for any $b\in R_\Phi$ we have $D_{-}m_\Phi([b]a)=\int_\Omega \chi_b(\omega)a(\omega)\varphi(\chi_b(\omega)a(\omega),\omega)\,d\mu=\int_\Omega \chi_b(\omega)a(\omega)\varphi(a(\omega),\omega)\,d\mu=\int_\Omega ([b]a)(\omega)\varphi(a(\omega),\omega)\,d\mu$, where χ_b is a characteristic function of the set $\{\omega;\,b(\omega)\neq 0\}$, we see $(T_{-}a)(\omega)=\varphi(a(\omega),\omega)$ for $a\in M_{-}$.

Especially, if $\Phi(\xi,\omega)=\frac{1}{p}\xi^p$ for some $p\geqq 1$, that is, R_Φ is L_p-space, then $M_{-}=R_\Phi^+$ and $T_{-}a=a^{p-1}$ for $a\in R_\Phi^+$.

§2. **Conjugate modulars.** Let $R(m)$ be a modulared semi-ordered linear space and $\bar{R}^m(\bar{m})$ be its modular conjugate space. On account of the results in §1 we can find conjugately similar transformations (T, M) on R and $(\bar{T},\bar{M})$ on $\bar{R}^m$ such that $m=m_T$ and $\bar{m}=m_{\bar{T}}$. Especially if $R(m)$ is on dimensional, then $\bar{T}$ is a non-decreasing function and $\bar{T}$ is the inverse function of T. In this section this relation between T and $\bar{T}$ will be generalized to the general case and at this point of view we shall construct directly $(\bar{T},\bar{M})$ from (T, M). However, we shall assume that $R(m)$ is monotone complete.

For a conjugately similar transformation (T, M) on R_T m_T is monotone complete if and only if the following condition is satisfied:

(16) If $M\ni a_\lambda\uparrow_{\lambda\in\Lambda}$ and $\sup_{\lambda\in\Lambda}(a_\lambda, Ta_\lambda)<+\infty$, then there exists a in R such that $a=\bigcup_{\lambda\in\Lambda} a_\lambda$.

Because, from the expression $m_T(a_\lambda)=\int_0^1 (a_\lambda, T\xi a_\lambda)\,d\xi$ we have easily $\left(\frac{1}{2}a_\lambda, T\frac{1}{2}a_\lambda\right)\leqq m(a_\lambda)\leqq(a_\lambda, Ta_\lambda)$. (T, M) satisfying (16) is called also *monotone completes*.

Through this section we shall assume R is semi-regular[11] and (T, M) is monotone complete and satisfies (C_{-}). According to the assumption that R is semi-regular R can be embedded isomorphically into $\bar{\bar{R}}$ and R is a semi-normal manifold of $\bar{\bar{R}}$ (Nakano's theorem about reflexibility). And from the assumption that (T, M) is monotone complete we have $\bar{R}^{m_T}=\bar{R}$ ([1], p. 173).

11) For any $a>0$, $a\in R$ we can find some $\bar{a}\in\bar{R}$ such as $(a,\bar{a})\neq 0$ ([1], p. 92).

We introduce a notation $\bar{a} \succ \bar{b}$ for $\bar{a}, \bar{b} \in \bar{R}^+$ having the following meaning:

(17) For all $[N]$ we have $\bar{a}[N] > \bar{b}[N]$ or $\bar{a}[N] = \bar{b}[N] = 0$.

We see easily for $\bar{a}, \bar{b}, \bar{c} \in \bar{R}^+$

1) if $\bar{a} \succ \bar{b}$, then $\bar{a}[N] \succ \bar{b}[N]$ for all $[N]$,
2) if $\bar{a} \succ \bar{b} \geqq \bar{c}$ or $\bar{a} \geqq \bar{b} \succ \bar{c}$, then $\bar{a} \succ \bar{c}$,
3) if $\bar{a} \succ \bar{b}$ and $\bar{a} \succ \bar{c}$, then $\bar{a} \succ \bar{b} \smile \bar{c}$,

(18) 4) if $\bar{b} \succ \bar{a}$ and $\bar{c} \succ \bar{a}$, then $\bar{b} \frown \bar{c} \succ \bar{a}$,
5) we have always $\bar{a} \succ 0$ and $\alpha\bar{a} \succ \bar{a}$ for all $\alpha > 1$,
6) if $\bar{a} \succ \bar{b}$, then $[\bar{a} - \bar{b}]^R = [\bar{a}]^R$,[12]
7) if $\bar{c} = (\bar{a} - \bar{b})^+$, then $\bar{a}[\bar{c}]^R \succ \bar{b}[\bar{c}]^R$.

For instance, the proof of 3): If $\bar{a}[N] = (\bar{b} \smile \bar{c})[N]$, putting $(\bar{b} - \bar{c})^+ = \bar{d}$, we have $(\bar{b} \smile \bar{c})[\bar{d}]^R = \bar{b}[\bar{d}]^R$ and $(\bar{b} \smile \bar{c})(1 - [\bar{d}]^R) = \bar{c}(1 - [\bar{d}]^R)$, therefore $\bar{a}[N][\bar{d}]^R = b[N][\bar{d}]^R = 0$ and $\bar{a}[N](1 - [\bar{d}]^R) = \bar{c}[N](1 - [\bar{d}]^R) = 0$, hence $\bar{a}[N] = (\bar{b} \smile \bar{c})[N] = 0$. Similarly we can see 4). The proof of 6): Since $0 \leqq \bar{a} - \bar{b} \leqq \bar{a}$, we have $[\bar{a} - \bar{b}]^R \leqq [\bar{a}]^R$. If we put $[N] = [\bar{a}]^R - [\bar{a} - \bar{b}]^R$, then $[N][\bar{a}]^R = [N]$ and $[N][\bar{a} - \bar{b}]^R = 0$. $[N][\bar{a} - \bar{b}]^R = 0$ implies $\bar{a}[N] = \bar{b}[N]$, therefore $\bar{a}[N] = 0$ from the assumption. Hence $[\bar{a}]^R[N] = [N] = 0$, that is, $[\bar{a}]^R = [\bar{a} - \bar{b}]^R$. The proof of 7): Evidently $\bar{a}[\bar{c}]^R \geqq \bar{b}[\bar{c}]^R$. If $\bar{a}[\bar{c}]^R[N] = \bar{b}[\bar{c}]^R[N]$, then $(\bar{a} - \bar{b})[\bar{c}]^R[N] = \bar{c}[N] = 0$, therefore $[\bar{c}]^R[N] = 0$ and $\bar{a}[\bar{c}]^R[N] = \bar{b}[\bar{c}]^R[N] = 0$.

For (T, M) we define a mapping $\bar{T}$ from the domain $\bar{M} \subset \bar{R}^+$ into R^+: $\bar{M} \ni \bar{a}$ if and only if $\bar{a} \geqq 0$ and $\{[\bar{a}]^R a;\ \bar{a} \succ Ta\}$ is bounded in R,

(19) $$\bar{T}\bar{a} = \bigcup_{\bar{a} \succ Ta} [\bar{a}]^R a.$$

Lemma 5.

i) *$T(M) \subset \bar{M}$ and $\bar{T}Ta \leqq a$ for all $a \in M$.*

ii) *$\bar{T}(\bar{M}) \subset M$ and $T\bar{T}\bar{a} \leqq \bar{a}$ for all $\bar{a} \in \bar{M}$.*

i): If $Ta \succ Tb$, then we see $[Ta]^R a \geqq [Ta]^R b$. Because, if $[Ta]^R a \ngeqq [Ta]^R b$, then there exists $[N]$ such as $[Ta]^R[N]a < [Ta]^R[N]b$, therefore $(Ta)[N] = T([Ta]^R[N]a) \leqq T([Ta]^R[N]b) \leqq T([N]b) = (Tb)[N]$ and hence $(Ta)[N] = (Tb)[N] = 0$, therefore $[Ta]^R[N] = 0$ and hence

12) If we put $N = \{a;\ (|a|, |\bar{a}|) = 0\}$ for a fixed $\bar{a}$, then N is a normal manifold of R. We denote $[\bar{a}]^R = [N]$ ([1]).

$[Ta]^R[N]a=[Ta]^R[N]b=0$, this is a contradiction. Therefore $Ta\in\overline{M}$ and $\overline{T}Ta=\bigcup_{Ta\succ Tb}[Ta]^Rb\leqq[Ta]^Ra\leqq a$ for $a\in M$.

ii): As $\overline{T}\overline{a}=\bigcup_{\overline{a}\succ Ta}[\overline{a}]^Ra$ for $\overline{a}\in\overline{M}$, we have $\sup_{\overline{a}\succ Ta}([\overline{a}]^Ra,\ T[\overline{a}]^Ra)\leqq(\overline{T}\overline{a},\ \overline{a})$ $<+\infty$, therefore from (16) $\bigcup_{\overline{a}\succ Ta}[\overline{a}]^Ra\in M$ and $T\overline{T}\overline{a}=\bigcup_{\overline{a}\succ Ta}T[\overline{a}]^Ra\leqq\overline{a}$ (8′)(−) of (C_-)).

Lemma 6. *We have* $(a,\ \overline{a})\leqq(a,\ Ta)+(\overline{T}\overline{a},\ \overline{a})$ *for* $a\in M$ *and* $\overline{a}\in\overline{M}$.

Because: If we put $\overline{b}=(Ta-\overline{a})^+$, then $\overline{a}[\overline{b}]^R\leqq(Ta)[\overline{b}]^R$, hence we have $([\overline{b}]^Ra,\ \overline{a})=(a,\ \overline{a}[\overline{b}]^R)\leqq(a,\ (Ta)[\overline{b}]^R)$. On the other hand we can decompose $1-[\overline{b}]^R$ into $[N_1]$ and $[N_2]$ such that $1-[\overline{b}]^R=[N_1]+[N_2]$, $(Ta)[N_1]=\overline{a}[N_1]$ and $(Ta)[N_2]\prec\overline{a}[N_2]\leqq\overline{a}$, and we have $\overline{T}\overline{a}\geqq[N_2][\overline{a}]^Ra$. Therefore $([N_1]a,\ \overline{a})=(a,\ \overline{a}[N_1])=(a,\ (Ta)[N_1])$ and $([N_2]a,\ \overline{a})=([\overline{a}]^R[N_2]a,\ \overline{a})\leqq(\overline{T}\overline{a},\ a)$. And we have $(a,\ \overline{a})=([\overline{b}]^Ra,\ \overline{a})+([N_1]a,\ \overline{a})+([N_2]a,\ \overline{a})\leqq(a,\ (Ta)[\overline{b}]^R)+(a,\ (Ta)[N_1])+(\overline{T}a,\ \overline{a})\leqq(a,\ Ta)+(\overline{T}\overline{a},\ \overline{a})$.

Theorem 4. $(\overline{T},\ \overline{M})$ *is a monotone complete conjugately similar transformation on* $\overline{R}$ *and satisfies* (C_-).

Proof. 1), if $\overline{M}\ni\overline{a}\geqq\overline{b}\geqq0$, then evidently $\overline{b}\in\overline{M}$.

2), if $\overline{M}\ni\overline{a},\ \overline{b}$, then for any $a\in M$ such as $\overline{a}\smile\overline{b}\succ Ta$ we have $\overline{a}[(\overline{a}-\overline{b})^+]^R$ $=(\overline{a}\smile\overline{b})[(\overline{a}-\overline{b})^+]^R\succ(Ta)[(\overline{a}-\overline{b})^+]^R$, therefore we have $[\overline{a}]^R[(\overline{a}-\overline{b})^+]^Ra\leqq\overline{T}\overline{a}$. Similarly we have $[\overline{b}]^R[(\overline{b}-\overline{a})^+]^Ra\leqq\overline{T}\overline{b}$. Hence $([\overline{a}]^R\smile[\overline{b}]^R)a=[\overline{a}\smile\overline{b}]^Ra$ $\leqq\overline{T}\overline{a}\smile\overline{T}\overline{b}$, therefore $\overline{a}\smile\overline{b}\in\overline{M}$.

3), for any $\overline{a}\in\overline{R}^+$ there exists $\alpha>0$ such that $\alpha\overline{a}\in\overline{M}$. Because: We can find $\alpha>0$ such that $D_+\overline{m}_T(\alpha\overline{a})<+\infty$. If $\alpha\overline{a}\succ Ta$, then we have $D_+\overline{m}_T(Ta)\leqq D_+\overline{m}_T(\alpha\overline{a})$. On the other hand $m_T(a)+\overline{m}_T(Ta)=(a,\ Ta)$ from theorem 3 and hence in the same way in lemma 4 we obtain $(a,\ Ta)$ $\leqq D_+\overline{m}_T(\alpha\overline{a})$. Therefore $\sup_{\alpha\overline{a}\succ Ta}(a,\ Ta)\leqq D_+\overline{m}_T(\alpha\overline{a})<+\infty$. On account of (16) there exists $\bigcup_{\alpha\overline{a}\succ Ta}[\overline{a}]^Ra$, therefore $\alpha\overline{a}\in\overline{M}$.

4), if $\overline{M}\ni\xi\overline{a}$ and $\overline{T}\xi\overline{a}=0$ for all $\xi\geqq0$, then $\overline{a}=0$. Because: From lemma 5 we have $(a,\ \xi\overline{a})\leqq(a,\ Ta)$ for all $a\in M$ and $\xi\geqq0$, therefore $(a,\ \overline{a})=0$ for all $a\in M$. Hence $\overline{a}=0$.

5), if $\overline{M}\ni\overline{a}\geqq\overline{b}\geqq0$, then evidently $\overline{T}\overline{a}\geqq\overline{T}\overline{b}\geqq0$.

6), for any $\overline{a}\in\overline{M}$ and $[N]$ we have $\overline{T}(\overline{a}[N])=[N](\overline{T}\overline{a})$. Because: If $\overline{a}\succ Ta$, then $\overline{a}[N]\succ(Ta)[N]=T([N]a)$, therefore $\overline{T}(\overline{a}[N])\geqq[N](\overline{T}\overline{a})$.

Conversely if $\bar{a}[N]\succ Tb$, then $\bar{a}\succ Tb$ and $[\bar{a}]^R b\leqq \overline{T}\bar{a}$ and $[N][\bar{a}]^R b\leqq [N](\overline{T}\bar{a})$, therefore $\overline{T}(\bar{a}[N])\leqq[N](\overline{T}\bar{a})$.

7), if $\overline{M}\ni\bar{a}_\lambda\uparrow_{\lambda\in\Lambda}$ and $\sup_{\lambda\in\Lambda}(\overline{T}\bar{a}_\lambda, \bar{a}_\lambda)<+\infty$, then we find $\bar{a}\in\overline{M}$ such that $\bar{a}=\bigcup_{\lambda\in\Lambda}\bar{a}_\lambda$. Because: For any $a\in R^+$ there exists $\alpha>0$ such that $\alpha a\in M$ and from lemma 5 we have $(\alpha a, \bar{a}_\lambda)\leqq(\alpha a, T\alpha a)+(\overline{T}\bar{a}_\lambda, \bar{a}_\lambda)$, therefore $\sup_{\lambda\in\Lambda}(a, \bar{a}_\lambda)<+\infty$ for all $a\in R^+$, hence there exists $\bar{a}\in\overline{R}$ such that $\bar{a}_\lambda\uparrow_{\lambda\in\Lambda}\bar{a}$. We may put $\bar{a}>0$ and prove $\bar{a}\in\overline{M}$. From lemma 6 $(\overline{T}\bar{a}_\lambda, T\overline{T}\bar{a}_\lambda)\leqq(\overline{T}\bar{a}_\lambda, \bar{a}_\lambda)$, hence $\sup_{\lambda\in\Lambda}(\overline{T}\bar{a}_\lambda, T\overline{T}\bar{a}_\lambda)<+\infty$, on account of (16) we have $\overline{T}\bar{a}_\lambda\uparrow_{\lambda\in\Lambda}a\in M$. If $\bar{a}\succ Tb$ and $[\bar{a}]^R b\nleqq a$, then there exists $[N]$ such that $0\neq[N]\leqq[\bar{a}]^R[b]$ and $[N]b\succ[N]a$. As $\bar{a}[N]\succ T([N]b)$ and $\bar{a}_\lambda[N]\uparrow_{\lambda\in\Lambda}\bar{a}[N]$ there exist $\bar{a}_{\lambda_0}$ and $[N_0]$ such that $0\neq[N_0]\leqq[N]$ and $0\neq\bar{a}_{\lambda_0}[N_0]\succ T([N_0]b)$, therefore $\overline{T}(\bar{a}_{\lambda_0}[N_0])\geqq[N_0][\bar{a}_{\lambda_0}]^R b\succ[N_0][\bar{a}_{\lambda_0}]^R a\geqq[N_0][\bar{a}_{\lambda_0}]^R\overline{T}\bar{a}_{\lambda_0}=\overline{T}(\bar{a}_{\lambda_0}[N])$, and hence $\overline{T}(\bar{a}_{\lambda_0}[N_0])=[N_0][\bar{a}_{\lambda_0}]^R b=0$ and $\bar{a}_{\lambda_0}[N_0]=0$. This is a contradiction. From the above if $\bar{a}\succ Tb$, then $[\bar{a}]^R b\leqq a$, therefore $\bar{a}\in\overline{M}$.

8), if $0\leqq\bar{a}_\lambda\uparrow_{\lambda\in\Lambda}\bar{a}$ and $\bar{a}\in\overline{M}$, then $\overline{T}\bar{a}_\lambda\uparrow_{\lambda\in\Lambda}\overline{T}\bar{a}$. Because: Evidently $\bigcup_{\lambda\in\Lambda}\overline{T}\bar{a}_\lambda\leqq\overline{T}\bar{a}$. Let be $\bar{a}\succ Ta$, putting $\bar{b}_\lambda=(\bar{a}_\lambda-Ta)^+$, we have $\bar{a}_\lambda\geqq\bar{a}_\lambda[\bar{b}_\lambda]^R\succ(T[\bar{b}_\lambda]^R a)$ from (18) 7). Therefore $\overline{T}\bar{a}_\lambda\geqq[\bar{b}_\lambda]^R a$ and $\bigcup_{\lambda\in\Lambda}\overline{T}\bar{a}_\lambda\geqq\bigcup_{\lambda\in\Lambda}[\bar{b}_\lambda]^R a$. On the other hand from $\bar{b}_\lambda\uparrow_{\lambda\ni\Lambda}\bar{a}-Ta$ we have $\bigcup_{\lambda\in\Lambda}[\bar{b}_\lambda]^R=[\bar{a}-Ta]^R=[\bar{a}]^R$ ((18), 6), therefore $\bigcup_{\lambda\in\Lambda}[\bar{b}_\lambda]^R a=[\bar{a}]^R a\leqq\bigcup_{\lambda\in\Lambda}\overline{T}\bar{a}_\lambda$, hence $\overline{T}\bar{a}\leqq\bigcup_{\lambda\in\Lambda}\overline{T}\bar{a}_\lambda$. Q.E.D.

In the above theorem it is more desirable to show 3) directly from the property of (T, M). However, we did not success in it and we used the property of $\overline{m}_T$

Lemma 7.

i) $m_T(a)+\overline{m}_T(Ta)=(a, Ta)$ *for all* $a\in M$,

ii) $m_T(\overline{T}\bar{a})+m_T(\bar{a})=(\overline{T}\bar{a}, \bar{a})$ *for all* $\bar{a}\in\overline{M}$.

i) was proved in lemma 4. The proof of ii): At first we prove $T([p](\overline{T}\bar{a}))\leqq\bar{a}[p]\leqq T_+([p](\overline{T}\bar{a}))$ for all $[p]$ such that $[p]\leqq[\overline{T}\bar{a}]$ and $[p](\overline{T}\bar{a})\in M_+$. $T([p](\overline{T}\bar{a}))=T(\overline{T}(\bar{a}[p]))\leqq\bar{a}[p]$ was proved in lemma 6. For $\alpha>1$ we have $\bar{a}[p]\leqq T(\alpha[p](\overline{T}\bar{a}))$. Because, if $\bar{a}[p]\nleqq T(\alpha[p](\overline{T}\bar{a}))$, then from (18) 7) there exists $[p_0]$ such that $0\neq[p_0]\leqq[p]$ and $\bar{a}[p_0]\succ T(\alpha[p_0](\overline{T}\bar{a}))$, therefore $\overline{T}(\bar{a}[p_0])\geqq\alpha[p_0](\overline{T}\bar{a})=\alpha\overline{T}(\bar{a}[p_0])$ and hence $\overline{T}(\bar{a}[p_0]$

$=[p_0](\overline{T}\overline{a})=0$ and hence $[p_0]=0$, this is a contradiction. Therefore $\overline{a}[p] \leqq T(\alpha[p](\overline{T}\overline{a}))$ for all $\alpha>1$ and $\overline{a}[p]\leqq \bigcap_{\alpha>1} T(\alpha[p](\overline{T}\overline{a}))=T_+([p](\overline{T}\overline{a}))$. From the above for all $[p]$ we obtain $D_-m_T([p](\overline{T}\overline{a}))=([p](\overline{T}\overline{a}),\ T([p](\overline{T}\overline{a}))) \leqq([p]\overline{T}\overline{a},\ \overline{a}[p])\leqq([p](\overline{T}\overline{a}),\ T_+([p](\overline{T}\overline{a})))=D_+m_T([p](\overline{T}\overline{a}))$. On account of lemma 4 we have $m_T(\overline{T}\overline{a})+\overline{m}_T(\overline{a})=(\overline{T}\overline{a},\ \overline{a})$.

Theorem 5. *The modular $m_{\overline{T}}$ on $\overline{R}$ generated by $(\overline{T}, \overline{M})$ is the conjugate modular $\overline{m}_T$ of m_T, that is, $\overline{m}_T=m_{\overline{T}}$.*

Proof. For $\overline{a}\in\overline{M}$ we have $\overline{m}_T(\overline{a})=m_{\overline{T}}(\overline{a})$; Because, from the above lemma 7 $m_T(\overline{T}\overline{a})+\overline{m}_T(\overline{a})=(\overline{T}\overline{a},\ \overline{a})$ and from the definition of $\overline{m}_T$ ((2)) we have for all $\xi\geqq 0$ $m_T(\overline{T}\overline{a})+\overline{m}_T(\xi\overline{a})\geqq(\overline{T}\overline{a},\ \xi\overline{a})$. Therefore $\overline{m}_T(\xi\overline{a})-\overline{m}_T(\overline{a}) \geqq(\xi-1)(\overline{T}\overline{a},\ \overline{a})$ for all $\xi\geqq 0$ and we see $D_-\overline{m}_T(\overline{a})\leqq(\overline{T}\overline{a},\ \overline{a})\leqq D_+\overline{m}_T(a)$, hence $D_{-\xi}\overline{m}_T(\xi\overline{a})\leqq(\overline{T}\xi\overline{a},\ \overline{a})\leqq D_{+\xi}\overline{m}_T(\xi\overline{a})$ for all $0\leqq\xi\leqq 1$ and $m_{\overline{T}}(\overline{a})=\int_0^1(\overline{T}\xi\overline{a},\ \overline{a})\,d\xi =\overline{m}_T(\overline{a})$. Next if $m_{\overline{T}}(\overline{a})<+\infty$, then $\alpha\overline{a}\in\overline{M}$ for all $0\leqq\alpha<1$, therefore $m_{\overline{T}}(\overline{a})=\sup_{0\leqq\alpha<1} m_{\overline{T}}(\alpha\overline{a})=\sup_{0\leqq\alpha<1}\overline{m}_T(\alpha\overline{a})=\overline{m}_T(\overline{a})$. Remembering the proof of 3) in theorem 4 we see $\overline{a}\in\overline{M}$ for all $D_+\overline{m}_T(\overline{a})<+\infty$, therefore for $D_+\overline{m}_T(\overline{a}) <+\infty$ we see $\overline{m}_T(\overline{a})=m_{\overline{T}}(\overline{a})$. If $\overline{m}_T(\overline{a})<+\infty$, then $D_+\overline{m}_T(\alpha\overline{a})<+\infty$ for all $0\leqq\alpha<1$ and hence $\overline{m}_T(\alpha\overline{a})=m_{\overline{T}}(\alpha\overline{a})$ for all $0\leqq\alpha<1$, therefore $\overline{m}_T(\overline{a}) =\sup_{0\leqq\alpha<1}\overline{m}_T(\alpha\overline{a})=\sup_{0\leqq\alpha<1} m_{\overline{T}}(\alpha\overline{a})=m_{\overline{T}}(\overline{a})$. $\overline{m}_T=m_{\overline{T}}$ has been proved. Q.E.D.

Essentially theorem 5 has been proved independently from the results of theorem 4. And it is ease to prove theorem 4 from theorem 5. However, it seems to be interesting for us to show theorem 4 independently from theorem 5.

Theorem 6. *We have $\overline{\overline{M}}=M$ and $\overline{\overline{T}}=T$, that is,*

i) $M\ni a$ *if and only if* $a\geqq 0$ *and* $\{\overline{a}[a];\ a\succ\overline{T}\overline{a}\}$ *is order bounded in* $\overline{R}$,

ii) $Ta=\bigcup_{a\succ\overline{T}\overline{a}}\overline{a}[a]$ *for* $a\in M$.

Proof. This fact is evident from theorem 5 and Nakano's theorem about the reflexibility of a modular ([1], p. 175). But in the following we shall prove this theorem directly without using the reflexibility of a modular.

i): We put $\overline{\overline{M}}$ the totality of a such that $a\geqq 0$ and $\{\overline{a}[a];\ a\succ\overline{T}\overline{a}\}$ is order bounded in $\overline{R}$ and $\overline{\overline{T}}a=\bigcup_{a\succ\overline{T}\overline{a}}\overline{a}[a]$. If $a\in M$ and $a\succ\overline{T}\overline{a}$, then we can see $Ta \geqq \bar{a}[a]$ and hence $a \in \bar{\bar{M}}$ namely $M \subset \bar{\bar{M}}$. Because, if $Ta \ngeqq \bar{a}[a]$, then there exists $[p]$ such that $0 \neq [p] \leqq [a]$ and $T[p]a \prec \bar{a}[p]$, therefore $\bar{T}(\bar{a}[p]) \geqq [p]a \succ \bar{T}\bar{a}[p]$ and hence $[p]a=0$ namely $[p]=0$, this is a contradiction. Next for $a \in \bar{\bar{M}}$ and $[p]a \in M$ we have $([p]a, T[p]a) \leqq ([p]a, \bar{\bar{T}}[p]a) \leqq (a, \bar{\bar{T}}a)$ ($Ta=\bar{\bar{T}}a$ for all $a \in M$ will be proved after in ii)), therefore $\sup\limits_{[p]a \in M} ([p]a, T[p]a) < +\infty$. From (16), putting $[N] = \bigcup\limits_{[p]a \in M} [p]$, we have $[N]a \in M$. Further we can proved $[N]a=a$ and therefore $a \in M$ namely $\bar{\bar{M}} \subset M$. Because, let be $a_0 = a - [N]a > 0$, as $[a_0]M \ni x$ implies $x \leqq a$, $[a_0]M$ is upper bounded. If we put $b = \bigcup\limits_{x \in [a_0]M} x$, we see easily $b \prec a_0$. On account of the assumption that R is semi-regular there exists $\bar{a} \in \bar{R}^+$ such as $(a_0, \bar{a}) > 0$ and $[\bar{a}]^R \leqq [a_0] = [a] - [N][a]$. Therefore for any $\xi \geqq 0$, if $\xi\bar{a} \succ Tx$, then $[\bar{a}]^R x \leqq [a_0]x \leqq b$, hence $\bar{M} \ni \xi\bar{a}$ and $\bar{T}\xi\bar{a} \leqq b \prec a_0$ for all $\xi \geqq 0$. Therefore $a_0 \notin \bar{\bar{M}}$. This contradicts $a_0 \leqq a \in \bar{\bar{M}}$.

ii): For any $a \in M$ we have $a \in \bar{\bar{M}}$ from the first part of i) and evidently $Ta \geqq \bar{\bar{T}}a$ from the definition of $\bar{\bar{T}}$. Conversely for $0 \leqq \alpha < 1$ we have $\bar{\bar{T}}T\alpha a \leqq \alpha a \prec a$, therefore $T\alpha a \leqq \bar{\bar{T}}a$ and hence $Ta = \bigcup\limits_{0 \leqq \alpha < 1} T\alpha a \leqq \bar{\bar{T}}a$. Q.E.D.

3. Types of conjugately similar transformations. Through this section (T, M) is a conjugately similar transformation on R satisfying (C_-). And we assume that the following definitions about types of modulars and classifications of modulars are known. We shall state in the following the relation between types of a modular m_T and types of (T, M), where $m_T(a) = \int_0^1 (|a|, T\xi|a|)d\xi$ for all $a \in R$, putting $(|a|, T\xi|a|) = +\infty$ for $\xi|a| \notin M$.

1) m_T is *singular*[13] $\equiv$ $T(M)=\{0\}$.

If m_T is singular, then $D_- m_T(a) < +\infty$ implies $D_- m_T([p]a)=0$ for all $[p]$, and hence $([p]|a|, T|a|)=0$ for all $[p]$, that is, $T|a|=0$. Conversely if $T(M)=\{0\}$, then $m_T(a) = \int_0^1 (|a|, T\xi|a|)d\xi = 0$ or $+\infty$ for all $a \in R$.

a) m_T is *semi-simple*[14] $\equiv$ for any $a>0$ we can find $\alpha>0$ and $[p]$ such as $T\alpha[p]a>0$.

If m_T is semi-simple, then for any $a>0$ there exist $\alpha>0$ and $[p]$ such that $0<m_T(\alpha[p]a)<+\infty$ and $\alpha[p]a$ is domestic ($m_T(\xi\alpha[p]a)<+\infty$

13) For any $a \in R(m)$ $m(a)=0$ or $+\infty$ ([1], p. 157).

14) For any $a>0$ there exist $\xi>0$ and $[p]$ such that $0<m(\xi[p]a)<+\infty$ ([1], p. 156).

for some $\xi>1$), therefore we have $\alpha[p]a\in M$ and $T\alpha[a]>0$. Conversely if m_T is not semi-simple, there exists $a>0$ such that $[a]R$ is singular, therefore for any $\alpha>0$ and $\alpha[p]a\in M$ implies $T\alpha[p]a=0$.

We see easily a is a simple domestic element if and only if $|a|\in M_+$ and $T[p]|a|>0$ for all $0\neq[p]\leqq[a]$.

3) m_T is *linear*[15] $\equiv$ $M=R^+$ and $T\xi a=Ta$ for all $\xi>0$ and $a\in R^+$.

If a is a positive linear element: $m_T(\xi a)=\xi m_T(a)$ for all $\xi\geqq 0$, then we have $D_{-\xi}m_T(\xi[p]a)=D_-m_T([p]a)<+\infty$ for all $\xi>0$ and $[p]$, and hence $([p]a, T\xi a)=([p]a, Ta)$ for all $\xi>0$ and $[p]$, hence $T\xi a=Ta$ for all $\xi>0$. Conversely if $T\xi a=Ta$ for all $\xi>0$, then $m_T(\xi a)=m_T(a)$ for all $\xi\geqq 0$ namely a is a linear element.

If m_T is linear, then $[a]=[b]$ implies $T|a|=T|b|$. Since $[a]|b|=[b]|b|=|b|$ we have $|b|\frown\nu|a|\uparrow_{\nu=1}^{\infty}|b|$, and hence $T(|b|\frown\nu|a|)\uparrow_{\nu=1}^{\infty}T|b|$ and $T(|b|\frown\nu|a|)\leqq T\nu|a|=T|a|$, therefore $T|b|\leqq T|a|$. Similarly $T|a|\leqq T|b|$.

Since m_T is non-linear if and only if there exists no linear element except 0,

4) m_T it *non-linear*[16] $\equiv$ if $M\ni\xi a$ for all $\xi\geqq 0$ and $T\xi a=Ta$, then $a=0$.

Since $m_T(a)=0$ if and only if $T|a|=0$,

5) m_T is *simple*[17] $\equiv$ $Ta=0$ implies $a=0$.

6) m_T is *semi-singular*[18] $\equiv$ $\{a;\ Ta=0\}$ is complete in R.

7) m_T is *monotone*[19] $\equiv$ $\bigcap_{\alpha>0} T\alpha a=0$ for all $a\geqq 0$.

For any $a\in M$ we have $(\alpha a, \bigcap_{\alpha>0} T\alpha a)\leqq m_T(\alpha a)\leqq(\alpha a, T\alpha a)$ $(0\leqq\alpha\leqq 1)$, therefore $\lim_{\alpha\to 0}\dfrac{m_T(\alpha a)}{\alpha}=(a, \bigcap_{\alpha>0}T\alpha a)$. Hence $\lim_{\alpha\to 0}\dfrac{m_T(\alpha a)}{\alpha}=0$ if and only if $\bigcap_{\alpha>0}T\alpha a=0$.

We have also,

8) m_T is *assending*[20] $\equiv$ $\bigcap_{\alpha>0}T\alpha a>0$ for all $a>0$.

a is a finite element if and only if $M\ni\xi|a|$ for all $\xi\geqq 0$. Therefore

9) m_T is *finite*[7] $\equiv$ $M=R^+$.

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15) $m(\xi a)=\xi m(a)$ for all $a\in R(m)$ and $\xi\geqq 0$ ([1], p. 183).

16) If $m(\xi a)=\xi m(a)$ for all $\xi\geqq 0$, then $a=0$, ([1], p. 183).

17) $m(a)=0$ if and only if $a=0$ ([1], p. 187))

18) The set of zero units ([1], p. 125) is complete ([1], p. 187).

19) $\inf_{\xi>0}\dfrac{m(\xi a)}{\xi}=0$ for all $a\in R(m)$. ([1], p. 189).

20) $\inf_{\xi>0}\dfrac{m(\xi a)}{\xi}>0$ for all $a>0$ ([1], p. 188).

10) m_T is *almost finite*[21] $\equiv \{a; M \ni \xi a$ for all $\xi \geqq 0\}$ is complete in R.

11) m_T is *infinite*[22] $\equiv$ for any $a>0$ there exists $\alpha>0$ such as $\alpha a \notin M$.

Since $\left(\frac{1}{2}\xi|a|,\ T\frac{1}{2}\xi|a|\right) \leqq m_T(\xi a) \leqq (\xi|a|,\ T\xi|a|)$, $\lim\limits_{\xi\to+\infty} \frac{m_T(\xi a)}{\xi} = \sup\limits_{\xi\geqq 0} (|a|,\ T\xi|a|)$. Therefore a is a infinitely linear element if and only if $\sup\limits_{\xi\geqq 0}(|a|,\ T\xi|a|)<+\infty$.

12) m_T is *infinitely linear*[23] $\equiv \{a; \sup\limits_{\xi\geqq 0}(a,\ T\xi a)<+\infty\}$ is complete in R.

13) m_T is *increasing*[24] (not infinitely linear) $\equiv \sup\limits_{\xi\geqq 0}(a,\ T\xi a)=+\infty$ for $a>0$.

m_T is called *strictly convex* if for any $a>0$ $m_T(\xi a)$ is a strictly convex function of $\xi \geqq 0$.

Evidently we see

14) m_T is *strictly convex* $\equiv T\xi_1 a > T\xi_1 a$ for $0<a\in M$ and $0\leqq\xi_2<\xi_1\leqq 1$.

14′) T is one to one mapping on M if and only if m_T is strictly convex.

Because, if m_T is strictly convex, for any $a, b\in M$ and $a \neq b$ there exist $[p]$ and $0\leqq\alpha<1$ such that $[p]a>\alpha[p]a\geqq[p]b$ (or $[p]a\leqq\alpha[p]b<[p]b$), therefore $T[p]a>T\alpha[p]a\geqq T[p]b$ and hence $T[p]a\neq T[p]b$ and then $Ta\neq Tb$.

15) m_T is *concave type*[25] $\equiv M=R^+$ and $T(\alpha\xi_1+\beta\xi_2)a\geqq\alpha T\xi_1 a+\beta T\xi_2 a$ for any $a\geqq 0$, $\xi_1\geqq\xi_2\geqq 0$ and $\alpha+\beta=1$, $\alpha, \beta\geqq 0$.

We see m_T is concave type if and only if for any $a\geqq 0$ $D_{-\xi}m_T(\xi a)$ is a finite concave function of ξ on $0<\xi<+\infty$, therefore if and only if $M=R^+$ and $T(\alpha\xi_1+\beta\xi_2)a\geqq\alpha T\xi_1 a+\beta T\xi_2 a$ for any $a\geqq 0$, $\xi_1\geqq\xi_2>0$ and $\alpha+\beta=1$, $\alpha, \beta\geqq 0$. Further from this we can prove $T(\alpha\xi_1+\beta\xi_2)a\geqq\alpha T\xi_1 a+\beta T\xi_2 a$ for $\xi_1\geqq\xi_2\geqq 0$ and $\alpha+\beta=1$, $\alpha, \beta\geqq 0$.

16) m_T is *convex type*[26] $\equiv T(\alpha\xi_1+\beta\xi_2)a\leqq\alpha T\xi_1 a+\beta T\xi_2 a$ for any $a\in M$, $1\geqq\xi_1\geqq\xi_2\geqq 0$ and $\alpha+\beta=1$, $\alpha, \beta\geqq 0$.

We see m_T is convex type if and only if m_T is monotone and

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21) The set of finite elements ($m(\xi a)<+\infty$ for all $\xi\geqq 0$) is complete in R ([1], p. 194).

22) There exists no finite element except zero ([1], p. 197).

23) The set of infinitely linear elements $\left(\sup\limits_{\xi<0}\frac{m(\xi a)}{\xi}<+\infty\right)$ is complete in R ([1], p. 200).

24) $\sup\limits_{\xi>0}\frac{m(\xi a)}{\xi}=+\infty$ for all $a>0$ ([1], p. 200).

25) $D_{+\xi}m(\xi a)$ is a concave function of ξ on $[0, +\infty)$ for all $a\in R$ ([1], p. 224).

26) For all $a\in R$ $\inf\limits_{\xi>0}\frac{m(\xi a)}{\xi}=0$ and $D_{+\xi}m(\xi a)$ is a convex function of ξ on $[0, +\infty)$.

$D_{-\xi}m_T(\xi a)$ is a convex function of ξ on $0<\xi\leqq 1$, therefore if and only if $\bigcap_{\xi>0} T\xi a=0$ and $T(\alpha\xi_1+\beta\xi_2)a\leqq\alpha T\xi_1 a+\beta T\xi_2 a$ for any $a\in M$ and $0<\xi_2\leqq\xi_1\leqq 1$ and $\alpha+\beta=1$, $\alpha, \beta\geqq 0$. These two conditions are equivalent to $T(\alpha\xi_1+\beta\xi_2)a\leqq\alpha T\xi_1 a+\beta T\xi_2 a$ for any $a\in M$ and $0\leqq\xi_2\leqq\xi_1\leqq 1$ and $\alpha+\beta=1$, $\alpha, \beta\geqq 0$.

17) m_T is *upper bounded*[27] $\equiv$ there exist $\gamma_1, \gamma_2>1$ such that for all $a\in M$ we have $\gamma_1 a\in M$ and $T\gamma_1 a\leqq\gamma_2 Ta$.

If m_T is upper bounded, then from the definition there exists $\gamma>2$ such that $m_T(4a)\leqq\gamma m_T(a)$ for all $a\in R$. Therefore $D_- m_T\left(\frac{1}{2}4a\right)\leqq m_T(4a)\leqq\gamma m_T(a)\leqq\gamma D_- m_T(a)$. Hence $a\in M$ implies $2a\in M$ and we have $([p]a, T2a)=\frac{1}{2}D_- m_T(2[p]a)\leqq\frac{\gamma}{2}D_- m_T([p]a)=\frac{\gamma}{2}([p]a, Ta)$ for all $[p]$ and $a\in M$, that is, $T2a\leqq\frac{\gamma}{2}Ta$ for all $a\in M$. Conversely, if we have $\gamma_1 a\in M$ for all $a\in M$, then $M=R^+$, and $T\gamma_1 a\leqq\gamma_2 Ta$ implies $(\gamma_1 a, T\xi\gamma_1 a)\leqq\gamma_1\gamma_2(a, T\xi a)$ for all $\xi\geqq 0$ and $a\in R^+$, therefore $m_T(\gamma_1 a)=\int_0^1(\gamma_1|a|, T\xi\gamma_1|a|)d\xi\leqq\gamma_1\gamma_2\int_0^1(|a|, T\xi|a|)d\xi=\gamma_1\gamma_2 m_T(a)$ for all $a\in R$.

18) m_T is *lower bounded*[28] $\equiv$ there exist $0<\gamma_1, \gamma_2<1$ such that $T\gamma_1 a\leqq\gamma_2 Ta$ for all $a\in M$.

If m_T is lower bounded, then there exist $\gamma>\alpha>1$ such that $m_T(\alpha a)\geqq\gamma m_T(a)$ for all $a\in R$. Since $m_T(\alpha^\nu a)\geqq\gamma^\nu m_T(a)$ for all $\nu=1,2,\cdots$, we can put $\frac{\alpha}{\gamma}<\frac{1}{2}$. Therefore $\frac{1}{\gamma}D_- m_T(a)\geqq\frac{1}{\gamma}m_T(a)\geqq m_T\left(\frac{1}{\alpha}a\right)\geqq D_- m_T\left(\frac{1}{2\alpha}a\right)$ for all $a\in R$, hence we have $\frac{2\alpha}{\gamma}Ta\geqq T\frac{1}{2\alpha}a$ for all $a\in M$. Conversely, if $T\gamma_1 a\leqq\gamma_2 Ta$ for all $a\in M$, then we have $m_T(\gamma_1 a)\leqq\gamma_1\gamma_2 m_T(a)$ for all $a\in R$, that is, $m_T\left(\frac{1}{\gamma_1}a\right)\geqq\frac{1}{\gamma_1\gamma_2}m_T(a)$ for all $a\in R$.

§4. Some types of modulars.

In this section we shall define some new types of a modular and decide these conjugate types.

4.1. We shall state some conditions about bounded modulars.

Definition. A modular is said to be *d-upper bounded*, if there exists a number $p>1$ such that $D_- m(\xi a)\leqq\xi^p D_- m(a)$ for all $a\in R$ and $\xi\geqq 1$.

Easily we see

27) There exist $\alpha, \gamma>1$ such that $m(\alpha x)\leqq\gamma m(x)$ for all $x\in R$ ([1], p. 214).

28) There exist $\gamma, \alpha>1$ such that $m(\alpha x)\geqq\gamma m(x)$ for all $x\in R$ ([1], p. 215).

1) m_T is d-upper bounded $\equiv$ $M=R^+$ and there exists $\alpha>0$ such that $T\xi a \leqq \xi^\alpha Ta$ for all $a\in R^+$ and $\xi\geqq 1$.

If $m(a)$ $(a\in R)$ is d-upper bounded, then evidently $m(a)$ $(a\in R)$ is upper bounded. However, generally the converse is not true. For example, let R be one dimensional and $m(\xi)=\xi$ $(0\leqq\xi\leqq 1)$ and $m(\xi)=2\xi-1$ $(\xi\geqq 1)$, then m is upper bounded, but not d-upper bounded.

Lemma 8. *If m is d-upper bounded: $D_-m(\xi a)\leqq\xi^p D_-m(a)$ for all $a\in R$ and $\xi\geqq 1$, then we have for any $0\leqq y\leqq x$ and $\xi\geqq 1$*

$$m(\xi x)+\xi^p m(y)\leqq \xi^p m(x)+m(\xi y). \tag{20}$$

From the assumption easily we have $m(\xi a)\leqq\xi^p m(a)$ for all $a\in R$ and $\xi\geqq 1$, therefore m is finite. If $y=\eta x$ for some $0\leqq\eta\leqq 1$, then $m(\xi x)-m(\xi y)$ $=\int_{\xi\eta}^{\xi}\frac{D_-m(tx)}{t}dt=\int_{\eta}^{1}\frac{D_-m(\xi tx)}{t}dt\leqq\int_{\eta}^{1}\xi^p\frac{D_-m(tx)}{t}=\xi^p\{m(x)-m(\eta x)\}=\xi^p$ $\{m(x)-m(y)\}$. Next for any $0\leqq y\leqq x$ we can find $0\leqq y_n\leqq x$ such that $y_n\uparrow_{n=1}^{\infty}y$ and $y_n=\sum_{\nu=1}^{\kappa_n}\xi_{\nu,n}[p_{\nu,n}]x$ and $0\leqq\xi_{\nu,n}\leqq 1$ $(\nu=1,2,\cdots,\kappa_n;\ n=1,2,\cdots)$. From the above we obtain $m(\xi x)+\xi^p m(y_n)\leqq\xi^p m(x)+m(\xi y_n)$ $(n=1,2,\cdots)$, therefore $m(\xi x)+\xi^p m(y)=\lim_{n\to\infty}\{m(\xi x)+\xi^p m(y_n)\}\leqq\lim_{n\to\infty}\{\xi^p m(x)+m(\xi y_n)\}=$ $\xi^p m(x)+m(\xi y)$.

Definition. A modular is said to be *d-lower bounded,* if there exists a number $p>1$ such that $D_-m(\xi a)\geqq\xi^p D_-m(a)$ for all $a\in R$ and $\xi\geqq 1$.

2) m_T is d-lower bounded $\equiv$ there exists $\alpha>0$ such that $T\xi a\leqq\xi^\alpha Ta$ for all $a\in M$ and $0\leqq\xi\leqq 1$.

Similarly in lemma 8 we obtain

Lemma 9. *If m is d-lower bounded, then for any $0\leqq y\leqq x$ and $\xi\geqq 1$ we have*

$$m(\xi x)+\xi^p m(y)\geqq\xi^p m(x)+m(\xi y). \tag{21}$$

Theorem 7. *If a modular $m(a)$ $(a\in R)$ is d-upper bounded, then its conjugate modular $\overline{m}(\overline{a})$ $(\overline{a}\in\overline{R}^m)$ is d-lower bounded. And if $m(a)$ $(a\in R)$ is d-lower bounded, then $\overline{m}(\overline{a})$ $(\overline{a}\in\overline{R}^m)$ is d-upper bounded.*

Proof. Let $m(a)$ $(a\in R)$ be d-upper bounded. Then there exists $p>1$ such that $D_-m(\xi a)\leqq\xi^p D_-m(a)$ for all $a\in R$ and $\xi\geqq 1$. For any $0\leqq\overline{a}\in\overline{R}^m$, $x, y\in R$, $\xi\geqq 1$ and $0\leqq\eta\leqq 1$ we can prove

$$\xi^p(x,\overline{a})-\xi^p m(x)+\xi^p\eta(y,\overline{a})-m(\xi\eta)\leqq\overline{m}(\xi^{p-1}\overline{a})+\xi^p\overline{m}(\eta\overline{a}).$$

Because: For $x, y\geqq 0$ we can find $[N]$ such that $[N]y\leqq[N]x$ and $[N^\perp]x\leqq[N^\perp]y$. Therefore from lemma 8 we have

$$\xi^p m([N]x)+m(\xi[N]y)\geqq\xi^p m([N]y)+m(\xi[N]x).$$

And, since $[N^{\perp}]x\leqq[N^{\perp}]y$ and $0\leqq\eta\leqq 1$ imply $[N^{\perp}]x+\eta[N^{\perp}]y\leqq\eta[N^{\perp}]x+[N^{\perp}]y$, we have

$$\xi^p([N^{\perp}]x,\bar{a})+\xi^p\eta([N^{\perp}]y,\bar{a})\leqq\xi^p\eta([N^{\perp}]x,\bar{a})+\xi^p([N^{\perp}]y,\bar{a}).$$

Hence we obtain

$$\begin{aligned}&\xi^p(x,\bar{a})-\xi^p m(x)+\xi^p\eta(y,\bar{a})-m(\xi y)\\ &\quad\leqq\xi^p([N]x+[N^{\perp}]y,\bar{a})-m(\xi[N]x+\xi[N^{\perp}]y)\\ &\qquad+\xi^p\eta([N]y+[N^{\perp}]x,\bar{a})-\xi^p m([N]y+[N^{\perp}]x)\\ &\quad\leqq\overline{m}(\xi^{p-1}\bar{a})+\xi^p\overline{m}(\eta\bar{a}).\end{aligned}$$

Therefore

$$\begin{aligned}\xi^p\overline{m}(\bar{a})+\overline{m}(\xi^{p-1}\eta\bar{a})&=\sup_{x,y\in R^+}\{\xi^p(x,\bar{a})-\xi^p m(x)+\xi^{p-1}\eta(y,\bar{a})-m(y)\}\\ &=\sup_{x,y\in R^+}\{\xi^p(x,\bar{a})-\xi^p m(x)+\xi^p\eta(y,\bar{a})-m(\xi y)\}\\ &\leqq\overline{m}(\xi^{p-1}\bar{a})+\xi^p\overline{m}(\eta\bar{a}).\end{aligned}$$

This inequality implies for any $\xi\geqq 1$, $\bar{a}\in\bar{R}^m$ and $0\leqq\eta\leqq 1$, putting $q=\frac{p}{p-1}$,

$$\xi^q\overline{m}(\bar{a})+\overline{m}(\xi\eta\bar{a})\leqq\overline{m}(\xi\bar{a})+\xi^q\overline{m}(\eta\bar{a}).$$

Therefore $D_{-}\overline{m}(\xi\bar{a})\geqq\xi^q D_{-}\overline{m}(\bar{a})$ for all $\bar{a}\in\bar{R}^m$ and $\xi\geqq 1$. In the same way we can prove the dual relation. Q.E.D.

4.2. We shall state some types which are related to continuous modulars and totally discontinuous modulars.

Definition. An element $a\geqq 0$ is said a *d-discontinuous unit*, if $D_{-}m(a)<+\infty$, and if $D_{-}m(x)<+\infty$ implies $[a]x\leqq a$.

By definition we have

1) if a is a d-discontinuous unit, then a is a discontinuous unit,[29]
2) for a d-discontinuous unit a $[N]a$ is also a d-discontinuous unit,
3) for d-discontinuous units a_1 and a_2 we have $[a_1]a_2=[a_2]a_1=a_1\frown a_2$,
4) for any system $a_\lambda\uparrow_{\lambda\in\Lambda}$ of discontinuous units, if $a_\lambda\uparrow_{\lambda\in\Lambda}a$ and $D_{-}m(a)<+\infty$, then a is also a d-discontinuous unit.

1): Evidently $m(a)\leqq D_{-}m(a)<+\infty$, and if $m(x)<+\infty$, then $D_{-}m(\alpha x)<+\infty$ for all $0\leqq\alpha<1$, therefore $[a]\alpha x\leqq a$ for all $0\leqq\alpha<1$ and $[a]x\leqq a$. 2) is evident. 3): By definition $[a_1]a_2\leqq a_1$, hence $[a_1]a_2\leqq a_1\frown a_2$. However, $a_1\frown a_2\leqq[a_1]a_2$. Therefore $[a_1]a_2=a_1\frown a_2$. 4): If $D_{-}m(x)<+\infty$, then $[a_\lambda]|x|\leqq a_\lambda$ $(\lambda\in\Lambda)$, therefore $[a]|x|=\bigcup_{\lambda\in\Lambda}[a_\lambda]|x|\leqq\bigcup_{\lambda\in\Lambda}a_\lambda=a$.

29) $a\geqq 0$ is called a discountinuous unit when $m(a)<+\infty$ and $m(x)<+\infty$ implies $[a]x\leqq a$ ([1], p. 191).

Definition. A modulared semi-ordered linear space $R(m)$ is said to be *totally d-discontinuous,* if the set of all d-discontinuous units is complete in R. And $R(m)$ is said to be *d-continuous,* if there exists no d-discontinuous unit except 0.

Obviously for any $R(m)$ there exists uniquely a normal manifold R_1 such that $R_1(m)$ is totally d-discontinuous and $R_1^{\perp}(m)$ is d-continuous.

Theorem 8. *In order that $R(m)$ is d-continuous, it is necessary and sufficient that* $D_-m(a)=\sup_{\substack{0\leqq x\leqq a \\ D_-m(x)<+\infty}} D_-m(x)$ *for all* $a\in R^+$.

Proof. Necessity: Let $R(m)$ be d-continuous and for some $a>0$ $D_-m(a)=+\infty$ and $\sup_{\substack{0\leqq x\leqq a \\ D_-m(x)<+\infty}} D_-m(x)<+\infty$. Then if we put x_λ $(\lambda\in\Lambda)$ all elements such as $0\leqq x\leqq a$ and $D_-m(x)<+\infty$, we have $x_\lambda\uparrow_{\lambda\in\Lambda}b\leqq a$. As $D_-m(b)=\sup_{\lambda\in\Lambda} D_-m(x_\lambda)<+\infty$, we see $b<a$. If we put $[a-b]b=d$, then we can see $d>0$ and d is a d-discontinuous unit. Because, we can find $\alpha>0$ such as $D_-m(\alpha a)<+\infty$, therefore $\alpha a\leqq b$ and $0<\alpha[a-b]a\leqq[a-b]b=d$, further if $D_-m(x)<+\infty$, then $x\frown a\leqq b$, and hence $(x-b)\frown(a-b)\leqq 0$ and $(x-b)^+\frown(a-b)=0$, therefore $[a-b]x\leqq[a-b]b=d$. This contradicts that $R(m)$ is d-continuous.

Sufficiency: Let a be $a>0$ and d-discontinuous unit, then obviously $D_-m(\alpha a)=+\infty$ for all $\alpha>1$. However, if $2a\geqq x\geqq 0$ and $D_-m(x)<+\infty$, then $x=[a]x\leqq a$. Therefore $\sup_{\substack{0\leqq x\leqq 2a \\ D_-m(x)<+\infty}} D_-m(x)=D_-m(a)<+\infty=D_-m(2a)$. Therefore the sufficiency is clear. Q.E.D.

Evidently the property to be totally d-discontinuous is weaker than to be singular[13] and stronger than to be totally discontinuous.[30] And the property to be d-continuous is weaker than to be continuous[31] and stronger than to be semi-simple.[14]

In the following we shall decide the conjugate type of a totally d-discontinuous $R(m)$.

Definition. An element $a\in R$ is called a *semi-linear element,* if there exist positive numbers ξ_0 and η_0 such that $m(\xi_0 a)<+\infty$ and $m(\xi a)=(\xi-\xi_0)\eta_0+m(\xi_0 a)$ for all $\xi\geqq\xi_0$.

By definition easily we have

1) if a is a semi-linear element, then a is an asymptotically linear

30) The set of all discontinuous units is complete in R ([1], p. 193).

31) There exists no discontinuous unit except zero ([1], p. 193).

element.[32]

2) for semi-linear elements a_1 and a_2 such as $|a_1|\frown|a_2|=0$ a_1+a_2 is also a semi-linear element.

3) for semi-linear element a $[N]a$ is also a semi-linear element

1) and 2) are evident. 3): As $\eta_0=\lim_{\xi\to+\infty}\frac{m(\xi a)}{\xi}$, we have $\eta_0=\lim_{\xi\to+\infty}\frac{m(\xi[N]a)}{\xi}+\lim_{\xi\to+\infty}\frac{m(\xi[N^\perp]a)}{\xi}=\eta_1+\eta_2$ from the additivity of a modular, and by the convexity of $m(\xi x)$ we have $m(\xi[N]a)\leqq(\xi-\xi_0)\eta_1+m(\xi_0[N]a)$ and $m(\xi[N^\perp]a)\leqq(\xi-\xi_0)\eta_2+m(\xi_0[N^\perp]a)$ for all $\xi\geqq\xi_0$. Therefore $m(\xi[N]a)=(\xi-\xi_0)\eta_1+m(\xi_0[N]a)$ and $m(\xi[N^\perp]a)=(\xi-\xi_0)\eta_2+m(\xi_0[N^\perp]a)$ for all $\xi\geqq\xi_0$.

Definition. A modulared semi-ordered linear space $R(m)$ is called *semi-linear*, if the set of all semi-linear elements is complete in R. And $R(m)$ is called *non-semi-linear*, if there exists no semi-linear element except 0.

Obviously for any $R(m)$ there exists uniquely a normal manifold R_1 such that $R_1(m)$ is semi-linear and $R_1^\perp(m)$ is non-semi-linear.

The property to be semi-linear is weaker than to be linear[15] and stronger than to be asymytotically linear.[33] And the property to be non-semi-linear is weaker than to be increasing[24] and stronger than to be non-linear.[16]

Theorem 9. *If $R(m)$ is totally d-discontinuous, then its modular conjugate space $\overline{R}^m(\overline{m})$ is semi-linear.*

Proof. We can represent R as direct sum of two normal manifolds R_1 and R_2 such that $R_1(m)$ is singular and $R_2(m)$ is semi-simple. As $\overline{R}^m=\overline{R}_1^m\oplus\overline{R}_2^m$ and $\overline{R}_1^m(\overline{m})$ is linear, we may assume further $R(m)$ is semi-simple.

Let $a>0$ be d-discontinuous unit, on account of theorem 2 we find $\overline{a}\in\overline{R}^m$ such that $\overline{m}(\overline{a})+m(a)=(a,\overline{a})$ and $([p]a,\overline{a})=D_-m([p]a)$ for all $[p]$. And for any $\xi\geqq1$ we see $D_-m([p]a)\leqq([p]a,\xi\overline{a})\leqq D_+m([p]a)$ for all $[p]$, because a is a d-discontinuous unit and we see $D_+m([p]a)=+\infty$ for $[p]a\neq0$. Therefore from lemma 4 we have $\overline{m}(\xi\overline{a})+m(a)=(a,\xi\overline{a})$ for all $\xi\geqq1$ and hence $\overline{m}(\xi\overline{a})=(a,\xi\overline{a})-m(a)=(\xi-1)(a,\overline{a})+\overline{m}(\overline{a})$ for all $\xi\geqq1$. Therefore $\overline{a}$ is a semi-linear element. On the other hand, if $[p]a>0$, then $D_-m([p]a)>0$. Because: If $D_-m([p]a)=0$ and $[p]a>0$, then we

32) a is called asymptotically linear when $\sup_{\xi>0}\frac{m(\xi a)}{\xi}=\gamma<+\infty$ and $\sup_{\xi>0}\{\xi\gamma-m(\xi a)\}<+\infty$.

33) The set of all asymptocally linear elements is complete in R ([1], p. 203).

have $D_{-}m(\xi[q]a)=0$ or $+\infty$ for all $[q]\leqq[p]$ and $\xi\geqq 0$, therefore $[[p]a]R(m)$ is singular. This contradicts $R(m)$ is semi-simple. And as $\int_{[a]}\left(\frac{x}{a}, \mathfrak{p}\right) D_{-}m(d\mathfrak{q}a)=(x, \bar{a})$ (theorem 2), we see $[\bar{a}]^{R}=[a]$. Therefore the set of all semi-linear elements in $\bar{R}^{m}(\bar{m})$ is complete. Q.E.D.

Theorem 10. *If $R(m)$ is semi-linear, then $\bar{R}^{m}(\bar{m})$ is totally d-discontinuous.*

Proof. Let $a>0$ be semi-linear and $m(\xi a)=(\xi-1)\xi+m(a)$ for all $\xi\geqq 1$ and $m(a)<+\infty$. Evidently $\eta=D_{+}m(a)$ and $m(\xi[p]a)=(\xi-1)D_{+}m([p]a)+m([p]a)$ for all $[p]$ and $\xi\geqq 1$. Therefore $D_{+}m([p]a)=0$ implies $[p]a=0$, for $D_{+}m([p]a)=0$ implies $m(\xi[p]a)=m([p]a)$ for all $\xi\geqq 1$, namely $[p]a=0$. From theorem 2 we can find $0\leqq\bar{a}\in\bar{R}^{m}$ such that $\bar{m}(\bar{a})+m(a)=(a, \bar{a})$ and $([p]a, \bar{a})=D_{+}m([p]a)$ for all $[p]$. Further we see $[\bar{a}]^{R}=[a]$, because $(x, \bar{a})=\int_{[a]}\left(\frac{x}{a}, \mathfrak{p}\right) D_{+}m(d\mathfrak{q}a)$ and $D_{+}m([p]a)=0$ implies $[p]a=0$. From the equality $\bar{m}(a)+m(a)=(a, \bar{a})$ we have $D_{-}\bar{m}(\bar{a})\leqq(a, \bar{a})<+\infty$. And for any $\xi>1$ and $\bar{a}[p]>0$ we see

$$\bar{m}(\xi\bar{a}[p])=\sup_{x\in R}\{\xi([p]x, \bar{a})-m([p]x)\}\geqq\sup_{\rho>1}\{\xi(\rho[p]a,\bar{a})-m(\rho[p]a)\}=\sup_{\rho\geqq 1}\{\rho(\xi-1)D_{+}m([p]a)\}+D_{+}m([p]a)-m([p]a)\geqq\sup_{\rho\geqq 1}\{\rho(\xi-1)D_{+}m([p]a)\}=+\infty,$$

because $\bar{a}[p]>0$ implies $[\bar{a}]^{R}[p]>0$, therefore $[a][p]>0$ namely $[p]a>0$ and $D_{+}m([p]a)>0$. Therefore $\bar{a}$ is a d-discontinuous unit: If $D_{-}\bar{m}(\bar{x})<+\infty$ and $\bar{x}[\bar{a}]^{R}\lneqq\bar{a}$, then we can find $\xi>1$ and $[p]$ such that $\bar{x}[\bar{a}]^{R}[p]\geqq\xi\bar{a}[p]>0$, hence $+\infty>D_{-}\bar{m}(\bar{x})\geqq D_{+}\bar{m}(\bar{x}[\bar{a}]^{R}[p])\geqq D_{+}\bar{m}(\xi\bar{a}[p])\geqq\bar{m}(\xi\bar{a}[p])=+\infty$, this is a contradiction. As $[\bar{a}]^{R}=[a]$, the set of all d-discontinuous units in $\bar{R}^{m}$ is complete. Q.E.D.

Finally we state one theorem concerning a d-continuous modular.

If we put $F=\{a;\ a\geqq 0,\ m(a)<+\infty\}$, then evidently we see $M_{+}\subset M_{-}\subset F$. When $M_{+}=F$, namely $m(a)<+\infty$ implies $D_{+}m(a)<+\infty$, T. Andō named that type *domestic* and he proved the interesting theorem: A modular is is domestic if and only if a modular is continuous and its modular norm[34] is continuous.[35] In this result the most interesting point is that $M_{+}=F$ implies the continuity of a modular norm. Recently further he showed

34) $|||a|||=\inf_{m(\xi a)\leqq 1}\frac{1}{|\xi|}$ $(a\in R)$ ([2], p. 212).

35) We have $\lim_{\nu\to\infty}|||a_{\nu}|||=0$ for any $a_{\nu}\downarrow_{\nu=1}^{\infty}0$ ([1], p. 127)

the weaker condition than $M_+=F$ implies the continuity of a modular norm. By the similar method we can see $M_+=M_-$ implies the continuity of a modular norm. Therefore in the following we shall prove

Theorem 11. *Let $R(m)$ be a modulared semi-ordered linear space. Then $M_+=M_-$ if and only if the modular norm is continuous and the modular is d-continuous.*

At first we remark that the modular norm $|||a|||$ $(a\in R)$ is continuous if and only if for any system $[p_\nu]\downarrow_{\nu=1}^{\infty}0$ and $a\in R$ we have $\inf_{\nu\geq 1} D_-m([p_\nu]a)=0$. Because; if the norm $|||a|||$ $(a\in R)$ is continuous then for $[p_\nu]\downarrow_{\nu=1}^{\infty}0$ and $a\in R$ we can find ν_n $(n=1,2\cdots)$ such that $|||2n[p_{\nu_n}]a|||\leqq 1$, hence $D_-m([p_{\nu_n}]a)\leqq m(2[p_{\nu_n}]a)\leqq\frac{1}{n}m(2n[p_{\nu_n}]a)\leqq\frac{1}{n}$ for every $n=1,2,\cdots$, therefore $\inf_{\nu\geq 1}D_-m([p_\nu]a)=0$. Conversely, if we have $\inf_{\nu\geq 1}D_-m([p_\nu]a)=0$ for every $[p_\nu]\downarrow_{\nu=1}^{\infty}0$ and $a\in R$, then we find ν_n $(n=1,2,\cdots)$ such that $m(n[p_{\nu_n}]a)\leqq D_-m(n[p_{\nu_n}]a)\leqq 1$, hence $|||[p_{\nu_n}]a|||\leqq\frac{1}{n}$ for every $n=1,2,\cdots$, therefore $\inf_{\nu\geq 1}|||[p_\nu]a|||=0$. This implies the continuity of the norm $|||a|||$ $(a\in R)$ ([1], p. 128).

Next we assume the following lemma that was proved by Andō ([5]):

Lemma. *Let $R(m)$ be a modulared semi-ordered linear space. If the modular norm is not continuous, then we can find a closed subspace S of R satisfying following conditions: $S(m)$ is a monotone complete modulared semi-ordered linear space and there exist normal manifolds N_ν of S $(\nu=1,2,\cdots)$ such that N_ν $(\nu=1,2,\cdots)$ are orthogonal each other and the modular norm is not continuous on all N_ν $(\nu=1,2,\cdots)$.*

Proof of theorem 10. Let $R(m)$ be d-continuous and its modular norm is continuous. For $a\in M_-$ let $[p_\lambda]$ $(\lambda\in\Lambda)$ be all projectors such that $[p_\lambda]a\in M_+$, we see easily $[p_\lambda]\uparrow_{\lambda\in\Lambda}$. If we put $b=\bigcup_{\lambda\in\Lambda}[p_\lambda]a$, then we can find $\lambda_n\in\Lambda$ $(n=1,2,\cdots)$ $[p_{\lambda_n}]a\uparrow_{n=1}^{\infty}b$ ([1], p. 128), therefore $[p_{\lambda_n}]\uparrow_{n=1}^{\infty}[b]$. We have $[a]=[b]$, because, if $[b]<[a]$, then for every $[q]$ such that $0<[q]\leqq[a]-[b]$ we have $[q]a\notin M_+$, therefore $([a]-[b])a$ is a d-discontinuous unit and non-zero. This fact contradicts the assumption. We put $[q_n]=[a]-[p_{\lambda_n}]$, then we have $[q_n]\downarrow_{n=1}^{\infty}0$, hence from the continuity of the modular norm we have $\inf_{n\geq 1}D_-m(2[q_n]a)=0$. Therefore we can find n_0 such that $D_-m(2[q_{n_0}]a)<+\infty$, hence $[q_{n_0}]a\in M_+$ and $a=[q_{n_0}]a+[p_{\lambda_{n_0}}]a\in M_+$. Thus we have $M_+=M_-$.

Conversely, let be $M_+=M_-$. Evidently if a is non-zero d-discontinuous unit, then $a \in M_-$ and $a \notin M_+$, therefore $M_+ = M_-$ implies that the modular is d-continuous. Next we see the continuity of the modular norm. If the norm is not continuous, then without loss of generality from the above lemma we can assume that $R(m)$ is monotone complete and there exist normal manifolds N_ν $(\nu=1,2,\cdots)$ of R such that the norm is not continuous on every N_ν $(\nu=1,2\cdots)$. From the property that the norm is not continuous on N_1 we can find $[p_{1,\mu}]\downarrow_{\mu=1}^{\infty}0$ and $0 \leqq a_1 \in N_1$ such that $[N_1] \geqq [p_{1,\mu}]$ and $D_-m([p_{1,\mu}]a_1) = +\infty$ for every $\mu=1,2,\cdots$. We put ξ_1 the infimum of $\xi \geqq 0$ such that $\inf_{\mu\geqq 1} D_-m(\xi[p_{1,\mu}]a_1) = +\infty$, then evidently $0<\xi_1 \leqq 1$ and we have $\inf_{\mu\geqq 1} D_-m(\xi_1[p_{1,\mu}]a_1) = +\infty$, because, if $D_-m(\xi_1[p_{1,\mu_0}]a_1) < +\infty$ for some μ_0, then $\xi_1[p_{1,\mu_0}]a_1 \in M_- = M_+$, hence we find $\xi' > \xi_1$ such that $\xi'[p_{1,\mu_0}]a_1 \in M_-$, therefore $\inf_{\mu\geqq 1} D_-m(\xi'[p_{1,\mu}]a_1) = 0$. This implies $\xi' \leqq \xi_1$, it is a contradiction. Thus we can find $[p_{\nu,\mu}]\downarrow_{\mu=1}^{\infty}0$ and $0<a_\nu \in N_\nu$ $(\nu=1,2,\cdots)$ such that $[N_\nu] \geqq [p_{\nu,\mu}]$ and $D_-m([p_{\nu,\mu}]a_\nu) = +\infty$ for every $\mu=1,2,\cdots$, $\nu=1,2,\cdots$ and $\inf_{\mu\geqq 1} D_-m(\xi[p_{\nu,\mu}]a_\nu) = 0$ for every $0 \leqq \xi < 1$ and $\nu=1,2,\cdots$. For a sequence of positive numbers such that $\alpha_1 < \alpha_2 < \cdots < 1$ and $\lim_{\nu\to\infty} \alpha_\nu = 1$ we can find μ_ν $(\nu=1,2,\cdots)$ such that $D_-m(\alpha_\nu[p_{\nu,\mu_\nu}]a_\nu) \leqq \frac{1}{2\nu}$. Then we have $\sum_{\nu=1}^{\infty} D_-m(\alpha_\nu[p_{\nu,\mu_\nu}]a_\nu) \leqq 1$, therefore from the monotone completeness there exists $a = \sum_{\nu=1}^{\infty} \alpha_\nu[p_{\nu,\mu_\nu}]a_\nu$ and $D_-m(a) \leqq 1$, hence $a \in M_-$. However, for any $\alpha > 1$ we find α_{ν_0} such that $\alpha\alpha_{\nu_0} > 1$, therefore $D_-m(\alpha a) \geqq D_-m(\alpha\alpha_{\nu_0}[p_{\nu_0,\mu_{\nu_0}}]a_{\nu_0}) = +\infty$, hence $a \notin M_+$. This is a contradiction. Q.E.D.

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References

[1] H. NAKANO: Modulared semi-ordered linear spaces, Tokyo Math. Book Ser. Vol. I (1950).

[2] H. NAKANO: Topology and linear topological spaces, Tokyo Math. Book Ser. Vol. III (1951).

[3] M. MIYAKAWA and H. NAKANO: Modulars on semi-ordered linear spaces I, Jour. Fac. Sci. Hokkaido Univ. Ser. I Vol. XIII (1956).

[4] S. YAMAMURO: Exponents of modulared semi-ordered linear spaces, Jour. Fac. Sci. Hokkaido Univ. Ser. I Vol. XII (1953).

[5] ANDO: On the continuity of the norm by a modular, Monograph Ser. of the Research Inst. of Applied Electricity, No. 7 (1959).