# SUBSPACE THEORY OF AN $n$-DIMENSIONAL SPACE WITH AN ALGEBRAIC METRIC

By

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**Introduction.** Let $F_n^{(p)}$ be an $n$-dimensional Finsler space with the metric given by the differential form of order $p$: $ds^p = a_{\alpha_1 \cdots \alpha_p} dy^{\alpha_1} \cdots dy^{\alpha_p}$ ($\alpha$'s run over 1, 2, $\cdots$, $n$), $a_{{}_1\alpha \cdots \alpha_p}$ being function of $y$'s. Suppose that one has a homogeneous polynomial of order $p$ in $\xi$'s:

$$(0.1) \qquad a = a_{\alpha_1 \cdots \alpha_p} \xi^{\alpha_1} \cdots \xi^{\alpha_p}$$

which is defined in an $n$-dimensional projective space $E_n$ attached to at a point ($y$). When we put $a_\alpha = \frac{1}{p} \frac{\partial a}{\partial \xi^\alpha}$ the resultant of the $n$ forms $a_\alpha$ ($\alpha = 1, \cdots, n$) is named the discriminant of the form $a$, and denoted by $\mathfrak{A}$. It is well known [1][1)] that $\mathfrak{A}$ is a homogeneous polynomial of order $n(p-1)^{n-1}$ in the coefficients $a_{\alpha_1 \cdots \alpha_p}$, and that $\mathfrak{A} = 0$ is the necessary and sufficient condition in order that $n$ hypersurfaces $a_\alpha = 0$ in $E_n$ have common point. Consequently, $\mathfrak{A}$ is a scalar density of weight $\omega = p(p-1)^{n-1}$. The differential geometry in $F_n^{(p)}$ was studied for $F_2^{(p)}$ by A. E. Liber [2] and for $F_3^{(3)}$ by the present author [3] and Yu. I. Ermakov [4]. Moreover, Yu. I. Ermakov [5] has established the foundation of differential geometry in general case: $F_n^{(p)}$ ($p > 3$) by introducing the affine connection $\Gamma_{\beta\gamma}^\alpha$. The principal purpose of the present paper is to discuss the theory of subspace immersed in $F_n^{(p)}$ ($p \geqq 3$). §1 is devoted to the abridgment of the method of determination of the affine connection which was studied by Yu. I. Ermakov. §2 is offered to introduce the projection factor $B_\alpha^i$ and the normal vectors $\overset{p}{C}{}^\alpha$ and $\overset{p}{C}{}_\alpha$ to the subspace which will play the important roles in the theory of subspace. §3 and §4 are devoted to discuss the curvatures of a curve in the subspace and the Gauss and Codazzi equations for the subspace.

Furthermore we can discuss other many theories of the subspace making use of the projection factors and the normal vectors as well as the subspace in the Riemannian space. However we will omitt those discussions in this paper.

1) Numbers in brackets refer to the references at the end of the paper.

§ 1. Let $\xi^{\alpha}$ and $\xi^{\alpha'}$ be two systems of affine coordinates in $E_n$, and related by $\xi^{\alpha'}=X^{\alpha'}_{\alpha}\xi^{\alpha}$ or $\xi^{\alpha}=X^{\alpha}_{\alpha'}\xi^{\alpha'}$. Since $a_{\alpha'_1\cdots\alpha'_p}=X^{\alpha_1}_{\alpha'_1}\cdots X^{\alpha_p}_{\alpha'_p}a_{\alpha_1\cdots\alpha_p}$, we have

(1.1) $$\mathfrak{A}'(a_{\alpha'_1\cdots\alpha'_p})=|X^{\alpha}_{\alpha'}|^{\omega}\mathfrak{A}(a_{\alpha_1\cdots\alpha_p}),$$

$\omega$ being $p(p-1)^{n-1}$. Differentiating (1.1) by $X^{\alpha}_{\alpha'}$ and putting $X^{\alpha}_{\alpha'}=\delta^{\alpha}_{\alpha'}$, we have

(1.2) $$p\frac{\bar{\partial}\mathfrak{A}}{\bar{\partial}a_{\gamma\alpha_2\cdots\alpha_p}}a_{\beta\alpha_2\cdots\alpha_p}=\omega\mathfrak{A}\delta^{\gamma}_{\beta},$$

where $\dfrac{\bar{\partial}\mathfrak{A}}{\bar{\partial}a_{\alpha_1\cdots\alpha_p}}=\dfrac{l_1!\cdots l_s!}{p!}\dfrac{\partial\mathfrak{A}}{\partial a_{\alpha_1\cdots\alpha_p}}$ when $\alpha_1,\cdots,\alpha_p$ consist of $l_1,\cdots,l_s$ blocks of the same indices. Accordingly we have

(1.3) $$A^{\gamma\alpha_2\cdots\alpha_p}a_{\beta\alpha_2\cdots\alpha_p}=\delta^{\gamma}_{\beta},$$

putting $\dfrac{p}{\omega\mathfrak{A}}\dfrac{\bar{\partial}\mathfrak{A}}{\bar{\partial}a_{\gamma\alpha_2\cdots\alpha_p}}=A_{\gamma\alpha_2\cdots\alpha_p}$.

Let $\Gamma^{\lambda}_{\mu\nu}$ be the coefficient of an affine connection, it follows that

$$\Delta_{\mu}a_{\nu\alpha_2\cdots\alpha_p}=\partial_{\mu}a_{\nu\alpha_2\cdots\alpha_p}-\Gamma^{\omega}_{\mu\nu}a_{\omega\alpha_2\cdots\alpha_p}-\sum_{i=2}^{p}\Gamma^{\omega}_{\mu\alpha_i}a_{\nu\alpha_2\cdots\alpha_{i-1}\omega\alpha_{i+1}\cdots\alpha_p}.$$

Multiplying by $A^{\lambda\alpha_2\cdots\alpha_p}$ and summing for $\alpha_2,\cdots,\alpha_p$ one obtains

(1.4) $$A^{\lambda\alpha_2\cdots\alpha_p}\nabla_{\mu}a_{\nu\alpha_2\cdots\alpha_p}=A^{\lambda\alpha_2\cdots\alpha_p}\partial_{\mu}a_{\nu\alpha_2\cdots\alpha_p}-\Gamma^{\lambda}_{\mu\nu}-(p-1)\Gamma^{\omega}_{\mu\tau}N^{\lambda\tau}_{\nu\omega},$$

putting $N^{\lambda\tau}_{\nu\omega}=A^{\lambda\tau\alpha_3\cdots\alpha_p}a_{\nu\omega\alpha_3\cdots\alpha_p}$. Moreover, if we put $B^{\lambda}_{\mu\nu,\alpha}{}^{\beta\gamma}=\delta^{\lambda}_{\alpha}\delta^{\beta}_{(\mu}\delta^{\gamma}_{\nu)}+(p-1)\delta^{\beta}_{(\mu}N^{\gamma\lambda}_{\nu)\alpha}$ we have from (1.4)

(1.5) $$A^{\lambda\alpha_2\cdots\alpha_p}\nabla_{(\mu}a_{\nu)\alpha_2\cdots\alpha_p}=A^{\lambda\alpha_2\cdots\alpha_p}\partial_{(\mu}a_{\nu)\alpha_2\cdots\alpha_p}-B^{\lambda}_{\mu\nu,\alpha}{}^{\beta\gamma}\Gamma^{\alpha}_{\beta\gamma}.$$

When the polynomial (0.1) is of the special form: $a=\sum_{\alpha=1}^{n}a_{\alpha}(\xi^{\alpha})^p$ that is $a_{\alpha_1\cdots\alpha_p}=\sum_{\omega=1}^{n}a_{\omega}\delta^{\omega}_{\alpha_1}\cdots\delta^{\omega}_{\alpha_p}$ we have $\mathfrak{A}=(a_1\cdots a_n)^{\frac{\omega}{p}}$ so that $A^{\alpha_1\cdots\alpha_p}=\sum_{\tau=1}^{n}\dfrac{l_1!\cdots l_s!}{p!}\dfrac{1}{a_{\tau}}\delta^{\alpha_1}_{\tau}\cdots\delta^{\alpha_p}_{\tau}$. Hence one has $B^{\lambda}_{\mu\nu,\alpha}{}^{\beta\gamma}=\delta^{\lambda}_{\alpha}\delta^{\beta}_{(\mu}\delta^{\gamma}_{\nu)}+(p-1)\sum_{\omega=1}^{n}\delta^{\gamma}_{\omega}\delta^{\lambda}_{\omega}\delta^{\beta}_{(\mu}\delta^{\omega}_{\nu)}\delta^{\omega}_{\alpha}$ from which it follows that the elements in the principal diagonal of the determinant $|B^{\lambda}_{\mu\nu,\alpha}{}^{\beta\gamma}|$ are different from zero and others are zero, and consequently $|B^{\lambda}_{\mu\nu,\alpha}{}^{\beta\gamma}|\neq0$. Hence it may be assumed that $|B^{\lambda}_{\mu\nu,\alpha}{}^{\beta\gamma}|$ does not vanish in generally. Assuming $|B^{\lambda}_{\mu\nu,\alpha}{}^{\beta\gamma}|\neq0$, we can determine a tensor $P^{\omega}_{\rho\tau,\lambda}{}^{\mu\nu}$ such that $P^{\omega}_{\rho\tau,\lambda}{}^{\mu\nu}B^{\lambda}_{\mu\nu,\alpha}{}^{\beta\gamma}=\delta^{\omega}_{\alpha}\delta^{\beta}_{(\rho}\delta^{\gamma}_{\tau)}$. Now multiplying (1.5) by $P^{\omega}_{\rho\tau,\lambda}{}^{\mu\nu}$ and summing for $\lambda$, $\mu$, $\nu$ it follows that

$$\Gamma^{\omega}_{\rho\tau}=P^{\omega}_{\rho\tau,\lambda}{}^{\mu\nu}A^{\lambda\alpha_2\cdots\alpha_p}\left(\partial_{(\mu}a_{\nu)\alpha_2\cdots\alpha_p}-\nabla_{(\mu}a_{\nu)\alpha_2\cdots\alpha_p}\right).$$

Under the condition $A^{\lambda\alpha_2\cdots\alpha_p}\nabla_{(\mu}a_{\nu)\alpha_2\cdots\alpha_p}=0$, we have

$$\Gamma^{\omega}_{\rho\tau} = P^{\omega\ \mu\nu}_{\rho\tau,\lambda}A^{\lambda\alpha_2\cdots\alpha_p}\partial_{(\mu}a_{\nu)\alpha_2\cdots\alpha_p}\,. \tag{1.6}$$

§ **2.** An $m$-dimensional subspace of $F_n^{(p)}$ may be represented parametrically by the equations

$$y^{\alpha} = y^{\alpha}(x^i) \qquad (\alpha=1,\cdots,n)\,, \tag{2.1}$$

where one suppose that the variables $x^i$ $(i=1,\cdots,m)$ $(m<n)$ form a coordinate system of the subspace. Furthermore throughout this paper we shall assume that the functions (2.1) are of class $C^4$, and introducing the notation $B^{\alpha}_i=\dfrac{\partial y^{\alpha}}{\partial x^i}$, we shall also assume that the matrix of $B^{\alpha}_i$ is of rank $m$. If $dy^{\alpha}$ is a small displacement tangent to the subspace (2.1), it follows that $dy^{\alpha}=B^{\alpha}_i dx^i$, $dx^i$ being the same displacement in term of the coordinate $x^i$ of the subspace. Thus, the $ds=(a_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p})^{1/p}$ represents the distance between near two points $x^i$ and $x^i+dx^i$ in the subspace, putting $a_{i_1\cdots i_p}=a_{\alpha_1\cdots\alpha_p}B^{\alpha_1}_{i_1}\cdots B^{\alpha_p}_{i_p}$. Assuming that the discriminant $\mathfrak{A}'$ of the polynomial in $dx$'s : $a=a_{i_1\cdots i_p}(x)dx^{i_1}\cdots dx^{i_p}$ be different from zero, we can derive a tensor $A^{ki_2\cdots i_p}$ in the subspace such that

$$A^{ki_2\cdots i_p}a_{ji_2\cdots i_p} = \delta^k_j\,. \tag{2.2}$$

If we put

$$A^{ki_2\cdots i_p}a_{\alpha\beta_2\cdots\beta_p}B^{\beta_2}_{i_2}\cdots B^{\beta_p}_{i_p} = B^k_{\alpha}\,,$$

in vertue of (2.2) it follows that

$$B^k_{\alpha}B^{\alpha}_j = \delta^k_j\,. \tag{2.3}$$

A covariant vector $C_{\alpha}$ is said to be nermal to the subspace (2.1), if it satisfies the equations

$$B^{\alpha}_i C_{\alpha} = 0 \qquad (i=1,\cdots,m)\,. \tag{2.4}$$

These are $m$ equations for the determination of $n$ functions $C_{\alpha}$ $(\alpha=1,\cdots,n)$. Since the rank of the matrix $\|B^{\alpha}_i\|$ was assumed to be $m$, there exist $(n-m)$ linearly independent vectors $\overset{p}{C}_{\alpha}$ $(p=m+1,\cdots,n)$ normal to the subspace and these may be chosen in a multiply infinite number of ways: $B^{\alpha}_i\overset{p}{C}_{\alpha}=0$. Hence $n$ covariant vectors $B^i_{\alpha}$ $(i=1,\cdots,m)$, $\overset{p}{C}_{\alpha}$ $(p=m+1,\cdots,n)$ are linearly independent, so that we may chose a set of $n-m$ contravariant vectors $\underset{q}{C}^{\alpha}$ $(q=m+1,\cdots,n)$ satisfying the relations

$$B^i_\alpha \underset{q}{C^\alpha} = 0 \,, \quad \overset{p}{C}_\alpha \underset{q}{C^\alpha} = \delta^p_q \,.$$

The vectors $\underset{p}{C^\alpha}$ is said to be contravariant normal to the subspace. Now, consider the tensor

$$\varphi^\alpha_\beta = \delta^\alpha_\beta - B^\alpha_i B^i_\beta \,. \tag{2.5}$$

Multiplying $B^\beta_j$ and summing for $\beta$ one has $\varphi^\alpha_\beta B^\beta_j = 0$ from which it follows that $\varphi^\alpha_\beta$ is a linear combination of $n-m$ vectors $\overset{p}{C}_\alpha$ $(p=m+1, \cdots, n)$, that is

$$\varphi^\alpha_\beta = \sum_{p=m+1}^{n} \underset{p}{\lambda^\alpha} \overset{p}{C}_\beta \,. \tag{2.6}$$

Multiplying (2.5) and (2.6) by $\underset{t}{C^\beta}$ and summing for $\beta$ we have respectively $\varphi^\alpha_\beta \underset{t}{C^\beta} = \underset{t}{C^\alpha}$ and $\varphi^\alpha_\beta \underset{t}{C^\beta} = \underset{t}{\lambda^\alpha}$ so that $\underset{t}{\lambda^\alpha} = \underset{t}{C^\alpha}$. Hence from (2.6) we have $\varphi^\alpha_\beta = \sum_{p=m+1}^{n} \underset{p}{C^\alpha} \overset{p}{C}_\beta$, and consequently it follows that

$$B^\alpha_i B^i_\beta + \sum_{p=m+1}^{n} \underset{p}{C^\alpha} \overset{p}{C}_\beta = \delta^\alpha_\beta \,. \tag{2.7}$$

Putting $B^\alpha_i B^i_\beta = B^\alpha_\beta$ and $\sum_{p=m+1}^{n} \underset{p}{C^\alpha} \overset{p}{C}_\beta = C^\alpha_\beta$ we have from (2.7) $B^\alpha_\beta + C^\alpha_\beta = \delta^\alpha_\beta$.

§ 3. Let us consider a vector field $v^\alpha(s)$ tangent to the subspace (2.1) along a curve in the subspace. The covariant derivative of $v^\alpha$ with respect to the arc length $s$ along the curve is defined as follows:

$$\frac{\delta v^\alpha}{\delta s} = \frac{dv^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} v^\beta \frac{dy^\gamma}{ds} \,. \tag{3.1}$$

If $v^i$ $(i=1, \cdots, m)$ are the components of the vector $v^\alpha$ in the coordinate system $x^i$, we have

(3.2 *a*) $\quad v^\alpha = B^\alpha_i v^i \quad$ or $\quad$ (3.2 *b*) $\quad v^i = B^i_\alpha v^\alpha \,.$

According to the usual definition of induced derivative we shall define the covariant derivative of $v^i$ along the curve

$$\frac{\delta v^i}{\delta s} = B^i_\alpha \frac{\delta v^\alpha}{\delta s} \tag{3.3}$$

that is

$$\frac{dv^i}{ds}+\Gamma^i_{jk}v^j\frac{dx^k}{ds}=B^i_\alpha\left(\frac{dv^\alpha}{ds}+\Gamma^\alpha_{\beta\gamma}v^\beta\frac{dy^\gamma}{ds}\right).$$

Hence we have

$$B^\alpha_{jk}B^i_\alpha+\Gamma^\alpha_{\beta\gamma}B^\beta_jB^\gamma_kB^i_\alpha-\Gamma^i_{jk}=0\,, \tag{3.4}$$

putting $B^\alpha_{jk}=\dfrac{\partial}{\partial x^k}B^\alpha_j$.

Substituting (3.2 $a$) in the right hand member of (3.1) one obtains

$$\frac{\delta v^\alpha}{\delta s}=B^\alpha_{j,k}v^j\frac{dx^k}{ds}+B^\alpha_i\frac{\delta v^i}{\delta s}\,, \tag{3.5}$$

where we put

$$B^\alpha_{j,k}=B^\alpha_{jk}+\Gamma^\alpha_{\beta\gamma}B^\beta_jB^\gamma_k-B^\alpha_i\Gamma^i_{jk}\,. \tag{3.6}$$

In virtue of (3.4) it follows that $B^i_\alpha B^\alpha_{j,k}=0$. Consequently we have $n-m$ symmetric tensor in the subspace: $\overset{p}{\omega}_{jk}$ $(p=m+1,\cdots,n)$ such that

$$B^\alpha_{j,k}=\sum_{p=m+1}^{n}-\overset{p}{\omega}_{jk}\underset{p}{C}^\alpha\,. \tag{3.7}$$

Substituting this expression in (3.5) one has

$$\frac{\delta v^\alpha}{\delta s}=B^\alpha_i\frac{\delta v^i}{\delta s}-\sum_{p=m+1}^{n}\overset{p}{\omega}_{jk}v^j\frac{dx^k}{ds}\underset{p}{C}^\alpha\,. \tag{3.8}$$

When we put $v^\alpha=\dfrac{dy^\alpha}{ds}$, (3.8) becomes

$$\frac{\delta^2y^\alpha}{\delta s^2}=B^\alpha_i\frac{d^2x^i}{ds^2}-\sum_{p=m+1}^{n}\overset{p}{\omega}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\underset{p}{C}^\alpha\,. \tag{3.9}$$

Hence it is easily seen that a path of $F^{(p)}_n$ lies in a subspace is a path of the subspace. Moreover we see that a necessary and sufficient condition that every path of a subspace be a path of the enveloping space: $F^{(p)}_n$ is that $\overset{p}{\omega}_{jk}=0$.

In (3.9) we understand that $\dfrac{\delta^2y^\alpha}{\delta s^2}$ and $B^\alpha_i\dfrac{\delta^2x^i}{\delta s^2}$ are the principal normal of the curve in $F^{(p)}_n$ and the subspace respectively, and $\overset{p}{\omega}_{jk}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}$ is the component of the curvature of the curve in the direction $\underset{p}{C}^\alpha$.

**§ 4.** In order to find the conditions of integrability of (3.7) we denote by $\underset{p}{h}{}^i_k$ the tensor in the subspace derived from the tensor $\nabla_\gamma \underset{p}{C}{}^\alpha$ and denote by $\underset{p}{\overset{q}{h}}{}_k$ the vector in the subspace derived from the tensor $\overset{q}{C}_\alpha \nabla_\gamma \underset{p}{C}{}^\alpha$, i.e.

$$(4.1)\qquad B^i_\alpha B^\gamma_k \nabla_\gamma \underset{p}{C}{}^\alpha = \underset{p}{h}{}^i_k\,, \qquad\qquad (4.2)\qquad B^\gamma_k \overset{q}{C}_\alpha \nabla_\gamma \underset{p}{C}{}^\alpha = \underset{p}{\overset{q}{h}}{}_k\,.$$

After some calculation, (4.1) can be written in the form

$$(4.3)\qquad \underset{p}{C}{}^\alpha_{,k} = \underset{p}{h}{}^i_k B^\alpha_i + \underset{q}{C}{}^\alpha \underset{p}{\overset{q}{h}}{}_k\,,$$

where $\underset{p}{C}{}^\alpha_{,k}$ represents $B^\gamma_k \nabla_\gamma \underset{p}{C}{}^\alpha$. Now, from (3.7) we have

$$(4.4)\qquad B^\alpha_{i,jk} - B^\alpha_{i,kj} = -\underset{p}{C}{}^\alpha(\overset{p}{\omega}_{ij,k} - \overset{p}{\omega}_{ik,j}) - (\overset{p}{\omega}_{ij}\underset{p}{C}{}^\alpha_{,k} - \overset{p}{\omega}_{ik}\underset{p}{C}{}^\alpha_{,j})\,,$$

where the symbol $X_{,j}$ represent a covariant derivative of $X$ induced to the subspace. Since $B^\nu_{i,jk} - B^\alpha_{i,kj} = B^\alpha_m K^m_{ijk} - B^\beta_i B^\gamma_j B^\delta_k K^\alpha_{\beta\gamma\delta}$, in consequence of (4.3) the equation (4.4) is reducible to

$$(4.5)\qquad B^\alpha_m K^m_{ijk} - B^\beta_i B^\gamma_j B^\delta_k K^\alpha_{\beta\gamma\delta} = -\underset{p}{C}{}^\alpha(\overset{p}{\omega}_{ij,k} - \overset{p}{\omega}_{ik,j} + \overset{q}{\omega}_{ij}\underset{q}{\overset{p}{h}}{}_k - \overset{q}{\omega}_{ik}\underset{q}{\overset{p}{h}}{}_j) + B^\alpha_m(\overset{q}{\omega}_{ik}\underset{q}{h}{}^m_j - \overset{q}{\omega}_{ij}\underset{q}{h}{}^m_k)\,,$$

where $K^m_{ijk}$ and $K^\alpha_{\beta\gamma\delta}$ are the same with the Riemannian Symboles for the coefficients of connection: $\Gamma^i_{jk}$ and $\Gamma^\alpha_{\beta\gamma}$ respectively.

If this equation be multiplied by $B^l_\alpha$ and $\overset{s}{C}_\alpha$ and $\alpha$ be summed we have respectively

$$(4.6)\qquad K^l_{ijk} = B^l_\alpha B^\beta_i B^\gamma_j B^\delta_k K^\alpha_{\beta\gamma\delta} + \overset{q}{\omega}_{ik}\underset{q}{h}{}^l_j - \overset{q}{\omega}_{ij}\underset{q}{h}{}^l_k$$

and

$$(4.7)\qquad \overset{s}{\omega}_{ij,k} - \overset{s}{\omega}_{ik,j} = K^\alpha_{\beta\gamma\delta} B^\beta_i B^\gamma_j B^\delta_k \overset{s}{C}_\alpha + \overset{q}{\omega}_{ik}\underset{q}{\overset{s}{h}}{}_j - \overset{q}{\omega}_{ij}\underset{q}{\overset{s}{h}}{}_k\,.$$

On the other hand, the conditions of integrability of (4.3) are obtained from

$$\underset{p}{C}{}^\alpha_{,[kj]} = \underset{p}{h}{}^i_{[k,j]} B^\alpha_i - \underset{p}{h}{}^i{}_{[k} B^\alpha_{|i|,j]} + \underset{q}{C}{}^\alpha_{,[j}\underset{p}{\overset{q}{h}}{}_{k]} + \underset{q}{C}{}^\alpha \underset{p}{\overset{q}{h}}{}_{[k,j]}\,,$$

that is

$$\underset{p}{C}{}^{\alpha}_{,[kj]} = \underset{p}{h}{}^{i}_{[k,j]} B^{\alpha}_{i} - \overset{q}{\omega}_{ij} \underset{p}{h}{}^{i}_{k} \underset{q}{C}{}^{\alpha} + \underset{q}{h}{}^{i}_{[j} \overset{q}{\underset{p}{h}}_{k]} B^{\alpha}_{i} + \overset{q}{\underset{p}{h}}_{[k,j]} \underset{q}{C}{}^{\alpha} .$$

From this we have

$$K^{\alpha}_{\beta\gamma\delta} B^{\beta}_{k} B^{\gamma}_{j} \underset{p}{C}{}^{\delta} = (\underset{p}{h}{}^{i}_{j,k} + \underset{q}{h}{}^{i}_{j} \overset{q}{\underset{p}{h}}_{k} - \underset{q}{h}{}^{i}_{k} \overset{q}{\underset{p}{h}}_{j}) B^{\alpha}_{i}$$
$$+ (\overset{q}{\omega}_{ik} \underset{p}{h}{}^{i}_{j} - \overset{q}{\omega}_{ij} \underset{p}{h}{}^{i}_{k} + \overset{q}{\underset{p}{h}}_{k,j} - \overset{q}{\underset{p}{h}}_{j,k}) \underset{q}{C}{}^{\alpha} .$$

From this equation it follows that

$$K^{\alpha}_{\beta\gamma\delta} B^{\beta}_{k} B^{\gamma}_{j} B^{l}_{\alpha} \underset{p}{C}{}^{\delta} = \underset{p}{h}{}^{l}_{k,j} - \underset{p}{h}{}^{l}_{j,k} + \underset{q}{h}{}^{l}_{j} \overset{q}{\underset{p}{h}}_{k} - \underset{q}{h}{}^{l}_{k} \overset{q}{\underset{p}{h}}_{j} \tag{4.8}$$

and

$$K^{\alpha}_{\beta\gamma\delta} B^{\beta}_{k} B^{\gamma}_{j} \underset{p}{C}{}^{\delta} \overset{q}{C}_{\alpha} = \overset{q}{\underset{p}{h}}_{k,j} - \overset{q}{\underset{p}{h}}_{j,k} + \overset{q}{\omega}_{ik} \underset{p}{h}{}^{i}_{j} - \overset{q}{\omega}_{ij} \underset{p}{h}{}^{i}_{k} .$$

We call (4.6) the equation of Gauss and (4.7), (4.8) the equations of Codazzi.

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