ON THE DISTRIBUTION OF ALMOST PRIMES IN AN ARITHMETIC PROGRESSION

By

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1. Introduction. An almost prime is a positive integer the number of whose prime divisors is bounded by a certain constant. The purpose of this paper is to deal with an existence problem of almost primes in a short arithmetic progression of integers. We shall prove the following

Theorem. *Let k and l be two integers with $k\geqq 1$, $0\leqq l\leqq k-1$, $(k, l)=1$. There exists a numerical constant $c_1>0$ such that for every real number $x\geqq c_1k^{3.5}$ there is at least one integer n satisfying*

$$x<n\leqq 2x\,, \qquad n\equiv l \pmod k\,, \qquad V(n)\leqq 2\,,$$

where $V(n)$ denotes the total number of prime divisors of n. In particular, if we write $a(k, l)$ for the least positive integer n (>1) satisfying

$$n\equiv l \pmod k\,, \qquad V(n)\leqq 2\,,$$

then we have

$$a(k, l)<c_2k^{3.5}$$

with some absolute constant $c_2>0$.

It is of some interest to compare our results presented above, though they are not the best possible, with a recent result of T. Tatuzawa [5] on the existence of a prime number p satisfying $x<p\leqq 2x$, $p\equiv l \pmod k$ and a celebrated theorem of Yu. V. Linnik concerning the upper bound for the least prime $p\equiv l \pmod k$ (cf. [3: X]).

Our proof of the theorem is based essentially upon the general sieve methods due to A. Selberg. The deepest result which we shall refer to is:

$$\pi(x) = \mathrm{li}\, x + O\left(x \exp(-c_3(\log x)^{1/2})\right)$$

with a positive constant c_3, where $\pi(x)$ denotes, as usual, the number of primes not exceeding x (in fact, a slightly weaker result will suffice for our purpose). Apart from this, the proof is entirely elementary.

Notations. Throughout in the following, k represents a fixed positive integer, l an integer with $0 \leqq l \leqq k-1$, $(k, l)=1$. The letters p, q are used to denote prime numbers and, d, m, n, r to denote positive integers. The functions $\mu(n)$ and $\varphi(n)$ are Möbius' and Euler's functions, respectively. The function $g(n)$ is defined as follows: $g(1)=1$ and for $n>1$ $g(n)=$the greatest prime divisor of n.

s, t, u, v, w, x, y, z will be used to denote real numbers, constant or variable. c represents positive constants, not depending on k and l, which are not necessarily the same in each occurrence; the constants implied in the symbol O are either absolute or else uniform in k and l.

2. Preliminaries. There needs the following lemma for later calculations:

Lemma 1. *We have*

$$\sum_{p \leqq t} \frac{1}{p} = \log\log t + c_4 + O\left(\frac{1}{\log t}\right),$$

where c_4 is a constant;

$$\sum_{p \leqq t} \frac{\log p}{p} = \log t + O(1);$$

$$\prod_{p \leqq t} \left(1-\frac{1}{p}\right)^{-1} = e^C \log t + O(1),$$

where C is Euler's constant; and

$$\varphi(m) > c\frac{m}{\log\log 3m}.$$

These results are well known. For a proof see [3: I, Theorems 3.1, 4.1 and 5.1].

Let $M \geqq 0$, $N \geqq 2$ be arbitrary but fixed integers and put

$$y = 2k(N+1), \qquad w = y^{\frac{1}{2}-\varepsilon},$$

where $0<\varepsilon<\frac{1}{4}$: we shall fix $\varepsilon=\frac{1}{7}$ later on. Further we put

$$z = y^{\frac{1}{\alpha}}, \qquad z_1 = y^{\frac{1}{\beta}}, \qquad z_2 = y^{\frac{1}{\gamma}},$$

where α, β, γ are fixed real numbers satisfying

$$10 \geqq \gamma \geqq 4 \geqq \alpha > 2 \geqq \beta > 1.$$

First we wish to evaluate from below the number S_1 of those integers of the form $kn+l$ $(M<n\leqq M+N)$ which are not divisible by any prime $p\leqq z$. Applying the 'lower' sieve of A. Selberg (see [2] and [7]), we find that

$$S_1\geqq(1-Q)\,N-R_1\,,$$

where

$$Q=\sum_{\substack{p\leqq z\\(p,k)=1}}\frac{1}{pZ_p}\qquad\text{with}\quad Z_p=\sum_{\substack{n\leqq w/\sqrt{p}\\ g(n)<p\\(n,k)=1}}\frac{\mu^2(n)}{\varphi(n)}$$

and

$$R_1=O\left(w^2\sum_{p\leqq z}\frac{1}{pZ_p^2}\right).$$

It will be shown later that

$$Z_p>c\frac{\varphi(k)}{k}\log p\qquad\text{for all}\quad p\leqq z,$$

and so we have, by Lemma 1,

$$R_1=O\left(w^2(\log\log 3k)^2\right).$$

We put

$$H_p=\prod_{\substack{q<p\\(q,k)=1}}\left(1-\frac{1}{q}\right)^{-1}\qquad(p\leqq z).$$

Then it is easily verified that

$$1-Q=\prod_{\substack{p\leqq z\\(p,k)=1}}\left(1-\frac{1}{p}\right)-\sum_{\substack{p\leqq z\\(p,k)=1}}\frac{H_p-Z_p}{pH_pZ_p}.$$

Lemma 2. *We have*

$$S_1\geqq\frac{kN}{\varphi(k)}\prod_{p\leqq z}\left(1-\frac{1}{p}\right)-N\sum_{\substack{p\leqq z\\(p,k)=1}}\frac{H_p-Z_p}{pH_pZ_p}$$
$$+O\left(\frac{N(\log\log 3k)^3}{z\log z}\right)+O\left(w^2(\log\log 3k)^2\right).$$

Proof. We have only to prove that

$$\prod_{\substack{p\leqq z\\(p,k)=1}}\left(1-\frac{1}{p}\right)=\frac{k}{\varphi(k)}\prod_{p\leqq z}\left(1-\frac{1}{p}\right)+O\left(\frac{(\log\log 3k)^3}{z\log z}\right)$$

or

$$(1) \qquad \prod_{\substack{p\leqq z\\ p\nmid k}}\left(1-\frac{1}{p}\right)^{-1}=\frac{k}{\varphi(k)}+O\left(\frac{(\log\log 3k)^3}{z}\right).$$

Now we have

$$0\leqq \prod_{\substack{p\leqq z\\ p\nmid k}}\left(1-\frac{1}{p}\right)-\frac{\varphi(k)}{k}=\prod_{\substack{p\leqq z\\ p\mid k}}\left(1-\frac{1}{p}\right)-\prod_{p\mid k}\left(1-\frac{1}{p}\right)$$

$$\leqq \sum_{\substack{p>z\\ p\mid k}}\sum_{\substack{d\mid k\\ d\equiv 0\,(p)}}\frac{\mu^2(d)}{d}=\sum_{\substack{p>z\\ p\mid k}}\frac{1}{p}\sum_{\substack{d\mid k/p\\ (d,p)=1}}\frac{\mu^2(d)}{d}$$

$$=O\left(\frac{1}{z}\,\frac{\log k}{\log z}\log\log 3k\right)=O\left(\frac{\log\log 3k}{z}\right),$$

from which follows (1) at once.

Let q be a prime number in the interval $z<q\leqq z_1$ with $(q, k)=1$. We next evaluate from above the number $S(q)$ of those integers $kn+l$ $(M<n\leqq M+N)$ which are not divisible by any prime $p\leqq z$ and are divisible by the prime q. Applying the 'upper' sieve of A. Selberg (see the Appendix below), we find that

$$S(q)\leqq \frac{N}{qW_q}+R(q),$$

where

$$W_q=\sum_{\substack{n\leqq z^a\\ g(n)\leqq z\\ (n,k)=1}}\frac{\mu^2(n)}{\varphi(n)}$$

with

$$a=\frac{\alpha}{2}\left(1-2\varepsilon-\frac{\log q}{\log y}\right)$$

and

$$R(q)=O\left(\frac{z^{2a}}{W_q^2}\right)=O\left(\frac{w^2}{qW_q^2}\right).$$

Now, let $r\geqq 1$ be a fixed integer and let S_2 denote the number of those integers of the form $kn+l$ $(M<n\leqq M+N)$ which are not divisible by any prime $p\leqq z$ and are divisible by at least $r+1$ distinct primes q in the interval $z<q\leqq z_1$ with $(q, k)=1$. Clearly S_2 is not greater than

$$\frac{1}{r+1}\sum_{\substack{z<q\leqq z_1\\(q,k)=1}} S(q)\,.$$

Hence:

Lemma 3. *We have*

$$S_2 \leqq \frac{N}{r+1}\sum_{\substack{z<q\leqq z_1\\(q,k)=1}}\frac{1}{qW_q} + O\left(\frac{w^2(\log\log 3k)^2}{\log^2 y}\right).$$

Proof. It will later be shown that

$$W_q > c\frac{\varphi(k)}{k}\log y \qquad \text{for} \quad z<q\leqq z_1\,.$$

It follows that

$$\frac{1}{r+1}\sum_{\substack{z<q\leqq z_1\\(q,k)=1}} R(q) = O\left(\frac{w^2(\log\log 3k)^2}{\log^2 y}\sum_{z<q\leqq z_1}\frac{1}{q}\right)$$
$$= O\left(\frac{w^2(\log\log 3k)^2}{\log^2 y}\right),$$

since

$$\sum_{z<q\leqq z_1}\frac{1}{q} = \log\frac{\alpha}{\beta} + O(1) = O(1)\,.$$

3. Some lemmas. Here we collect some auxiliary results which will be needed in the next two sections.

Lemma 4. *We have*

$$\sum_{d|m}\frac{\mu^2(d)\log d}{d} = O\left((\log\log 3m)^2\right).$$

Proof. The left-hand side is equal to

$$\sum_{d|m}\frac{\mu^2(d)}{d}\sum_{p|d}\log p = \sum_{p|m}\log p \sum_{\substack{d|m\\d\equiv 0\ (p)}}\frac{\mu^2(d)}{d}$$
$$= \sum_{p|m}\frac{\log p}{p}\sum_{\substack{d|m/p\\(d,p)=1}}\frac{\mu^2(d)}{d}\,,$$

where we have

$$\sum_{\substack{d|m/p\\(d,p)=1}}\frac{\mu^2(d)}{d} \leqq \sum_{d|m}\frac{1}{d} = O(\log\log 3m)$$

and

$$\sum_{p|m}\frac{\log p}{p}=\sum_{\substack{p|m\\ p\le\log m}}\frac{\log p}{p}+O(1)$$
$$=O(\log\log 3m).$$

This proves Lemma 4.

Lemma 5. *We have*

$$\sum_{\substack{n\le t\\ (n,m)=1}}\frac{\mu^2(n)}{\varphi(n)}=\frac{\varphi(m)}{m}\log t+O(\log\log 3m).$$

Proof. H. N. Shapiro and J. Warga [4: Appendix I] have proved that

$$\sum_{\substack{n\le t\\ (n,m)=1}}\frac{\mu^2(n)}{n}=\frac{\varphi(m)}{m}\prod_{p\nmid m}\left(1-\frac{1}{p^2}\right)\log t+O(\log\log 3m).$$

Using this inequality we obtain

$$\sum_{\substack{n\le t\\ (n,m)=1}}\frac{\mu^2(n)}{\varphi(n)}=\sum_{\substack{n\le t\\ (n,m)=1}}\frac{\mu^2(n)}{n}\prod_{p|n}\left(1+\frac{1}{p-1}\right)$$
$$=\sum_{\substack{n\le t\\ (n,m)=1}}\frac{\mu^2(n)}{n}\sum_{d|n}\frac{1}{\varphi(d)}$$
$$=\sum_{\substack{d\le t\\ (d,m)=1}}\frac{\mu^2(d)}{d\varphi(d)}\sum_{\substack{n\le t/d\\ (n,dm)=1}}\frac{\mu^2(n)}{n}$$
$$=\sum_{\substack{d\le t\\ (d,m)=1}}\frac{\mu^2(d)}{d\varphi(d)}\left(\frac{\varphi(dm)}{dm}\prod_{p\nmid dm}\left(1-\frac{1}{p^2}\right)\log\frac{t}{d}+O(\log\log 3dm)\right)$$
$$=\frac{\varphi(m)}{m}\prod_{p\nmid m}\left(1-\frac{1}{p^2}\right)\sum_{\substack{d\le t\\ (d,m)=1}}\frac{\mu^2(d)}{d^2}\prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1}\log\frac{t}{d}$$
$$+O\left(\sum_{d\le t}\frac{\mu^2(d)}{d\varphi(d)}\log\log 3dm\right)$$
$$=\frac{\varphi(m)}{m}\prod_{p\nmid m}\left(1-\frac{1}{p^2}\right)\sum_{\substack{d=1\\ (d,m)=1}}^{\infty}\frac{\mu^2(d)}{d^2}\prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1}\log t$$
$$+O\left(\sum_{d>t}\frac{\mu^2(d)}{d^2}\prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1}\log d\right)$$
$$+O\left(\sum_{d\le t}\frac{\mu^2(d)}{d\varphi(d)}\log\log 3dm\right)$$

$$= \frac{\varphi(m)}{m} \log t + O(\log\log 3m),$$

since

$$\sum_{\substack{d=1 \\ (d,m)=1}}^{\infty} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1-\frac{1}{p^2}\right)^{-1} = \prod_{p\nmid m} \left(1+\frac{1}{p^2-1}\right) = \prod_{p\nmid m} \left(1-\frac{1}{p^2}\right)^{-1}.$$

Now, for $u>0$, $v\geqq 2$, let $G(u,v)$ denote the number of positive integers $n\leqq u$ with $g(n)\leqq v$.

Define the function $\rho(s)$ by the following properties:

$$(2) \qquad \begin{cases} \rho(s)=0 \quad (s<0); & \rho(s)=1 \quad (0\leqq s\leqq 1); \\ s\rho'(s)=-\rho(s-1) \quad (s>1); & \rho(s) \text{ continuous for } s>0. \end{cases}$$

Then the following result has been proved by N. G. de Bruijn [1]:

Lemma 6. *Let $u>0$, $v\geqq 2$, and put $t=(\log u)/\log v$. Then we have*

$$G(u,v) = O(ue^{-ct})$$

and, more precisely,

$$G(u,v) = u\rho(t)\left(1+O\left(\frac{\log(2+t)}{\log v}\right)\right) + O(1) + O\left(ut^2P(v)\right),$$

where $P(v)$ is a function satisfying

$$P(v)\downarrow 0 \quad (v\to\infty), \qquad P(v)>(\log v)/v \quad (v\geqq 2),$$

$$|\pi(v)-\mathrm{li}\, v| < vP(v)/\log v \qquad (v\geqq 2).$$

As to the function $\rho(s)$ itself, it is not difficult to prove the following result, which is known as a lemma of N. C. Ankeny:

Lemma 7. *For $s_1\geqq s_2\geqq 1$ we have*

$$\rho(s_1) \leqq \rho(s_2)e^{-(s_1-s_2)},$$

so that

$$\int_s^{\infty} \rho(t)\,dt \leqq \rho(s) \qquad (s\geqq 1).$$

For a proof of this result see [8].

4. Evaluation of S_1. We are now going to find an explicit lower bound for S_1 on the basis of Lemma 2.

First we have to evaluate Z_p and H_p-Z_p for $p\leqq z$. To accomplish this we distinguish three cases on the magnitude of the prime p.

It is clear that

$$T_p \overset{\text{def}}{=} \sum_{\substack{n>w/\sqrt{p}\\ g(n)\leqq p\\ (n,k)=1}} \frac{1}{n} \geqq H_p-Z_p\geqq 0\,.$$

Case 1: $2\leqq p\leqq \exp(\log y)^{\frac{2}{3}}$. By partial summation we get

$$T_p\leqq \sum_{\substack{n>w/\sqrt{p}\\ g(n)\leqq p}} \frac{1}{n} = \sum_{n>w/\sqrt{p}} \frac{G(n,p)}{n^2}+O(y^{-c_5})\,,$$

where $c_5=\dfrac{1}{2}-\varepsilon-\dfrac{1}{2\alpha}$. By Lemma 6 we have

$$\sum_{n>w/\sqrt{p}} \frac{G(n,p)}{n^2} = O\Big(\sum_{n>w/\sqrt{p}} n^{-(1+c/\log p)}\Big)$$
$$= O\left((\log y)^{\frac{2}{3}} \exp\left(-c(\log y)^{\frac{1}{3}}\right)\right)\,.$$

It follows that

$$H_p-Z_p=O\left(\frac{1}{\log^2 y}\right),\qquad Z_p>c\frac{\varphi(k)}{k}\log p\,,$$

since, by Lemma 1,

$$H_p=\prod_{\substack{q<p\\ (q,k)=1}} \left(1-\frac{1}{q}\right)^{-1} \geqq \frac{\varphi(k)}{k} e^C \log p+O(1)\,.$$

Case 2: $\exp(\log y)^{\frac{2}{3}}<p\leqq z_2$. We have

$$T_p=\sum_{\substack{n>w/\sqrt{p}\\ g(n)\leqq p}} \frac{1}{n}\sum_{d|(n,k)} \mu(d)$$
$$=\sum_{\substack{d|k\\ g(d)\leqq p}} \frac{\mu(d)}{d}\sum_{\substack{n>w/d\sqrt{p}\\ g(n)\leqq p}} \frac{1}{n}$$
$$=\sum_{\substack{d|k\\ g(d)\leqq p}} \frac{\mu(d)}{d}\sum_{\substack{n>w/\sqrt{p}\\ g(n)\leqq p}} \frac{1}{n}$$
$$+\sum_{\substack{d|k\\ g(d)\leqq p}} \frac{\mu(d)}{d}\sum_{\substack{w/\sqrt{p}\geqq n>w/d\sqrt{p}\\ g(n)\leqq p}} \frac{1}{n}$$

$$= \prod_{\substack{q|k \\ q \leqq p}} \left(1 - \frac{1}{q}\right) \sum_{\substack{n > w/\sqrt{p} \\ g(n) \leqq p}} \frac{1}{n} + O\left((\log\log 3k)^2\right),$$

since we have, by Lemma 4,

$$\sum_{\substack{d|k \\ g(d) \leqq p}} \frac{\mu(d)}{d} \sum_{\substack{w/\sqrt{p} \geqq n > w/d\sqrt{p} \\ g(n) \leqq p}} \frac{1}{n} = O\left(\sum_{d|k} \frac{\mu^2(d) \log d}{d}\right)$$

$$= O\left((\log\log 3k)^2\right).$$

Now, by partial summation, we have

$$\sum_{\substack{n > w/\sqrt{p} \\ g(n) \leqq p}} \frac{1}{n} = \sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O(y^{-c_5}).$$

Here, by Lemma 6, we find that

$$\sum_{n > \exp(\log y)^2} \frac{G(n, p)}{n^2} = O\left(\sum_{n > \exp(\log y)^2} n^{-(1+c/\log p)}\right)$$

$$= O\left(\log y \exp(-c \log y)\right)$$

$$= O\left((\log y)\, y^{-c}\right),$$

so that

$$\sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} = \sum_{\exp(\log y)^2 \geqq n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O\left(\frac{1}{\log^2 y}\right).$$

Let us write I for the interval $w/\sqrt{p} < n \leqq \exp(\log y)^2$. Then, by making use of the result in Lemma 6, we obtain

$$\sum_{n \in I} \frac{G(n, p)}{n^2} = \sum_{n \in I} \frac{1}{n} \rho\left(\frac{\log n}{\log p}\right) \left(1 + O\left(\frac{\log\log y}{\log p}\right)\right)$$

$$+ O\left(\sum_{n \in I} \frac{1}{n}\right) + O\left(\sum_{n \in I} \frac{1}{n} \left(\frac{\log n}{\log p}\right)^2 P(p)\right).$$

It is easily verified that

$$\sum_{n \in I} \frac{1}{n} \rho\left(\frac{\log n}{\log p}\right) = \log p \int_{t_p}^{\infty} \rho(t)\, dt + O(y^{-c_5}),$$

where $t_p = (\log w/\sqrt{p})/\log p$;

$$\sum_{n\in I}\frac{1}{n}=O(y^{-c_5})\,;\qquad \sum_{n\in I}\frac{1}{n}\left(\frac{\log n}{\log p}\right)^2P(p)=O\left(\frac{1}{\log^2 y}\right),$$

where we have taken $P(v)=c\exp(-c(\log v)^{\frac{1}{2}})$.

We thus have

$$T_p=\log p\int_{t_p}^{\infty}\rho(t)dt\left(1+O\left(\frac{\log\log y}{\log p}\right)\right)+O\left(\frac{1}{\log^2 y}\right).$$

Hence

$$H_p-Z_p\leqq\prod_{\substack{q\mid k\\ q\leqq p}}\left(1-\frac{1}{q}\right)\log p\int_{t_p}^{\infty}\rho(t)dt\left(1+O\left(\frac{\log\log y}{\log p}\right)\right)$$
$$+O\left((\log\log 3k)^2\right),$$

$$Z_p\geqq\prod_{\substack{q\mid k\\ q\leqq p}}\left(1-\frac{1}{q}\right)\left(e^C-\int_{t_p}^{\infty}\rho(t)dt\right)\log p+O(\log\log y)$$
$$+O\left((\log\log 3k)^2\right).$$

Case 3: $z_2<p\leqq z$. Put $t_p=(\log w/\sqrt{p})/\log p$, as before. If $0<t_p\leqq 1$ then we have

$$Z_p=\sum_{\substack{n\leqq w/\sqrt{p}\\ (n,k)=1}}\frac{\mu^2(n)}{\varphi(n)}$$
$$=\frac{\varphi(k)}{k}\log\frac{w}{\sqrt{p}}+O(\log\log 3k)$$
$$=\frac{\varphi(k)}{k}t_p\log p+O(\log\log 3k),$$

$$H_p-Z_p=\frac{\varphi(k)}{k}(e^C-t_p)\log p+O(\log\log 3k),$$

by Lemma 5. If $t_p>1$ then

$$Z_p\geqq\sum_{\substack{n\leqq w/\sqrt{p}\\ (n,k)=1}}\frac{\mu^2(n)}{\varphi(n)}-\sum_{\substack{p\leqq q\leqq w/\sqrt{p}\\ (q,k)=1}}\sum_{\substack{n\leqq w/\sqrt{p}\\ n\equiv 0\,(q)\\ (n,k)=1}}\frac{\mu^2(n)}{\varphi(n)}$$
$$=\sum_{\substack{n\leqq w/\sqrt{p}\\ (n,k)=1}}\frac{\mu^2(n)}{\varphi(n)}-\sum_{\substack{p\leqq q\leqq w/\sqrt{p}\\ (q,k)=1}}\frac{1}{\varphi(q)}\sum_{\substack{n\leqq w/q\sqrt{p}\\ (n,qk)=1}}\frac{\mu^2(n)}{\varphi(n)},$$

where, again by Lemma 5,

$$\sum_{\substack{p\leqq q\leqq w/\sqrt{p}\\(q,k)=1}}\frac{1}{\varphi(q)}\sum_{\substack{n\leqq w/q\sqrt{p}\\(n,qk)=1}}\frac{\mu^2(n)}{\varphi(n)}$$

$$=\sum_{\substack{p\leqq q\leqq w/\sqrt{p}\\(q,k)=1}}\frac{1}{\varphi(q)}\left(\frac{\varphi(qk)}{qk}\log\frac{w}{q\sqrt{p}}+O(\log\log 3qk)\right)$$

$$=\frac{\varphi(k)}{k}\sum_{\substack{p\leqq q\leqq w/\sqrt{p}\\(q,k)=1}}\frac{1}{q}\log\frac{w}{q\sqrt{p}}+O\left(\sum_{p\leqq q\leqq w/\sqrt{p}}\frac{\log\log 3qk}{\varphi(q)}\right)$$

$$=\frac{\varphi(k)}{k}\left(\log\frac{w}{\sqrt{p}}\log\frac{\log\frac{w}{\sqrt{p}}}{\log p}-\log\frac{w}{\sqrt{p}}+\log p\right)+O(\log\log y),$$

and hence

$$Z_p\geqq\frac{\varphi(k)}{k}(2t_p-1-t_p\log t_p)\log p+O(\log\log y).$$

Therefore

$$H_p-Z_p\leqq\frac{\varphi(k)}{k}\left(e^C-(2t_p-1-t_p\log t_p)\right)\log p+O(\log\log y).$$

Here we have, as in the proof of Lemma 2,

$$H_p=\prod_{\substack{q<p\\q\mid k}}\left(1-\frac{1}{q}\right)\prod_{q<p}\left(1-\frac{1}{q}\right)^{-1}$$

$$=\left(\frac{\varphi(k)}{k}+O\left(\frac{\log\log 3k}{p}\right)\right)\left(e^C\log p+O(1)\right)$$

$$=\frac{\varphi(k)}{k}e^C\log p+O(\log\log 3k).$$

We are now in position to be able to evaluate the sum

$$\sum_{\substack{p\leqq z\\(p,k)=1}}\frac{H_p-Z_p}{pH_pZ_p}.$$

Define:

$$A(t)=\begin{cases}\dfrac{e^C-t}{t} & (0<t\leqq 1),\\[2ex] \dfrac{e^C-(2t-1-t\log t)}{2t-1-t\log t} & (1<t<e^{c_6}),\end{cases}$$

where $t=e^{c_6}$ is the unique solution of

$$2t-1-t\log t=0\,, \qquad t>1\,,$$

so that $1.8<c_6<1.9$; and

$$B(t)=\frac{\int_t^\infty \rho(s)ds}{e^C-\int_t^\infty \rho(s)ds} \qquad \left(t>\frac{1}{4}\right).$$

Let us put, for the sake of brevity,

$$z_3=\exp(\log y)^{2/3}\,.$$

Then we have

$$\sum_{\substack{2\leqq p\leqq z_3\\(p,k)=1}}\frac{H_p-Z_p}{pH_pZ_p}=O\left(\frac{k}{\varphi(k)}\frac{\log\log 3k}{\log^2 y}\sum_p\frac{1}{p\log^2 p}\right)$$
$$=O\left(\frac{k}{\varphi(k)}\frac{\log\log 3k}{\log^2 y}\right),$$

and, noticing that

$$\prod_{\substack{q|k\\q\leqq p}}\left(1-\frac{1}{q}\right)^{-1}\leqq\frac{k}{\varphi(k)}$$

for every p,

$$\sum_{\substack{z_3<p\leqq z_2\\(p,k)=1}}\frac{H_p-Z_p}{pH_pZ_p}\leqq\frac{k}{\varphi(k)}e^{-C}\sum_{z_3<p\leqq z_2}\frac{B(t_p)}{p\log p}$$
$$+O\left(\frac{k}{\varphi(k)}\frac{(\log\log 3k)^3}{\log^{4/3}y}\right)+O\left(\frac{k}{\varphi(k)}\frac{\log\log y}{\log^{4/3}y}\right),$$

where we have used the inequality

$$\sum_{z_3<p\leqq z_2}\frac{1}{p\log^2 p}=O\left(\sum_{\pi(z_3)<n\leqq\pi(z_2)}\frac{1}{n\log^3 n}\right)=O\left(\frac{1}{\log^2 z_3}\right).$$

We now assume that γ, $4\leqq\gamma\leqq 10$, be integral. Write J_ν for the interval $y^{1/\nu+1}<p\leqq y^{1/\nu}(\nu\geqq\gamma)$. Then, since the function $B(t)$ is monotone decreasing,

$$\sum_{z_3<p\leqq z_2}\frac{B(t_p)}{p\log p}=\sum_{\gamma\leqq\nu<c(\log y)^{\frac{1}{3}}}\sum_{p\in J_\nu}\frac{B(t_p)}{p\log p}$$

$$\leqq \sum_{\gamma\leqq\nu<c(\log y)^{\frac{1}{3}}} \left(\sum_{p\in J_\nu}\frac{1}{p}\right)\max_{p\in J_\nu}\frac{B(t_p)}{\log p}$$

$$\leqq \sum_{\gamma\leqq\nu<c(\log y)^{\frac{1}{3}}} \log\frac{\nu+1}{\nu}\,\frac{\nu+1}{\log y}\,B\left(\left(\frac{1}{2}-\varepsilon\right)\nu-\frac{1}{2}\right)$$

$$+\,O\left(\sum_{\gamma\leqq\nu<c(\log y)^{\frac{1}{3}}}\frac{\nu}{\log^2 y}\right)$$

$$\leqq \frac{1}{\log y}\sum_{\nu=\gamma}^{\infty}(\nu+1)\log\frac{\nu+1}{\nu}B\left(\left(\frac{1}{2}-\varepsilon\right)\nu-\frac{1}{2}\right)+O\left(\frac{1}{\log^{4/3} y}\right).$$

Thus we obtain

$$\sum_{\substack{z_3<p\leqq z_2\\(p,\,k)=1}}\frac{H_p-Z_p}{pH_pZ_p}\leqq\frac{k}{\varphi(k)}\,\frac{e^{-C}}{\log y}\sum_{\nu=\gamma}^{\infty}(\nu+1)\log\frac{\nu+1}{\nu}B\left(\left(\frac{1}{2}-\varepsilon\right)\nu-\frac{1}{2}\right)$$

$$+O\left(\frac{k}{\varphi(k)}\,\frac{(\log\log 3k)^3}{\log^{4/3} y}\right)+O\left(\frac{k}{\varphi(k)}\,\frac{\log\log y}{\log^{4/3} y}\right).$$

We have similarly

$$\sum_{\substack{z_2<p\leqq z\\(p,k)=1}}\frac{H_p-Z_p}{pH_pZ_p}\leqq\frac{k}{\varphi(k)}e^{-C}\sum_{z_2<p\leqq z}\frac{A(t_p)}{p\log p}$$

$$+O\left(\frac{k}{\varphi(k)}\,\frac{(\log\log 3k)\log\log y}{\log^2 y}\right).$$

Put

$$n=[\log^{1/2} y]\,,\qquad u_j=\alpha+\frac{\gamma-\alpha}{n}j\qquad\qquad (j\geqq 0)\,,$$

and write K_j for the interval

$$y^{1/u_{j+1}}<p\leqq y^{1/u_j}\qquad\qquad (0\leqq j\leqq n-1)\,.$$

Now, the function $A(t)$ is continuous, monotone decreasing in the interval $0<t\leqq e$ and monotone increasing in the interval $e<t<e^{c_6}$. Thus, if we denote by m the integer for which

$$\left(\frac{1}{2}-\varepsilon\right)u_m-\frac{1}{2}\leqq e<\left(\frac{1}{2}-\varepsilon\right)u_{m+1}-\frac{1}{2}\,,$$

then

$$\sum_{z_2<p\leqq z}\frac{A(t_p)}{p\log p}=\sum_{j=0}^{n-1}\sum_{p\in K_j}\frac{A(t_p)}{p\log p}$$

$$\leqq\sum_{j=0}^{n-1}\left(\sum_{p\in K_j}\frac{1}{p}\right)\max_{p\in K_j}\frac{A(t_p)}{\log p}$$

$$\leqq\sum_{j=0}^{m-1}\log\frac{u_{j+1}}{u_j}\frac{u_{j+1}}{\log y}A\left(\left(\frac{1}{2}-\varepsilon\right)u_j-\frac{1}{2}\right)$$

$$+\log\frac{u_{m+1}}{u_m}\frac{u_{m+1}}{\log y}\max\left(A\left(\left(\frac{1}{2}-\varepsilon\right)u_m-\frac{1}{2}\right),\quad A\left(\left(\frac{1}{2}-\varepsilon\right)u_{m+1}-\frac{1}{2}\right)\right)$$

$$+\sum_{j=m+1}^{n-1}\log\frac{u_{j+1}}{u_j}\frac{u_{j+1}}{\log y}A\left(\left(\frac{1}{2}-\varepsilon\right)u_{j+1}-\frac{1}{2}\right)+O\left(\sum_{j=0}^{n-1}\frac{u_j}{\log^2 y}\right)$$

$$=\frac{1}{\log y}\int_\alpha^\gamma A\left(\left(\frac{1}{2}-\varepsilon\right)u-\frac{1}{2}\right)du+O\left(\frac{1}{\log^{3/2}y}\right),$$

where it should be noticed that we have uniformly

$$u_{j+1}\log\frac{u_{j+1}}{u_j}=\frac{\gamma-\alpha}{n}+O\left(\frac{1}{n^2}\right)\qquad(0\leqq j\leqq n-1).$$

Hence

$$\sum_{\substack{z_2<p\leqq z\\(p,k)=1}}\frac{H_p-Z_p}{pH_pZ_p}\leqq\frac{k}{\varphi(k)}\frac{e^{-C}}{\log y}\int_\alpha^\gamma A\left(\left(\frac{1}{2}-\varepsilon\right)u-\frac{1}{2}\right)du$$

$$+O\left(\frac{k}{\varphi(k)}\frac{(\log\log y)\log\log 3k}{\log^2 y}\right)+O\left(\frac{k}{\varphi(k)}\frac{1}{\log^{3/2}y}\right).$$

Collecting these results, we thus obtain, via Lemma 2, the following

Lemma 8. *We have*

$$S_1\geqq\frac{kN}{\varphi(k)}\frac{e^{-C}}{\log y}\left(\alpha-\int_\alpha^\gamma A\left(\left(\frac{1}{2}-\varepsilon\right)u-\frac{1}{2}\right)du\right.$$

$$\left.-\sum_{\nu=\gamma}^{\infty}(\nu+1)\log\frac{\nu+1}{\nu}B\left(\left(\frac{1}{2}-\varepsilon\right)\nu-\frac{1}{2}\right)\right)$$

$$+O\left(\frac{kN}{\varphi(k)}\frac{(\log\log 3k)^3}{\log^{4/3}y}\right)+O\left(\frac{kN}{\varphi(k)}\frac{\log\log y}{\log^{4/3}y}\right)$$

$$+O\left(\frac{N(\log\log 3k)^3}{y^{1/4}\log y}\right)+O\left(y^{1-2\varepsilon}(\log\log 3k)^2\right).$$

5. Evaluation of S_2. By virtue of Lemma 3, our present task is only to estimate the quantity

$$\sum_{\substack{z<q\leqq z_1\\(q,k)=1}}\frac{1}{qW_q}.$$

We set

$$C(t)=\begin{cases}\dfrac{\alpha}{a} & (0<a\leqq 1),\\[2ex] \dfrac{\alpha}{2a-1-a\log a} & (1<a\leqq 2),\end{cases}$$

where

$$a=\frac{\alpha}{2}\left(1-2\varepsilon-\frac{1}{t}\right).$$

Then, it is not difficult to verify, by Lemma 5, that, with $t=t_q=(\log y)/\log q$

$$W_q=\sum_{\substack{n\leqq z^\alpha\\ g(n)\leqq z\\(n,k)=1}}\frac{\mu^2(n)}{\varphi(n)}$$

$$\geqq\frac{\varphi(k)}{k}\frac{\log y}{C(t_q)}+O(\log\log 3k)\qquad (z<q\leqq z_1),$$

and consequently

$$\sum_{\substack{z<q\leqq z_1\\(q,k)=1}}\frac{1}{qW_q}\leqq\frac{k}{\varphi(k)}\frac{1}{\log y}\sum_{z<q\leqq z_1}\frac{C(t_q)}{q}+O\left(\frac{k}{\varphi(k)}\frac{(\log\log 3k)^2}{\log^2 y}\right).$$

Put

$$n=[\log^{1/2}y],\qquad u_j=\beta+\frac{\alpha-\beta}{n}j\qquad (0\leqq j\leqq n),$$

and write L_j for the interval

$$y^{1/u_{j+1}}<q\leqq y^{1/u_j}\qquad (0\leqq j\leqq n-1).$$

Then we have

$$\sum_{z<q\leqq z_1}\frac{C(t_q)}{q}=\sum_{j=0}^{n-1}\sum_{q\in L_j}\frac{C(t_q)}{q}\leqq\sum_{j=0}^{n-1}\left(\sum_{q\in L_j}\frac{1}{q}\right)\max_{q\in L_j}C(t_q)$$

$$\leqq\sum_{j=0}^{n-1}\left(\log\frac{u_{j+1}}{u_j}+O\left(\frac{1}{\log y}\right)\right)C(u_j)=\int_\beta^\alpha C(u)\frac{du}{u}+O\left(\frac{1}{\log^{1/2}y}\right),$$

since the function $C(t)$ is continuous and decreases monotonously for $t>(1-2\varepsilon)^{-1}$ and since we have uniformly

$$\log\frac{u_{j+1}}{u_j}=\frac{\alpha-\beta}{n}\frac{1}{u_j}+O\left(\frac{1}{n^2}\right) \qquad (0\leqq j\leqq n-1).$$

We thus have proved the following

Lemma 9. *We have*

$$S_2\leqq\frac{1}{r+1}\frac{kN}{\varphi(k)}\frac{1}{\log y}\int_\beta^\alpha\frac{C(u)}{u}du$$
$$+O\left(\frac{kN}{\varphi(k)}\frac{(\log\log 3k)^2}{\log^2 y}\right)+O\left(\frac{kN}{\varphi(k)}\frac{1}{\log^{3/2} y}\right)+O\left(\frac{y^{1-2\varepsilon}(\log\log 3k)^2}{\log^2 y}\right).$$

6. Numerical computations. We need the following easy lemma, a part of which has already been used in the proof of Lemmas 8 and 9.

Lemma 10. *The function*

$$f(s)=\frac{1}{2s-1-s\log s} \qquad (1<s<e^{c_6})$$

is positiv, convex, and monotone decreasing for $1<s\leqq e$ *and monotone increasing for* $e<s<e^{c_6}$.

Putting $f_1(s)=(f(s))^{-1}$, we see that $f_1(s)>0$, $f_1'(s)=1-\log s$ and $f_1''(s)=-1/s$, and the result follows at once.

We now choose $\varepsilon=\dfrac{1}{7}$ and take

$$\alpha=4,\quad \beta=2,\quad \text{and}\quad \gamma=10.$$

Our aim in this section is to compute numerically two integrals and a sum appearing in Lemmas 8 and 9.

(i) Computation of

$$\int_\alpha^\gamma A\left(\left(\frac{1}{2}-\varepsilon\right)u-\frac{1}{2}\right)du=\int_4^{10}A\left(\frac{5}{14}u-\frac{1}{2}\right)du.$$

The integral is equal to

$$(3)\qquad \int_4^{4.2}A\left(\frac{5}{14}u-\frac{1}{2}\right)du+\int_{4.2}^{10}A\left(\frac{5}{14}u-\frac{1}{2}\right)du,$$

where the first integral is found to be

$$= e^C \int_4^{4.2} \left(\frac{5}{14} u - \frac{1}{2} \right)^{-1} du - \int_4^{4.2} du$$

$$= e^C \frac{14}{5} \log \frac{14}{13} - 0.2 < 0.1696 ,$$

while the second is

$$= e^C \int_{4.2}^{10} F(u)\, du - \int_{4.2}^{10} du$$

with $F(u)=f(s(u))$, where $f(s)$ is the function defined in Lemma 10 and $s(u)=\frac{5}{14}u-\frac{1}{2}$. To estimate the integral of $F(u)$ over (4.2, 10) we proceed as follows.

We find:

$$\begin{aligned} &F(4.2) = 1.0000\,; && F(4.5) < 0.9080\,; \\ &F(5) < 0.8011\,; && F(6) < 0.6803\,; \\ &F(7) < 0.6197\,; && F(8) < 0.5907\,; \\ &F(9) < 0.5820\,; && F(10) < 0.5896\,. \end{aligned}$$

By Lemma 10, the function $F(u)$ is convex for $4.2 \leqq u \leqq 10$. Hence

$$\int_{4.2}^{10} F(u) du \leqq \frac{3}{20} \big(F(4.2)+F(4.5)\big) + \frac{1}{4} \big(F(4.5)+F(5)\big)$$

$$+ \frac{1}{2} \big(F(5)+F(10)\big) + \big(F(6)+F(7)+F(8)+F(9)\big)$$

$$< 3.8817 ,$$

and the second integral in (3) is less than

$$3.8817\, e^C - 5.8 < 1.1137 .$$

Thus we have

$$\int_4^{10} A\left(\frac{5}{14}u-\frac{1}{2}\right) du < 0.1696 + 1.1137 = 1.2833 .$$

(ii) Computation of

$$\sum_{\nu=r}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B\left(\left(\frac{1}{2}-\varepsilon\right)\nu-\frac{1}{2}\right)$$

$$= \sum_{\nu=10}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B\left(\frac{5}{14}\nu-\frac{1}{2}\right).$$

By the definition (2), the function $\rho(s)$ is positive and monotone decreasing for $s>0$, and moreover

$$\rho(s)=1-\log s \qquad \text{for} \quad 1\leqq s\leqq 2 .$$

Put $s(\nu)=\frac{5}{14}\nu-\frac{1}{2}$. Then we have $s(10)=\frac{43}{14}>3$ and

$$\rho\left(s(10)\right)\leqq\rho(3)\leqq\rho(2)e^{-1}=(1-\log 2)e^{-1}<0.1129 ,$$

by Lemma 7. Now, using Lemma 7 again, we find that for $\nu\geqq 10$

$$B\left(s(\nu)\right)\leqq\frac{\rho(s(\nu))}{e^{C}-\rho(s(\nu))}\leqq\frac{\rho(s(10))}{e^{C}-\rho(s(10))}e^{-\frac{5}{14}(\nu-10)} .$$

Since $(\nu+1)\log((\nu+1)/\nu)$ decreases monotonously as $\nu\to\infty$, we thus obtain

$$\sum_{\nu=10}^{\infty}(\nu+1)\log\frac{\nu+1}{\nu}B\left(s(\nu)\right)$$

$$\leqq 11\log\frac{11}{10}\frac{\rho(s(10))}{e^{C}-\rho(s(10))}\frac{1}{1-e^{-5/14}}<0.2366 .$$

(iii) Computation of

$$\int_{\beta}^{\alpha}\frac{C(u)}{u}du=\int_{2}^{4}\frac{C(u)}{u}du .$$

For $2\leqq u\leqq 4$ we have

$$\frac{3}{7}\leqq a=2\left(\frac{5}{7}-\frac{1}{u}\right)\leqq\frac{13}{14} .$$

Hence

$$\int_{2}^{4}\frac{C(u)}{u}du=2\int_{2}^{4}\left(\frac{5}{7}u-1\right)^{-1}du$$

$$=\frac{14}{5}\log\frac{13}{3}<4.1058 .$$

7. Proof of the theorem. Let $1\leqq k<x$, $0\leqq l\leqq k-1$, $(k,l)=1$. Take

$$M=\left[\frac{x-l}{k}\right],\qquad N=\left[\frac{x}{k}\right],$$

and put

$$y=2k(N+1)\,,\quad z=y^{1/4}\,,\quad z_1=y^{1/2}\,,\quad w=y^{5/14}\,.$$

Then it is clear that $y>2x$ and that $M<n\leqq M+N$ implies $x<kn+l\leqq 2x$.

By $D(x;k,l)$ we denote the number of those integers of the form $kn+l$ $(M<n\leqq M+N)$ which are divisible by no primes $p\leqq z$, by at most two primes q in $z<q\leqq z_1$, and by no integers of the form q^2, q being a prime in $z<q\leqq z_1$: clearly such an integer $kn+l$ $(M<n\leqq M+N)$, if it exists, has at most two prime factors, i.e. $V(kn+l)\leqq 2$.

In order to estimate $D(x;k,l)$ from below, we apply Lemma 8 and Lemma 9 with $r=2$. Let us note that we have from the data in §6

$$e^{-C}\left(4-\int_4^{10}A\left(s(u)\right)du-\sum_{\nu=10}^{\infty}(\nu+1)\log\frac{\nu+1}{\nu}B\left(s(\nu)\right)\right)$$
$$>e^{-C}(4-1.2833-0.2366)>1.3923$$

and

$$\frac{1}{3}\int_2^4\frac{C(u)}{u}du<\frac{4.1058}{3}=1.3686\,.$$

Now, the number R_2 of those integers $kn+l$ $(M<n\leqq M+N)$ which are not divisible by any prime $p\leqq z$ and are divisible by some integer q^2 with q in $z<q\leqq z_1$ does not exceed

$$\sum_{z<q\leq z_1}\left(\frac{N}{q^2}+1\right)=O\left(\frac{N}{z}\right)+O(z_1)\,.$$

We find, therefore, that

$$D(x;k,l)\geqq S_1-S_2-R_2\geqq(1.3923-1.3686)\frac{kN}{\varphi(k)}\frac{1}{\log y}$$
$$+O\left(\frac{kN}{\varphi(k)}\frac{(\log\log 3k)^3}{\log^{4/3}y}\right)+O\left(\frac{kN}{\varphi(k)}\frac{\log\log y}{\log^{4/3}y}\right)$$
$$+O\left(\frac{N(\log\log 3k)^3}{y^{1/4}\log y}\right)+O\left(y^{5/7}(\log\log 3k)^2\right)+O\left(\frac{N}{y^{1/4}}\right)\,.$$

Since $N=\dfrac{x}{k}+O(1)$, $2x<y\leqq 4x$, it follows that

$$D(x;k,l)\geqq 0.0237\frac{1}{\varphi(k)}\frac{x}{\log x}$$
$$+O\left(\frac{1}{\varphi(k)}\frac{x(\log\log 3k)^3}{\log^{4/3}x}\right)+O\left(\frac{1}{\varphi(k)}\frac{x\log\log x}{\log^{4/3}x}\right)$$

$$+O\left(\frac{1}{k}\frac{x^{3/4}(\log\log 3k)^3}{\log x}\right)+O\left(x^{5/7}(\log\log 3k)^2\right)+O\left(\frac{1}{k}x^{3/4}\right).$$

Let $c_7>3.5$ be a fixed number. If $x\geqq k^{c_7}$ and k is sufficiently large, then all the error terms on the right-hand side of the above inequality for $D(x;k,l)$ are of negligible order of magnitude, with respect to the leading term. Thus, for all large enough k, $x\geqq k^{c_7}$ implies that

$$D(x;k,l)>0.0236\frac{1}{\varphi(k)}\frac{x}{\log x}>1.$$

Hence, by continuity argument, we conclude that there is a (finite) natural number k_0 such that, if $k\geqq k_0$ and $x\geqq k^{3.5}$ then we have $D(x;k,l)>0$. Therefore there exists an absolute constant $c_1>0$ such that

$$D(x;k,l)>0 \qquad \text{for all} \qquad x\geqq c_1k^{3.5}, \quad k\geqq 1.$$

This completes the proof of our theorem.

Appendix

ON THE 'UPPER' SIEVE OF A. SELBERG

Here we aim at generalizing the results obtained in [6].

Let $N>1$ and let $a_1, a_2, \cdots, a_N$ be rational integers not necessarily different from each other. Let S denote the number of those integers a_j $(1\leqq j\leqq N)$ which are not divisible by any prime number $p\leqq z$, where $z\geqq 2$. Suppose that for every positive integer d

$$S_d\overset{\text{def}}{=}\sum_{\substack{n\leqq N\\ a_n\equiv 0\ (d)}}1=\frac{\omega(d)}{d}N+R(d),$$

where $R(d)$ is the error term for S_d and $\omega(d)$ is a multiplicative function of d. We put

$$f(d)=\frac{d}{\omega(d)}$$

and suppose that $f(d)>1$ for all $d>1$.

Let w be an arbitrary but fixed real number such that $w\geqq 2$. We define for positive integers m and d

$$f_1(m)=\sum_{n\mid m}\mu(n)f\left(\frac{m}{n}\right),$$

$$W(d)=\sum_{\substack{r\leq w/d\\(r,\,d)=1}}\varepsilon_z(r)\frac{\mu^2(r)}{f_1(r)}\,,\qquad W=W(1)\,,$$

$$\lambda(d)=\varepsilon_z(d)\,\mu(d)\prod_{p|d}\left(1-\frac{1}{f(p)}\right)^{-1}\cdot\frac{W(d)}{W}\,,$$

where $\varepsilon_z(n)=0$ or 1 according as n has or has not a prime factor $>z$. Then we have, since $\lambda(1)=1$,

$$S\leqq\sum_{n\leq N}\left(\sum_{\substack{d\leqq w\\d|a_n}}\lambda(d)\right)^2=\sum_{d\leqq w^2}\left(\sum_{\substack{d_1,d_2\leqq w\\\{d_1,d_2\}=d}}\lambda(d_1)\,\lambda(d_2)\right)\frac{N}{f(d)}$$
$$+\sum_{d_1,d_2\leqq w}|\lambda(d_1)\,\lambda(d_2)\,R(\{d_1\,d_2\})|\,,$$

where $\{d_1, d_2\}$ denotes the least common multiple of d_1 and d_2.

Now

$$\sum_{d\leqq w^2}\left(\sum_{\substack{d_1,d_2\leqq w\\\{d_1,d_2\}=d}}\lambda(d_1)\,\lambda(d_2)\right)\frac{1}{f(d)}$$
$$=\sum_{r\leqq w}f_1(r)\left(\sum_{\substack{d\leqq w\\d\equiv 0\,(r)}}\frac{\lambda(d)}{f(d)}\right)^2$$
$$=\frac{1}{W^2}\sum_{r\leqq w}f_1(r)\left(\sum_{\substack{d\leqq w\\d\equiv 0\,(r)}}\varepsilon_z(d)\,\mu(d)\frac{1}{f_1(d)}\sum_{\substack{m\leqq w/d\\(m,d)=1}}\varepsilon_z(m)\frac{\mu^2(m)}{f_1(m)}\right)^2$$
$$=\frac{1}{W^2}\sum_{r\leqq w}f_1(r)\left(\varepsilon_z(r)\frac{\mu(r)}{f_1(r)}\sum_{\substack{n\leqq w/r\\(n,\,r)=1}}\varepsilon_z(n)\frac{\mu^2(n)}{f_1(n)}\sum_{d|n}\mu(d)\right)^2$$
$$=\frac{1}{W^2}\sum_{r\leqq w}\varepsilon_z(r)\frac{\mu^2(r)}{f_1(r)}=\frac{1}{W}\,.$$

We thus have proved the following

Theorem. *Under the notations and conditions described above we have*

$$S\leqq\frac{N}{W}+R$$

with

$$R=\sum_{d_1,d_2\leqq w}|\lambda(d_1)\,\lambda(d_2)\,R(\{d_1,\,d_2\})|\,.$$

This is a generalization of [3: II, Theorem 3.1].

To evaluate the remainder term R explicitly, let us suppose that for all positive integers d, d_1, d_2

$$|R(d)| \leqq B\omega(d), \qquad \omega(\{d_1, d_2\}) \leqq \omega(d_1)\,\omega(d_2),$$

where $B>0$ is a constant independent of d. These conditions imply

$$R \leqq B\Big(\sum_{d \leqq w} \lambda(d)\,\omega(d)\Big)^2.$$

Then, it is not difficult to show that we have, in general,

$$R = O\left(w^2 (\log\log w)^2\right),$$

and, in the special case where $\omega(p) \leqq 1$ for all primes $p \leqq z$,

$$R = O\left(\frac{w^2}{W^2}\right),$$

where the constants implied in the symbol O depend only on the constant B.

The proof of these estimates of the remainder term R can easily be carried out just in the same way as in [6], and we shall omit the details (cf. also [7]).

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