SYSTEMS OF n COMPLEX NUMBERS WITH VANISHING POWER SUMS

By

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1. **Introduction.** Let us denote by $Z(m, n)$ the set of all systems of n complex numbers $(z_1, z_2, \cdots, z_n)$ with the property

(1. 1) $$s_{m+1} = s_{m+2} = \cdots = s_{m+n-1} = 0 ,$$

where $s_\nu = z_1^\nu + z_2^\nu + \cdots + z_n^\nu$ and m is a non-negative integer. It will be almost evident that the set $Z(m,n)$ always contains a non-trivial system, i.e. a system other than $(0,0,\cdots,0)$: we write $Z^*(m,n)$ for the set of all non-trivial systems in $Z(m,n)$.

Vera T. Sós and P. Turán [1] have shown that 1) the systems in $Z(0,n)$ are given by the zeros of an equation

$$z^n + a = 0 \qquad (a \text{ arbitrary complex}),$$

2) the systems in $Z(1,n)$ are given by the zeros of an equation

$$z^n + \frac{a}{1!} z^{n-1} + \cdots + \frac{a^n}{n!} = 0 \qquad (a \text{ arbitrary complex})$$

and 3) the systems in $Z(2,n)$ are formed by the zeros of an equation

$$z^n + \frac{H_1(\lambda)}{1!} az^{n-1} + \cdots + \frac{H_n(\lambda)}{n!} a^n = 0 ,$$

where $H_\nu(t)$ is the ν-th HERMITE polynomial defined by

$$H_\nu(t) = (-1)^\nu e^{t^2} \frac{d^\nu}{dt^\nu} e^{-t^2} ,$$

λ denotes any zero of the equation $H_{n+1}(t) = 0$ and a is an arbitrary complex number.

A general characterization of the set $Z(m,n)$ for $m > 0$ has been given by the present author in [2], the details of which will be reproduced later on (see Theorem 3 below). However, as a matter of fact, it is not quite easy to find a special non-trivial system of $Z(m,n)$ even when $m=2$.

Now we introduce the notion of equivalence between non-trivial systems in $Z(m, n)$ which is substantially due to Professor P. TURÁN. Any two systems $(z_1, \cdots, z_n)$ and $(z'_1, \cdots, z'_n)$ in $Z^*(m, n)$ are said to be *equivalent*, if there exist a non-zero (complex) number λ and a permutation π on $1, 2, \cdots, n$ such that

$$z'_j = \lambda z_{\pi(j)} \qquad\qquad (j = 1, 2, \cdots, n) .$$

Clearly, this induces an equivalence relation in $Z^*(m, n)$ and we shall denote by $B(m, n)$ the number of inequivalent classes in $Z^*(m, n)$ according to the equivalence thereby defined. The number $B(m, n)$ is easily seen to be always finite. Thus, roughly speaking, $B(m, n)$ is the number of *basic* systems in $Z(m, n)$ from which others are obtained by stretching, with rotation, from the origin of the complex number plane.

According to the results of Sós and TURÁN mentioned above, we can see at once that

$$B(0, n) = 1 , \qquad B(1, n) = 1 .$$

However, the determination of $B(m, n)$ will be not so trivial when $m \geqq 2$. We shall prove:

Theorem 1. *We have*

$$(1.2) \qquad\qquad B(2, n) = \left[\frac{n}{2}\right] + 1 .$$

Theorem 2. *We have*

$$(1.3) \qquad\qquad B(3, n) = \left[\frac{n^2 + 3n}{6}\right] + 1 .$$

Incidentally we find that

$$(1.4) \qquad\qquad B(m, 2) = \left[\frac{m}{2}\right] + 1$$

and

$$(1.5) \qquad\qquad B(m, 3) = \left[\frac{m^2 + 3m}{6}\right] + 1 .$$

Thus we have $B(m, n) = B(n, m)$ for $m = 2, 3$. It is then natural to ask whether the relation $B(m, n) = B(n, m)$ will hold for every non-negative integer m. We can show that this is certainly the case, on interpreting that $B(m, 0) = 1$: indeed, we have the recurrence formula

$$(1.6) \qquad\qquad \sum_{d \mid (m, n)} a(d) B\left(\frac{m}{d}, \frac{n}{d}\right) = \frac{(m + n - 1)!}{m! \, n!} ,$$

where $a(k) = k^{-1} \prod_{p \mid k} (1-p)$, and, in particular, when $(m,n)=1$ this implies that

$$B(m,n) = \frac{(m+n-1)\,!}{m!\ n!} = B(n,m)\,.$$

This, together with (1.6), proves our assertion. Hence, if we assume the general relation $B(m,n) = B(n,m)$, then it turns out that, in order to establish our Theorems 1 and 2, we have only to prove (1.4) and (1.5): in effect, as will be seen later, it is considerably easier to prove (1.4) or (1.5) than to prove the corresponding relation (1.2) or (1.3). However, it will be of some interest to get direct proofs for the relations (1.2) and (1.3).

Our proof of (1.6) will be published elsewhere [3].

2. Characterization of $Z(m,n)$. In what follows we shall suppose that $m \geq 1$. We define polynomials $C_\nu = C_\nu(t_1, \cdots, t_m)$ $(\nu = 0,1,2,\cdots)$ in m indeterminates $t_1, \cdots, t_m$ by the relation

$$(2.1) \qquad \exp\left(-\sum_{\mu=1}^{m} \frac{1}{\mu}\, t_\mu x^\mu\right) = \sum_{\nu=0}^{\infty} \frac{C_\nu}{\nu\,!}\, x^\nu\,.$$

In the case of $m=2$, our polynomials $C_\nu(t_1, t_2)$ are closely connected with those of HERMITE, since there holds the relation

$$e^{2tx-x^2} = \sum_{\nu=0}^{\infty} \frac{H_\nu(t)}{\nu\,!}\, x^\nu\,,$$

and we have in fact

$$C_\nu(-2u, 2v^2) = v^\nu H_\nu\left(\frac{u}{v}\right) \qquad (\nu = 0,1,2,\cdots)\,.$$

In [2] we have proved the following

Theorem 3. *All the systems $(z_1, \cdots, z_n)$ of $Z(m,n)$, $m \geq 1$, are formed by the zeros of an equation*

$$(2.2) \qquad \sum_{\nu=0}^{n} \frac{C_\nu(\lambda_1, \cdots, \lambda_m)}{\nu\,!}\, z^{n-\nu} = 0\,,$$

where $(\lambda_1, \cdots, \lambda_m)$ is any solution of the simultaneous equations

$$(2.3) \qquad C_{n+\kappa}(t_1, \cdots, t_m) = 0 \qquad (\kappa = 1,2, \cdots, m-1)\,.$$

We note that, if $m > n$, then the values $\lambda_{n+1}, \cdots, \lambda_m$ among the λ_μ determined by (2.3) do not enter into the left-hand side of (2.2): in that case the values of $t_{n+1}, \cdots, t_m$ are (uniquely) determined by those

of $t_1, \cdots, t_n$. Also, if $m > n+1$, the first $m - n$ equations of (2.3) are superfluous.

As is easily seen, if the system $(z_1, \cdots, z_n)$ of $Z(m, n)$ is given by a solution $(\lambda_1, \cdots, \lambda_m)$ of (2.3), then we have

$$z_1^\mu + \cdots + z_n^\mu = \lambda_\mu \qquad (1 \leq \mu \leq m) .$$

Anyway, for a non-trivial system of $Z(m, n)$, we may assume that $\lambda_m \neq 0$.

3. A criterion of the equivalence. The following lemma is elementary :

Lemma. *Let* $(z_1, \cdots, z_n)$ *and* $(z_1', \cdots, z_n')$ *be any two systems in* $Z^*(m, n)$. *Write*

$$s_\nu = z_1^\nu + \cdots + z_n^\nu , \qquad s_\nu' = z_1'^\nu + \cdots + z_n'^\nu .$$

Then, $(z_1, \cdots, z_n)$ *and* $(z_1', \cdots, z_n')$ *are equivalent if and only if*

$$s_\nu' = \lambda^\nu s_\nu \qquad (\nu = 1, 2, \cdots, n)$$

for some non-zero λ .

Proof. Let a_i and a_i' be the i-th elementary symmetric functions of $z_1, \cdots, z_n$ and $z_1', \cdots, z_n'$, respectively. Then, by the definition, $(z_1, \cdots, z_n)$ and $(z_1', \cdots, z_n')$ are equivalent if and only if $a_i' = \lambda^i a_i \ (i = 1, 2, \cdots, n)$ for some non-zero λ. Hence, by the NEWTON-GIRARD formulae, $(z_1, \cdots, z_n)$ and $(z_1', \cdots, z_n')$ are equivalent if and only if $s_\nu' = \lambda^\nu s_\nu \ (\nu = 1, 2, \cdots, n)$ for some non-zero λ.

In the proofs of Theorems 1 and 2 we shall make use of this lemma as the criterion of the equivalence in $Z^*(m, n)$.

4. Proof of Theorem 1. By Theorem 3, any system of $Z(2, n)$ is given by the zeros of an equation

$$\sum_{\nu=0}^{n} \frac{C_\nu(\lambda_1, \lambda_2)}{\nu !} z^{n-\nu} = 0 ,$$

where (λ_1, λ_2) is a solution of

$$C_{n+1}(t_1, t_2) = 0 .$$

Now we have

$$C_{n+1}(-2u, 2v^2) = v^{n+1} H_{n+1}\left(\frac{u}{v}\right) .$$

Since each root is a simple one, there are exactly $n+1$ distinct (real)

roots of $H_{n+1}(t)=0$. We distinguish two cases.

(*Case* 1) n even: Then $H_{n+1}(t)$ is an odd function of t and we have $H_{n+1}(0)=0$. Hence, corresponding to $\lambda_1=0, \lambda_2 \neq 0$ we have a non-trivial system of $Z(2,n)$. By the Lemma, all such systems are mutually equivalent. Next, we take a non-zero root ρ of $H_{n+1}(t)=0$. Then $\lambda_1 = -2u\rho$, $\lambda_2 = 2u^2$ $(u \neq 0)$ gives a non-trivial system of $Z(2,n)$, all such systems again being mutually equivalent. On the other hand, we have

$$H_{n+1}(-\rho) = -H_{n+1}(\rho) = 0 ,$$

and hence $\lambda_1' = -2u'(-\rho)$, $\lambda_2' = 2u'^2 (u' \neq 0)$ gives another non-trivial system of $Z(2,n)$; but this is actually equivalent to a system corresponding to the λ_1, λ_2, since $\lambda_1' = -2(-u')\rho$, $\lambda_2' = 2(-u')^2$. Also, the different ρ's with different absolute values give inequivalent systems. ·Thus, there exist $\dfrac{n}{2}$ inequivalent classes in $Z^*(2,n)$ corresponding to n non-zero roots ρ of $H_{n+1}(t)=0$. Hence $B(2,n)=\dfrac{n}{2}+1$ in this case.

(*Case* 2) n odd: Then $H_{n+1}(t)$ is an even function of t and we have $H_{n+1}(0)\neq 0$. Hence, the $n+1$ roots of $H_{n+1}(t)=0$ may be paired according to their absolute values, and, by a similar argument to that of the (*Case* 1), we can prove that $B(2,n)=\dfrac{n+1}{2}$ in this case.

This completes the proof of Theorem 1.

5. Proof of Theorem 2. By Theorem 3, any system of $Z(3,n)$ is given by the zeros of an equation

$$\sum_{\nu=0}^{n} \frac{C_\nu (\lambda_1, \lambda_2, \lambda_3)}{\nu !} z^{n-\nu} = 0 ,$$

where $(\lambda_1, \lambda_2, \lambda_3)$ is a solution of

(5.1) $C_{n+1}(t_1, t_2, t_3) = C_{n+2}(t_1, t_2, t_3) = 0 .$

As was noted above, we may assume that $\lambda_3 \neq 0$ for a non-trivial system of $Z(3,n)$. Hence $C_n(\lambda_1, \lambda_2, \lambda_3) \neq 0$ for any solution $(\lambda_1, \lambda_2, \lambda_3)$ of (5.1) with $\lambda_3 \neq 0$. For, otherwise, at least one of the numbers z_j from the system $(z_1, \cdots, z_n)$ corresponding to $(\lambda_1, \lambda_2, \lambda_3)$ by Theorem 3 must be equal to zero, which would actually imply in turn that all the z_j are zero. This is impossible since $z_1^3 + \cdots + z_n^3 = \lambda_3 \neq 0$.

Now we put

' (5.2) $f_\nu (\tau_1, \tau_2, \tau_3) = C_\nu (\tau_1, \tau_2^2, \tau_3^3)$ $(\nu = 0, 1, 2, \cdots)$

and consider the curves defined by

(5. 3) $f_{n+1}(\tau_1, \tau_2, \tau_3) = 0$

and by

(5. 4) $f_{n+2}(\tau_1, \tau_2, \tau_3) = 0$

in a projective plane over the complex numberfield.

It can be easily verified that the number of points of intersection of those curves is finite and hence, by virtue of BÉZOUT's theorem, is equal to $(n+1)(n+2)$, the multiple points of the intersection being counted with their multiplicities. Moreover, if (τ_1, τ_2, τ_3) is a point of the intersection of (5.3) and (5.4) then we may assume that $\tau_3 \neq 0$.

By differentiation with respect to the τ_μ and to x in turn we obtain from (5.2) and (2.1) with $m=3$ that

(5. 5)
$$
\begin{cases}
\dfrac{\partial}{\partial \tau_1} f_\nu(\tau_1, \tau_2, \tau_3) = -\nu f_{\nu-1}(\tau_1, \tau_2, \tau_3) \\[2ex]
\dfrac{\partial}{\partial \tau_2} f_\nu(\tau_1, \tau_2, \tau_3) = -\nu(\nu-1)\tau_2 f_{\nu-2}(\tau_1, \tau_2, \tau_3) \\[2ex]
\dfrac{\partial}{\partial \tau_3} f_\nu(\tau_1, \tau_2, \tau_3) = -\nu(\nu-1)(\nu-2)\tau_3^2 f_{\nu-3}(\tau_1, \tau_2, \tau_3)
\end{cases}
$$

and

(5. 6)
$$
f_{\nu+1}(\tau_1, \tau_2, \tau_3) + \tau_1 f_\nu(\tau_1, \tau_2, \tau_3)
$$
$$
+ \nu \tau_2^2 f_{\nu-1}(\tau_1, \tau_2, \tau_3) + \nu(\nu-1)\tau_3^3 f_{\nu-2}(\tau_1, \tau_2, \tau_3) = 0 .
$$

Hence, if (τ_1, τ_2, τ_3) is a multiple point of the intersection of (5.3) and (5.4), then necessarily $\tau_2 = 0$. Thus, using (5.5) and the recurrence formula (5.6) with $\tau_2 = 0$, we can show that there are no multiple points of the intersection with multiplicities > 2, since $f_0(\tau_1, \tau_2, \tau_3) \equiv 1$.

We now write

$$
f_\nu(t) = f_\nu(t, 0, -3^{1/3}) .
$$

Then, by (2.1) with $m=3$, $t_1=t$, $t_2=0$, $t_3=-3$, we have

$$
f_\nu(t) = \sum_{0 \le \mu \le \frac{\nu}{3}} \frac{\nu!}{\mu!(\nu-3\mu)!} (-t)^{\nu-3\mu} ,
$$

which may be rewritten in the form

$$
f_\nu(t) = t^\kappa g_\nu(t^3) ,
$$

where $\nu \equiv \kappa$ (mod 3), $0 \leq \kappa \leq 2$, and $g_\nu(t)$ is a polynomial of degree $\left[\dfrac{\nu}{3}\right]$ in t such that $g_\nu(0) \neq 0$.

Thus we have proved that the point $(0, 0, \tau_3)$ is a double point of the intersection only if $n \equiv 0$ (mod 3).

Note that if (τ_1, τ_2, τ_3) is a point of the intersection of (5.3) and (5.4), then so is also each one of the points

$$(\tau_1, (-1)^\alpha \tau_2, \omega^\beta \tau_3) \qquad (\alpha = 0, 1; \ \beta = 0, 1, 2),$$

where $\omega = e^{2\pi i/3}$, and these points correspond to one and the same solution $(\lambda_1, \lambda_2, \lambda_3)$ of (5.1). Hence, denoting by N the number of possible double points $(\tau_1, 0, \tau_3)$ with $\tau_1 \neq 0$ of that intersection, we thus have if $3 \nmid n$,

$$B(3, n) = \frac{(n+1)(n+2) - 2N}{3!} + \frac{N}{3} = \frac{(n+1)(n+2)}{6}$$

and if $3 \mid n$,

$$B(3, n) = \frac{(n+1)(n+2) - 2 - 2N}{3!} + 1 + \frac{N}{3} = \frac{(n+1)(n+2) + 4}{6}.$$

The proof of Theorem 2 is now complete.

6. Case of $n = 2$. In this case we have only to consider the single equation

$$z_1^{m+1} + z_2^{m+1} = 0.$$

Since

$$z_1^{m+1} + z_2^{m+1} = \prod_{\kappa=1}^{m+1} (z_1 - \omega^{2\kappa-1} z_2) \qquad (\omega = e^{\pi i/(m+1)}),$$

any system in $Z(m, 2)$ is of the form

$$(z, \omega^{2\kappa-1} z),$$

where z is a complex number and $1 \leq \kappa \leq m+1$. The $m+1$ systems corresponding to these κ are not all inequivalent.

If m is even, the system $(z, -z)$ (with $\kappa = m/2 + 1$) is inequivalent to any other $(z, \omega^{2\kappa-1} z)$ with $\kappa \neq m/2 + 1$. We find easily that $(z, \omega^{2\kappa-1} z)$ and $(z', \omega^{2\kappa'-1} z')$ are equivalent if and only if $\kappa = \kappa'$ or $\kappa + \kappa' \equiv 1$ (mod $m+1$). Thus there are $B(m, 2) = \dfrac{m}{2} + 1$ inequivalent classes in this case.

On the other hand, if m is odd, we can show that $B(m, 2) = \dfrac{m+1}{2}$, by a similar argument, completing the proof of (1.4).

7. Case of $n = 3$. Any system (z_1, z_2, z_3) of $Z(m, 3)$ is determined

by the conditions

(7. 1) $$z_1^{m+1} + z_2^{m+1} + z_3^{m+1} = 0$$

and

(7. 2) $$z_1^{m+2} + z_2^{m+2} + z_3^{m+2} = 0 \; .$$

It is easy to show on applying BÉZOUT's theorem that there exist exactly $(m+1)(m+2)$ distinct points of intersection of the curves defined by (7.1) and (7.2) in a projective plane over the complex numberfield.

Now, let π be a permutation on $1, 2, 3$, which is not the identical one. Suppose that, for such a π, we have

$$(z_1, z_2, z_3) = (z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)})$$

as the points of the projective plane, where (z_1, z_2, z_3) is a point of the intersection of (7.1) and (7.2). Then there must be a $\lambda \neq 0, 1$ such that

(7. 3) $$z_{\pi(1)} = \lambda z_1, \quad z_{\pi(2)} = \lambda z_2, \quad z_{\pi(3)} = \lambda z_3 \; .$$

Hence π is necessarily cyclic, and it follows from (7.3) that λ is a cubic root of unity satisfying the condition

$$1 + \lambda^{m+1} + \lambda^{2(m+1)} = 1 + \lambda^{m+2} + \lambda^{2(m+2)} = 0 \; ,$$

which is impossible unless $3 \mid m$.

Since there are exactly six possible permutations on $1, 2, 3$, we thus have proved that

$$B(m, 3) = \begin{cases} \dfrac{(m+1)(m+2)}{6} & \text{if} \quad 3 \nmid m \; , \\[3mm] \dfrac{(m+1)(m+2) - 2}{6} + 1 & \text{if} \quad 3 \mid m \; . \end{cases}$$

This is equivalent to (1.5).

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