ON QUASI-INJECTIVE MODULES

A Generalization of the Theory of Completely Reducible Modules

By

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Contents

§0. Introduction. The main purpose of this paper is to extend some results on completely reducible modules to quasi-injective modules by replacing "irreducible submodules" with "uniform submodules". To this end, a number of concepts and results on quotient rings (which are given by Johnson, Utumi, Goldie, Lambek) will be needed. Let R be a non-zero ring with 1. A unital R-left module V is called R-uniform if every pair of its non-zero R-submodules has a non-zero intersection (Goldie [3]). Let $\{V_\lambda;\lambda\in\Lambda\}$, $\{W_\gamma;\gamma\in\Gamma\}$ be maximal independent sets of R-uniform submodules of a unital R-left module M. Then there exists a 1–1 mapping f of Λ onto Γ such that $V_\lambda\sim W_{f(\lambda)}$ for all λ, where $V_\lambda\sim W_{f(\lambda)}$ means that a non-zero R-submodule of V_λ is R-isomorphic to an R-submodule of $W_{f(\lambda)}$ (Th. 1.10.) This result generalizes the one on the rank of abelian groups as well as the one on completely reducible modules. M is called R-quasi-injective if every R-homomorphism of any R-submodule of M into M can be extended to an R-endomorphism of M (Johnson and Wong [6]). If R is a left Noetherian ring (with 1) and M is R-quasi-injective, then M is a direct sum of R-quasi-injective uniform submodules, and such a representation of M is unique up to R-isomorphism (Th. 4.6). If R is a ring such that for any non-zero left ideal I, R/I contains a minimal R-left submodule, then an R-left module M is R-injective if and only if every R-homomorphism of any maximal left ideal into M can be extended to an R-homomorphism of R into M. This is a corollary to Th. 6.1. Th. 6.1 generalizes also a result on neat subgroups of abelian groups. M is called an R-c.q.i-module if M is R-quasi-injective, and for any non-zero R-submodule A, $l_M(r_K(A))$ is the unique maximal submodule such that $l_M(r_K(A))\supseteq A$ and every non-zero R-submodule of $l_M(r_K(A))$ has a non-zero intersection with A (Cor. to Prop. 5.11), where $K=\mathrm{Hom}_R(M, M)$ acting on the right, $r_K(A)=\{\alpha\in K;\ A\alpha=0\}$ and $l_M(r_K(A))=\{u\in M;\ u\cdot r_K(A)=0\}$. Let M be a unital R-L-module, where L is a non-zero ring with 1. M is called an R-L-c.q.i-module if M is an $R\otimes_J L^0$-c.q.i-module, where L^0 is the opposite ring of L and J the ring of rational integers. If M is an R-c.q.i-module then there hold the following: (1) M is a Q_0-c.q.i.-module, where Q_0 is any intermediate ring of Q and R_0 of all (additive group) endomorphisms induced by R. (2) Every R-direct summand of M is an R-c.q.i-module. (3) Every R-K-submodule is an R-c.q.i-module and an R-K-c.q.i-module. (4) K is, as a K-left module, a K-c.q.i-module. And we can generalize some results on completely reducible modules in this situation. From these facts cited above, the center of any left injective ring with zero left singular ideal is also an injective ring with zero singular ideal.

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§ 1. Throughout the present paper, M will denote a unital R-left module $(\neq 0)$, where R is a non-zero ring with 1. $\mathfrak{M}$ and $\mathfrak{S}$ denote the set of all non-zero R-submodules of M and the set of all subsets of M properly containing $\{0\}$, respectively. For $S\in\mathfrak{S}$, we set $S^-=\{0\}\cup\{x\in M;\ Rx\cap S\neq 0\}$. And we set $0^-=0$. Then, to be easily seen, $S\subseteq T(S, T\in\mathfrak{S})$ implies $S^-\subseteq T^-$, $S\subseteq S^-=S^{--}$. And, $A\cap S=0$ $(A\in\mathfrak{M})$ implies $A\cap S^-=0$. Therefore, $S^-\subseteq S_1^-(S_1\in\mathfrak{S})$ is nothing but to say that $X\cap S\neq 0$ $(X\in\mathfrak{M})$ implies $X\cap S_1\neq 0$. $S\in\mathfrak{S}$ is said to be *dense* in $T\in\mathfrak{S}$, if $S\subseteq T\subseteq S^-$. If S is dense in T then $S^-=T^-$ obviously, so that if S and T are dense in T and U respectively, then so is S in U, where $U\in\mathfrak{S}$. And, to be easily seen, $A\in\mathfrak{M}$ is dense in $B\in\mathfrak{M}$ if and only if $A\subseteq B$ and $X\cap A\neq 0$ for all $X\in\mathfrak{M}$ with $X\subseteq B$. "submodule" and "homomorphism" without modifier mean always "R-submodule" and "R-homomorphism" respectively.

Proposition 1.1. *Let* $\{A_\lambda;\ \lambda\in\Lambda\}$ *and* $\{B_\lambda;\ \lambda\in\Lambda\}$ *be subsets of* $\mathfrak{M}$. *If* $A_\lambda^-=B_\lambda^-$ *for all* λ, *then* $\sum A_\lambda=\sum\oplus A_\lambda$ (*direct sum*) *if and only if* $\sum B_\lambda=\sum\oplus B_\lambda$.

Proof. It suffices to prove that if $A_i^-=B_i^-$ $(i=1,\cdots,n)$ and $A_1+\cdots+A_n=A_1\oplus\cdots\oplus A_n$ then $B_1+\cdots+B_n=B_1\oplus\cdots\oplus B_n$. In fact, $A_1\cap(A_2+\cdots+A_n)=0$ and $A_1^-=B_1^-$ yield $B_1\cap(A_2+\cdots+A_n)=0$, which means $A_2+\cdots+A_n+B_1=$

$A_2 \oplus \cdots \oplus A_n \oplus B_1$. Similarly we obtain $A_3 + \cdots + A_n + B_1 + B_2 = A_3 \oplus \cdots \oplus A_n \oplus B_1 \oplus B_2$, and eventually $B_1 + \cdots + B_n = B_1 \oplus \cdots \oplus B_n$.

Corollary. *Let $\{A_\lambda ; \lambda \in \Lambda\}$ and $\{B_\lambda ; \lambda \in \Lambda\}$ be subsets of $\mathfrak{M}$ such that $A_\lambda \subseteq B_\lambda$ and $\sum B_\lambda = \sum \oplus B_\lambda$. Then $\sum A_\lambda$ is dense in $\sum B_\lambda$ if and only if each A_λ is dense in B_λ.*

Proof. If $\sum A_\lambda$ is dense in $\sum B_\lambda$, then $0 \neq X \cap \sum A_\lambda = X \cap A_{\lambda_1}$ for all $X \in \mathfrak{M}$ with $X \subseteq B_{\lambda_1}$. Hence, each A_{λ_1} is dense in B_{λ_1}. Conversely, if $A_\lambda^- = B_\lambda^-$ for all λ, then $X \cap \sum A_\lambda = 0$ $(X \in \mathfrak{M})$ implies $X \cap \sum B_\lambda = 0$, by Prop. 1.1. Hence, $\sum A_\lambda$ is dense in $\sum B_\lambda$.

$V \in \mathfrak{M}$ is called *uniform* if every pair of non-zero submodules of V has a non-zero intersection (Goldie [3]).

Proposition 1.2. *Let V and W be uniform submodules of M. If $V \cap W \neq 0$, then $V^- = W^-$.*

Proof. As V is uniform, $X \cap (V \cap W) \neq 0$ for all $X \in \mathfrak{M}$ with $X \subseteq V$, so that $V \cap W$ is dense in V, and symmetrically in W. Hence, we obtain $V^- = (V \cap W)^- = W^-$.

The proof of the next proposition may be left to readers.

Proposition 1.3. *Let $\{A_\lambda ; \lambda \in \Lambda\}$ be a subset of $\mathfrak{M}$ such that $\sum A_\lambda = \sum \oplus A_\lambda$, and let A be a submodule of M such that $A \cap \sum A_\lambda \neq 0$. If $\{A_{\lambda_i} ; i = 1, \cdots, n\}$ is a minimal (finite) subset of $\{A_\lambda\}$ such that $A \cap \sum A_{\lambda_i} \neq 0$, then $A \cap \sum A_{\lambda_i}$ is isomorphically mapped into each A_{λ_i} by the projection to A_{λ_i}. In particular, if A is uniform, then $\{A_{\lambda_i}\}$ is uniquely determined by A.*

Proposition 1.4. *Let A_i, B_i $(i=1, 2)$ be in $\mathfrak{M}$. If $A_1^- \subseteq A_2^-$ and $B_1^- \subseteq B_2^-$, then $(A_1 \cap B_1)^- \subseteq (A_2 \cap B_2)^-$.*

Proof. If $X \cap (A_1 \cap B_1) \neq 0$ $(X \in \mathfrak{M})$, then $(X \cap A_1) \cap B_2 \neq 0$ by $B_1^- \subseteq B_2^-$, whence $X \cap (A_2 \cap B_2) = (X \cap B_2) \cap A_2 \neq 0$ by $A_1^- \subseteq A_2^-$.

Corollary. *Let $A, B \in \mathfrak{M}$. If $A^- = B^-$, then $A^- = (A \cap B)^- = B^-$. In particular, if A and B are dense submodules of M (i.e. dense in M), then so is $A \cap B$.*

Proposition 1.5. *Let M' be an R-left module, and φ an (R-) homomorphism of M' into M. If $S^- \subseteq T^-$ $(S, T \in \mathfrak{S})$, then $(S\varphi^{-1})^- \subseteq (T\varphi^{-1})^-$, where $S\varphi^{-1} = \{u \in M' ; u\varphi \in S\}$. In particular, if S is dense in M, then $S\varphi^{-1}$ is dence in M'.* (Johnson [5])

Proof. If $X' \cap T\varphi^{-1} = 0$ for some non-zero submodule X' of M', then $(X' \cap S\varphi^{-1}) \cap \mathrm{Ker}\,\varphi \subseteq X' \cap S\varphi^{-1} \cap T\varphi^{-1} = 0$. Hence, if $X' \cap S\varphi^{-1} \neq 0$, then $X'\varphi \cap S \neq 0$. However, as $S^- \subseteq T^-$, $X'\varphi \cap S \neq 0$ implies a contradiction $X'\varphi \cap T \neq 0$.

If a dense submodule of M is isomorphic to a dense submodule of an R-left module M', M is said to be *similar* to M', and denoted by $M \sim M'$. The similarity is an equivalence relation by Cor. to prop. 1.4. A subset $\{A_\lambda; \lambda \in \Lambda\}$ of $\mathfrak{M}$ is called homogeneous if $A_\lambda \sim A_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda$.

Proposition 1.6. *Let $\{A_\lambda\}$ be a maximal independent homogeneous set of uniform submodules of M, and let $A \in \mathfrak{M}$. In order that $A \cap \sum A_\lambda \neq 0$, it is necessary and sufficient that A contains a uniform submodule U such that $U \sim A_\lambda$.*

Proof. Since every non-zero submodule of A_λ is dense in A_λ, the necessity is a direct consequence of Prop. 1.3. And, the sufficiency follows from the maximality of $\{A_\lambda\}$.

Let A be a submodule of M. A *complement* A^c of A (in M) is a maximal submodule of M such that $A \cap A^c = 0$. And, a *double complement* A^{cc} of A is a complement of a complement of A such that $A^{cc} \supseteq A$. If $A \cap B = 0$ ($B \in \mathfrak{M}$), by Zorn's lemma we can take a complement A^c of A such that $A^c \supseteq B$. Evidently, $0^c = M$ and $M^c = 0$. If A is a complement of some submodule of M, A is called a *complemented submodule* (of M). To be easily seen, every direct summand is a complemented submodule. The many-to-many correspondence $A \rightarrow A^{cc}$ is called *the d.c-correspondence*, more precisely, *the R-d.c-correspondence* in M.

Proposition 1.7. *Let A be a submodule of M.*

(i) *A submodule X of M is a double complement of A if and only if X is a maximal submodule such that $A \subseteq X \subseteq A^-$. Accordingly, if C, D are arbitrary complement and double complement of A respectively, then C, D are complements of D, C respectively.*

(ii) *A is complemented if and only if A is a double complement of itself, that is, there exists no submodule X of M such that $A \subsetneqq X \subseteqq A^-$. Accordingly, if A is complemented, A^{cc} is unique and coincides with A.*

Proof. Evidently $0^{cc} = 0^- = 0$. If $A \neq 0$ is not dense in A^{cc}, then $A \oplus Y \subseteq A^{cc}$ for some $Y \in \mathfrak{M}$, and whence it follows $(A \oplus Y) \oplus A^c = A \oplus (Y \oplus A^c)$, where $A^{cc} = (A^c)^c$. This contradiction shows that A is dense in A^{cc}. And further, if $A^{cc} \subsetneqq W$ ($W \in \mathfrak{M}$) then $W \cap A^c \neq 0$, and so $W \not\subseteq A^-$. Conversely, let X be a maximal submodule such that $A \subseteq X \subseteq A^-$. Then $A^- \cap A^c = 0$ implies $X \cap A^c = 0$. We can take a double complement A^{cc} such that $A^{cc} \supseteq X$. Then, as $A^{cc} \subseteq A^-$, $A^{cc} = X$. Thus we have obtained the former assertion. Next, let $A = B^c$ ($B \in \mathfrak{M}$). We take a complement A^c of A such that $A^c \supseteq B$, and further we take a complement $(A^c)^c$ of A^c such that $(A^c)^c \supseteq A$. Then $A = (A^c)^c$, because $A^c \supseteq B$ and $A = B^c$.

Proposition 1.8. *If* $\{V_\lambda;\ \lambda\in\Lambda\}$ *and* $\{W_\gamma;\ \gamma\in\Gamma\}$ *are maximal independent homogeneous sets of uniform submodules of* M *such that* $V_\lambda\sim W_\gamma$, *then* $\sharp\Lambda=\sharp\Gamma$, *where* $\sharp\Lambda$ *denotes the cardinal number of* Λ.

Proof. We shall distinguish between two cases.

Case 1. $\sharp\Lambda<\infty$ or $\sharp\Gamma<\infty$. Without loss of generality, we may assume $\sharp\Gamma\leq\sharp\Lambda$. We set $\{W_\gamma\ ;\ \gamma\in\Gamma\}=\{W_1,\cdots,W_s\}$. Let $V'_{\lambda_1}=\sum_{\lambda\neq\lambda_1}V_\lambda$ for an arbitrary $\lambda_1\in\Lambda$. If $W'_i=V'_{\lambda_1}\cap W_i\neq 0$ for all i, then $W'^{-}_i=W^{-}_i$ by Prop 1.2. Since $V_{\lambda_1}\cap\sum_i\oplus W'_i=0$, Prop. 1.1 yields $V_{\lambda_1}\cap\sum\oplus W_i=0$, which contradicts the maximality of $\{W_i;\ i=1,\cdots,s\}$. Hence, for some W_i, say W_1, there holds $V'_{\lambda_1}\cap W_1=0$. We set here $V''_{\lambda_1}=V'_{\lambda_1}\oplus W_1$. Then, $V''_{\lambda_1}\cap V_{\lambda_1}\neq 0$ by the maximality of $\{V_\lambda\}$, and $\{V_\lambda;\ \lambda\neq\lambda_1\}\cup\{W_1\}$ is a maximal independent homogeneous set of uniform submodules of M. In fact, if $\{V_\lambda;\ \lambda\neq\lambda_1\}\cup\{W_1\}\cup\{U\}$ is an independent homogeneous set of uniform submodules then, as $(V''_{\lambda_1}\cap V_{\lambda_1})^{-}=V^{-}_{\lambda_1}$ by Prop. 1.2, $(V''_{\lambda_1}\cap V_{\lambda_1})+V'_{\lambda_1}+U=(V''_{\lambda_1}\cap V_{\lambda_1})\oplus V'_{\lambda_1}\oplus U$ yields $V_{\lambda_1}+V'_{\lambda_1}+U=V_{\lambda_1}\oplus V'_{\lambda_1}\oplus U=(\sum\oplus V_\lambda)\oplus U$, which contradicts the maximality of $\{V_\lambda\}$. Repeating the above argument, we obtain eventually $\sharp\Lambda=\sharp\Gamma$ $(=s)$.

Case 2. $\sharp\Lambda=\infty$ and $\sharp\Gamma=\infty$. By the maximality of $\{W_\gamma\}$ and Prop. 1.3, for each $V\in\{V_\lambda\}$ there corresponds the unique minimal (finite) subset $\{W_1,\cdots,W_n\}$ of $\{W_\gamma\}$ such that $V\cap\sum W_i\neq 0$. We shall prove that $\cup_V\{W_1,\cdots,W_n\}=\{W_\gamma\}$. To this end, let W be an arbitrary member of $\{W_\gamma\}$, and let $\{V_1,\cdots,V_m\}$ be the unique minimal (finite) subset of $\{V_\lambda\}$ such that $W\cap\sum V_i\neq 0$. And then, let $\{W_{i1},\cdots,W_{in_i}\}$ be the unique minimal subset of $\{W_\gamma\}$ such that $V'_i=V_i\cap\sum_j W_{ij}\neq 0$. Since each V_i is uniform, $V'^{-}_i=V^{-}_i$ by Prop. 1.2. Hence, there holds $W\cap\sum_{ij}W_{ij}\neq 0$, which means $W\in\{W_{ij}\}$. We have seen therefore that $\cup_V\{W_1,\cdots,W_n\}$ coincides with $\{W_\gamma\}$, whence it follows $\sharp\Gamma\leq\sharp\Lambda\cdot\aleph_0=\sharp\Lambda$. And, we have symmetrically $\sharp\Lambda\leq\sharp\Gamma$. Hence $\sharp\Lambda=\sharp\Gamma$.

The set of all uniform submodules of M can be classified with respect to the equivalence relation $\sim$. And, P will represent the set of all similar classes. The existense of maximal independent [homogeneous] set of uniform submodules is secured by Zorn's lemma.

Proposition 1.9. (i) *If* $\{V_\lambda;\ \lambda\in\Lambda\}$ *is a maximal independent set of uniform submodules of* M, *then for each* $\rho\in P$, $\{V_\lambda;\ \lambda\in\Lambda_\rho\}$ *is a maximal independent homogeneous subset of* ρ, *where* $\Lambda_\rho=\{\lambda\in\Lambda;\ V_\lambda\in\rho\}$.

(ii) *If for each* $\rho\in P$ *there corresponds a maximal independent homogeneous subset* $\{W_\gamma;\ \gamma\in\Gamma_\rho\}$ *of* ρ, *then* $\cup_\rho\{W_\gamma;\ \gamma\in\Gamma_\rho\}$ *is a maximal independent set of uniform submodules.*

Proof. (i). For any $U\in\rho, U\cap\sum V_\lambda\neq 0$, and further, $U\cap\sum_{\lambda\in\Lambda_\rho}V_\lambda\neq 0$ by

Prop. 1.3. Hence each $\{V_\lambda; \lambda\in\Lambda_\rho\}$ is a maximal independent subset of ρ. (ii). Under the same notations as in (i); $(\sum_{\gamma\in\Gamma_\rho}W_\gamma)^-=(\sum_{\lambda\in\Lambda_\rho}V_\lambda)^-$ by Prop. 1.6, so that $\sum_\rho(\sum_{\gamma\in\Gamma_\rho}W_\gamma)=\sum_\rho\oplus(\sum_{\gamma\in\Gamma_\rho}\oplus W_\gamma)$ by Prop. 1.1.

Combining Prop. 1.8 and Prop. 1.9 (i), we obtain the following fundamental theorem:

Theorem 1.10. *Let* $\{V_\lambda; \lambda\in\Lambda\}$[1] *and* $\{W_\gamma; \gamma\in\Gamma\}$ *be maximal independent sets of uniform* [*complemented uniform*] *submodules of* M. *Then, there exists a* 1–1 *mapping* f *from* Λ *onto* Γ *such that* $V_\lambda\sim W_{f(\lambda)}$ *for all* $\lambda\in\Lambda$.

This theorem extends the one on the rank of abelian groups (cf. Fuchs [1]) as well as the one on completely reducible modules, and will be treated again in §4. For any uniform submodule $U\in\mathfrak{M}$, we denote the class containing U by $\tilde{U}$.

Theorem 1.11. *Let* $\cup_\rho\{V_\lambda; \lambda\in\Lambda_\rho\}$ *be any maximal independent set of uniform submodules, where* $V_\lambda\in\rho$ $(\lambda\in\Lambda_\rho)$, *and let* P_0 *be any non-empty subset of* P.

(i) $(\sum_{\lambda\in\Lambda_{P_0}}V_\lambda)^-(\Lambda_{P_0}=\cup_{\rho\in P_0}\Lambda_\rho)$ *depends on* P_0 *only* (*and is independent of the choice of* $\{V_\lambda\}$).

(ii) $(\sum_{\lambda\in\Lambda_{P_0}}V_\lambda)^{c-}$ *depends on* P_0 *only* (*and is independent of the choice of* $\{V_\lambda\}$ *and complements*).

Proof. $\{V_\lambda; \lambda\in\Lambda_{P_0}\}$ is a maximal independent subset of $\cup_{\rho\in P_0}\rho$. For $X\in\mathfrak{M}$, $X\cap\sum'V_\lambda\neq 0$ if and only if X contains a uniform submodule U such that $\tilde{U}\in P_0$, where $\sum'V_\lambda=\sum_{\lambda\in\Lambda_{P_0}}V_\lambda$ (cf. the proof of Prop. 1.6). And, this is nothing but to say that $(\sum'V_\lambda)^-$ is independent of the choice of $\{V_\lambda\}$ and is uniquely determined by P_0. To prove (ii), we set $C_1=(\sum'V_\lambda)^c$, and take a complement C_2 of a sum of another maximal independent subset of $\cup_{\rho\in P_0}\rho$. If $C_1^-\neq C_2^-$, say $C_2^-\not\subseteq C_1^-$, then there is some $Y\in\mathfrak{M}$ such that $Y\subseteq C_2$ and $Y\cap C_1=0$. Then, since $Y\oplus C_1\supsetneqq C_1$, $(Y\oplus C_1)\cap\sum'V_\lambda\neq 0$. Therefore, by Prop. 1.3, there is a uniform submodule U such that $U\subseteq(Y\oplus C_1)\cap\sum'V_\lambda$ and $\tilde{U}\in P_0$. By the projection, $U(\subseteq Y\oplus C_1)$ is isomorphic to a submodule U' of Y. As $\tilde{U}\in P_0$ and $U\cong U'\subseteq Y\subseteq C_2$, we have $\tilde{U}'\in P_0$ and $U'\subseteq C_2$. But this contradicts that C_2 is a complement of a sum of a maximal independent subset of $\cup_{\rho\in P_0}\rho$.

By the validity of Th. 1.10, we can define dim M and ρ-dim M as $\sharp\Lambda$ and $\sharp\Lambda_\rho$ repectively, where $\{V_\lambda; \lambda\in\Lambda\}$ is an arbitrary maximal independent set of uniform submodules of M and $\Lambda_\rho=\{\lambda\in\Lambda; V_\lambda\in\rho\}$. Evidently, we have dim $M=\sum_\rho\rho$-dim M. For any $A\in\mathfrak{M}$, the set $\mathrm{P}(A)$ of the similar classes of uniform submodules of A may be regarded as a subset of P. And, for any $\rho\in P$, we

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1) For each V_λ, we take a double complement V_λ^{cc} of V_λ. Then, by Prop. 1.1. and Prop. 1.7, $\{V_\lambda^{cc}; \lambda\in\Lambda\}$ is also a (maximal) independent set of uniform submodules.

define $\rho(A)=\{X\in\rho\,;\,X\subseteq A\}$.

Proposition 1.12. *If* $\{A_\lambda;\lambda\in\Lambda\}$ *is an independent subset of* $\mathfrak{M}$, *then* $\dim\sum_\lambda A_\lambda=\sum_\lambda \dim A_\lambda$ *and* ρ-$\dim\sum_\lambda A_\lambda=\sum_\lambda\rho$-$\dim A_\lambda$ *for all* $\rho\in P$.

Proof. Let ρ be in P. For each A_λ, choose an arbitrary maximal independent subset $\{V_{\lambda\gamma};\gamma\in\Gamma_\lambda\}$ of $\rho(A_\lambda)$, and let B_λ be a complement of $V_\lambda=\sum_{\gamma\in\Gamma_\lambda}V_{\lambda\gamma}$ in A_λ. Then, each $B_\lambda\oplus V_\lambda$ being dense in A_λ, $\sum_\lambda(B_\lambda\oplus V_\lambda)$ in dense in $\sum A_\lambda$ by Cor. to Prop. 1.1. If U is in $\rho(\sum A_\lambda)$ then $U\cap(\sum_\lambda(B_\lambda\oplus V_\lambda))\neq 0$. so that $U\cap\sum V_\lambda\neq 0$ by Prop. 1.3, because each B_λ does not contain a submodule belonging to ρ. This proves evidently that $\cup_\lambda\{V_{\lambda\gamma}:\gamma\in\Gamma_\lambda\}$ is a maximal independent subset of $\rho(\sum A_\lambda)$. Hence, ρ-$\dim\sum_\lambda A_\lambda=\sum_\lambda\rho$-$\dim A_\lambda$. And then, as $\dim A=\sum_\rho\rho$-$\dim A$ for any $A\in\mathfrak{M}$, we obtain $\dim\sum_\lambda A_\lambda=\sum_\lambda\dim A_\lambda$.

Let A be a submobule of M, and A^c an arbitrary complement of A. Evidently, A and A^c may be regarded naturally as submodules of M/A^c and M/A respectively. Now, in this meaning, we have the following:

Proposition 1.13. *Let* A *be a submodule and* A^c *an arbitrary complement of* A. *If* A *is non-zero then* A *is dense in* M/A^c, *and if* A^c $(\neq 0)$ *is dense* M/A *then* A *is complemented. Consequently, if* $B,\ C\in M$ *have a common complement* E *then* $B\sim M/E\sim C$.

Proof. If A is non-zero, $A\cap X\neq 0$ for each $X\in\mathfrak{M}$ with $X\supsetneqq A^c$, and hence $(A\oplus A^c)/A^c\cap X/A^c=((A\cap X)\oplus A^c)/A^c\neq 0$. This implies that $(A\cong)$ $(A\oplus A^c)/A^c$ is dense in M/A^c. Next, if A^c is dense in M/A, then $(A^c\oplus A)/A\cap A^{cc}/A=((A^c\cap A^{cc})\oplus A)/A=0$ yields $A^{cc}=A$.

As any double complement of $A\in\mathfrak{M}$ is a complement of any complement of A (Prop. 1.7), by Prop. 1.13, complements of A are similar to each other. Thus $\dim A^c$ and ρ-$\dim A^c$ are uniquely determined by A, and we denote them by $\operatorname{codim} A$ and ρ-$\operatorname{codim} A$, respectively. Then, $\operatorname{codim} A=\sum_\rho\rho$-$\operatorname{codim} A$ is evident, and, as $A\oplus A^c$ is dense in M, $\dim A+\operatorname{codim} A=\dim M$ and ρ-$\dim A+\rho$-$\operatorname{codim} A=\rho$-$\dim M$ by Prop. 1.12.

Proposition 1.14. *Let* $V\in\mathfrak{M}$ *be uniform, and let* W *be a submodule containing* V. *Then* W *is uniform if and only if* V *is dense in* W.

Proof. If V is dense in W, then every non-zero submodule of W has a non-zero intersection with V, and hence W must be uniform. The "only if" part is evident.

Combining Prop. 1.14 with Prop. 1.7, we readily obtain

Corollary. *A complemented uniform submodule is a maximal uniform submodule* (*i.e. maximal as a uniform submodule*), *and conversely.*

§2. **Complemented submodules.** We shall begin this section with the following theorem (cf. [7], [8]).

Theorem 2.1. *Let N be a dense submodule of M. If C is a complemented submodule of M, then $C\cap N$ is a complemented submodule of N and C is a double complement of $C\cap N$ in M. And if Z is a complemented submodule of N then $Z^{cc}\cap N=Z$ for every double complement Z^{cc} of Z in M.*

Proof. Let $C\cap N\neq 0$ be dense in $X\in\mathfrak{M}$ with $X\subseteq N$. If $X\not\subseteq C$ then $(X+C)\cap C^c\neq 0$, and so $0\neq N\cap(X+C)\cap C^c=X\cap C^c$. Since $C\cap N$ is dense in X, we have a contradiction $(C\cap N)\cap(X\cap C^c)\neq 0$. Hence $X\subseteq C$, that is, $X=C\cap N$. This implies that $C\cap N$ is complemented in N (Prop. 1.7). Next, let Z be a non-zero complemented submodule of N. Then, as $Z^{cc}\supseteq Z^{cc}\cap N\supseteq Z$, Z is dense in $Z^{cc}\cap N$, and hence $Z^{cc}\cap N=Z$ by Prop. 1.7.

Let $A\subsetneqq B$ be submodules of M. If B/A is dense in M/A, then so is B in M. Because, if $B^c\neq 0$ then $(A\oplus B^c)/A\neq 0$ and $B/A\cap(A\oplus B^c)/A=(A\oplus(B\cap B^c))/A=0$, a contradiction. Next, if C is a complemented submodule of M containing A then C/A is a complemented submodule of M/A. For, if C/A is non-zero and dense in X/A for $X\in\mathfrak{M}$ containing C then the same argument as above yields that C is dense in X, and so $C=X$ by Prop. 1.7. These prove the half of the following:

Thorem 2.2. *Let C be a proper complemented submodule of M. Then, the set of all complemented submodules of M/C coincides with the set $\{C'/C; C'$ ranges over the complemented submodules of M containing $C\}$, and the set of all dense submodules of M/C coincides with the set $\{D/C; D$ ranges over the dense submodules of M containing $C\}$.*

Proof. Let D be a dense submodule of M such that $D\supsetneqq C$, and let X be any submodule of M properly containing C. Then, C being complemented, $X\supseteq C\oplus Y$ for some $Y\in\mathfrak{M}$ by Prop. 1.7. As D is dense in M, $D\cap Y\neq 0$ and so $D/C\cap X/C\supseteq D/C\cap(C\oplus Y)/C=(C\oplus(D\cap Y))/C\neq 0$, which implies that D/C is dense in M/C. Next, let B/C is complemented in M/C for $B\in\mathfrak{M}$ with $B\supsetneqq C$. Then, since B is dense in B^{cc} and C is complemented in B^{cc}, the preceding implies that B/C is dense in B^{cc}/C. On the other hand, B/C being complemented in M/C, $B/C=B^{cc}/C$, that is, $B=B^{cc}$.

Theorem 2.3. *If $A\supseteq B$ are submodules of M then for any A^{cc} there exists such a double complement B^{cc} that $A^{cc}\supseteq B^{cc}$.*

Proof. Let $\hat{M}$ be the injective envelope of M[2], and A' a double complement of A^{cc} in $\hat{M}$, which is evidently a double complement of A in $\hat{M}$, and

2) The R-injective envelope $\hat{M}$ of M is a (unital) injective R-module which contains M as a dense R-submodule.

$M\cap A'=A^{cc}$ by Th. 2.1. A' is then an (injective) direct summand of $\hat{\mathfrak{M}}$ (see Proof. 4.2). If B' is a double complement of B in A' then, A' being injective, B' is a direct summand of A', and therefore of $\hat{\mathfrak{M}}$. Since M is dense in $\hat{\mathfrak{M}}$, by Th. 2.1, $M\cap B'$ is a complemented submodule of M. Noting here that B is dense in $M\cap B'$, by Prop. 1.7 we see that $M\cap B'$ is a double complement of B (in M) requested.

Corollary 1. *If for any submodule X of M its double complement is uniquely determined, then the d.c-correspondence is a closure operation.*

Corollary 2. *For $V\in\mathfrak{M}$, the following conditions are equivalent*:

(i) *V is a minimal complemented (i.e. minimal as a complemented submodule $\neq 0$) submodule.* (ii) *V is a maximal uniform submodule.* (iii) *V is a complemented uniform submodule.*

Proof. By Cor. to Prop. 1.14, a maximal uniform submodule is nothing but a complemented uniform submodule. And, a complemented uniform submodule is evidently a minimal complemented submodule. Conversely, let V be a minimal complemented submodule, and let $V\supseteq A$ $(A\in\mathfrak{M})$. Since a double complement A^{cc} of A in M is contained in V by Th. 2,3, the minimality of V yields $V=A^{cc}$, and hence every non-zero submodule of V is dense in V. And this is nothing but to say that V is uniform.

A submodule A of M is said to be *meet irreducible* (in M) if A can not be represented as an intersection of two submodules of properly containing A. Evidently, for a proper submodule A of M, M/A is uniform if and only if A is meet irreducible, in particular, M is uniform if and only if $\{0\}$ is meet irreducible in M.

Proposition 2.4. *If a submodule B is properly contained in a non-dense submodule A of M, then B is meet reducible (i.e. not meet irreducible). Consequently, a non-dense meet irreducible submodule is a minimal meet irreducible submodule.*

Proof. In fact, $B=A\cap(B\oplus A^c)$ and $B\oplus A^c\supsetneqq B$.

Proposition 2.5. *Let A be a proper submodule of M. Then the following conditions are equivalent*:

(i) *A is a maximal complemented submodule.*

(ii) *A is a complemented submodule, and A^c is uniform.*

(iii) *A is a non-dense meet irreducible submodule.*

Proof. (i)$\Rightarrow$(ii). If A^c is not uniform, then there are non-zero B, C such that $A^c\supseteq B\oplus C$. Let B^c be a complement of B with $B^c\supseteq A\oplus C$. Then $M\supsetneqq B^c\supsetneqq A$, and this contradicts the maximality of A. (ii)$\Rightarrow$(iii). Since a uniform module A^c may be regarded as a dense submodule of M/A, M/A is uniform by Prop. 1. 14. (iii)$\Rightarrow$(i). Since $A=A^{cc}\cap(A\oplus A^c)$ and $A\subsetneqq A\oplus A^c$, we have $A=A^{cc}$. Assume that there exists a complemented submodule B such that $A\subsetneqq B\subsetneqq M$. Then, since $A\oplus X\subseteq B$ for some $X\in\mathfrak{M}$ and $B^c\neq 0$, we have a contradiction $(A\oplus X)\cap(A\oplus B^c)(=A+((A\oplus X)\cap B^c))=A$. Hence, A is a maximal complemented submodule.

Theorem 2. 6. (i) *Let $C\supseteq C_0$ be submodules of M. If C and C_0 are complemented in M and C repectively, then so is C_0 in M.*

(ii) *If $C\supsetneqq C_0$ are complemented submodules, then for each complement C_0^c of C_0 there exists a complement C^c of C such that $C^c\subsetneqq C_0^c$.*

Proof. (i) is an immediate consequence of Th. 2.3 and Prop. 1.7. Now, let X be a complement of $C\cap C_0^c$ $(\neq 0)$ in C_0^c. As $C\cap X\subseteq C\cap C_0^c$ and $(C\cap C_0^c)\cap X=0$, $C\cap X=0$. Since $(C\cap C_0^c)\oplus X$ is dense in C_0^c, $C_0\oplus((C\cap C_0^c)\oplus X)$ is dense in M (Cor. to Prop. 1. 1), and therefore $C\oplus X$ is dense in M. If we take a complement C^c with $C^c\supseteq X$, X is dense in C^c. Hence, as X is complemented in M by (i), we have $C^c=X(\subseteq C_0^c)$. And, as $C^-\supsetneqq C_0^-$, $C^c\subsetneqq C_0^c$ (Prop. 1.7 (i)).

M is said to be *locally uniform* if every non-zero submodule of M contains a uniform submodule. And M is said to be *finite-dimensional* if every independent subset of $\mathfrak{M}$ is finite (Goldie [2]). In the rest of this section, by making use of complemented submodules, we shall characterize these two types of modules.

Theorem 2. 7. *The following conditions are equivalent to one another:*

(i) *M is locally uniform.*

(ii) *Every non-zero complemented submodule contains a minimal complemented submodule.*

(iii) *Every proper complemented submodule is contained in a maximal complemented submodule.*

Proof. (i)$\Rightarrow$(ii) will be easily seen by Th. 2.3 and its corollary. (ii)$\Rightarrow$(iii). Let C be a proper complemented submodule. By assumption, $C^c\neq 0$ contains a minimal complemenied submodule V. If we take a complement V^c containing C then, by Prop. 2.5, V^c is a maximal complemented submodule. (iii)$\Rightarrow$(i). For any $A\in\mathfrak{M}$, A^c is contained in a maximal complemented submodule C. By Th. 2.6, A^{cc} contains a complement C^c of C, and $A\cap C^c\neq 0$, for A is dense in A^{cc}. Since C is a maximal complemented submodule, C^c is uniform by Prop. 2.5. Hence A contains a uniform submodule $A\cap C^c$.

The part (i)$\Longleftrightarrow$(iii) of the following theorem was given in [2; Lemma (1.1)].

Theorem 2.8. *The following conditions are equivalent to one another:*

(i) *M is finite-dimensional.*

(ii) *The descending chain condition holds for complemented submodules of M.*

(iii) *The ascending chain condition holds for complemented submodules of M.*

Proof. If $C \subsetneqq C'$ are complemented submodules then $C \oplus X \subseteq C'$ for some $X \in \mathfrak{M}$. From this fact, (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) will be easily seen. Next, if $C_1 \supsetneqq C_2 \supsetneqq C_3 \supsetneqq \cdots$ is an infinite descending chain of complemented submodules then for any C_i^c we can choose a complement C_{i+1}^c of C_{i+1} with $C_i^c \subseteq C_{i+1}^c$. Then, as $C_i^- \supsetneqq C_{i+1}^-$, we see $C_i^c \subsetneqq C_{i+1}^c$. Accordingly, we can find an infinite (strictly) ascending chain of complemented submodules, which proves (iii)$\Rightarrow$(ii). To prove (ii)$\Rightarrow$(i), we assume that there exists an infiinite independent set $\{X_i; i=1, 2, \cdots\} \subseteq \mathfrak{M}$, and set $Y_n = X_n \oplus X_{n+1} \oplus \cdots$. Then, by Th. 2.3, for any double complement Y_n^{cc} of Y_n we can find a double complement Y_{n+1}^{cc} with $Y_n^{cc} \supseteq Y_{n+1}^{cc}$. Evidently, there holds $Y_n^{cc-}(=Y_n^-) \supsetneqq Y_{n+1}^-$, so that $Y_n^{cc} \supsetneqq Y_{n+1}^{cc}$. Hence there exists an infinite descending chain of complemented submodules: $Y_1^{cc} \supsetneqq Y_2^{cc} \supsetneqq Y_3^{cc} \supsetneqq \cdots$. This proves (ii)$\Rightarrow$(i), completing the proof.

By Theorems 2.7 and 2.8, we readily obtain

Corollary. *M is finite-dimensional if and only if M is locally uniform and* $\dim M < \infty$.

Let $\{N_\lambda; \lambda \in \Lambda\}$ be a non-empty set of submodules of M. The meet $\cap N_\lambda$ is said to be *irredundant if* $\cap_{\lambda \neq \lambda_0} N_\lambda \supsetneqq \cap N_\lambda$ for every $\lambda_0 \in \Lambda$. And the meet $\cap N_\lambda$ is said to be *s-irredundant if* $(\cap_{\lambda \neq \lambda_0} N_\lambda)^- \supsetneqq (\cap N_\lambda)^-$ for every $\lambda_0 \in \Lambda$. Evidently, an s-irredundant meet is irredundant. If $\cap N_\lambda$ is irredundant [s-irredundant] then, for any non-empty subset Λ_0 of Λ, $\cap_{\lambda \in \Lambda_0} N_\lambda$ is also irredundant [s-irredundant]. To see these, assume first $\cap N_\lambda$ be irredundant. If $\cap_{\lambda \in \Lambda_0} N_\lambda = \cap_{\lambda \in \Lambda_0 - \{\lambda_0\}} N_\lambda$ for some $\lambda_0 \in \Lambda_0$, then $\cap N_\lambda = \cap_{\lambda \neq \lambda_0} N_\lambda$, a contradiction. Next, assume $\cap N_\lambda (\neq M)$ be s-irredundant and C a complement of $\cap N_\lambda$. Then, $(\cap_{\lambda \neq \lambda_0} N_\lambda) \cap C \neq 0$ for arbitrary λ_0, so that for $A = (\cap_{\lambda \in \Lambda - \Lambda_0} N_\lambda) \cap C$ we have $A \cap (\cap_{\lambda \in \Lambda_0} N_\lambda) = 0$ and $A \cap (\cap_{\lambda \in \Lambda_0 - \{\lambda_0\}} N_\lambda) \neq 0$ $(\lambda_0 \in \Lambda_0)$. If $\cap N_\lambda$ is a complemented submodule and irredundant then it is s-irredundant by Prop. 1.7.

Assume now that M is locally uniform, and let $\{V_\lambda : \lambda \in \Lambda\}$ be a maximal independent set of uniform submodules of M. For each V_{λ_0}, choose a complement $V_{\lambda_0}^c$ containing $\sum_{\lambda \neq \lambda_0} V_\lambda$. Then there holds $\cap V_\lambda^c = 0$. If not, non-zero $\cap V_\lambda^c$ contains a uniform submodule, and so by the maximality of $\{V_\lambda\}$, $(\cap V_\lambda^c) \cap (\sum V_\lambda) \neq 0$. Hence $V_{\lambda_1}^c \cap \cdots \cap V_{\lambda_n}^c \cap (V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}) \neq 0$ for some finite subset $\{V_{\lambda_i}\}$. On the oiher hand, by the modular law, we can show

$V^c_{\lambda_1}\cap\cdots\cap V^c_{\lambda_n}\cap(V_{\lambda_1}\oplus\cdots\oplus V_{\lambda_n})=0$. This contradiction proves $\cap V^c_\lambda=0$. Since $\cap_{\lambda\neq\lambda_0}V^c_\lambda\supseteq V_{\lambda_0}$ for all λ_0, $\cap V^c_\lambda=0$ is irredundant, and eace V^c_λ $(\neq M)$ is a maximal complemented submodule by Prop. 1. 13, Prop. 1.14 and Prop. 2.5. Next, let $\cap_{\nu\in N}C_\nu=0$ be an irredundant meet of maximal complemented submodules. Then, as $C_{\nu_0}\cap(\cap_{\nu\neq\nu_0}C_\nu)=0$ for all ν_0, each $\cap_{\nu\neq\nu_0}C_\nu$ is uniform by Prop. 2.5, and evidently $\{\cap_{\nu\neq\nu_0}C_\nu;\ \nu_0\in N\}$ is independent. The first assertion of the following theorem is thus an easy consequence of Th. 2.2.

Theorem 2.9. (i) *Every proper complemented submodule C of M with locally uniform M/C can be represented as an s-irredundant meet of maximal complemented submodules, and* codim C $(=\dim M/C)$ *coincides with the maximum of the number of maximal complemented submodules appearing in an s-irredundant representation of C.*

(ii) *If $A=C_1\cap\cdots\cap C_n$ is an s-irredundant finite meet of maximal complemented submodules then A is complemented, M/A is finite-dimensional, and n coincides with* codim A $(=\dim M/A)$.

Proof. If is only left to prove (ii). As $(\cap_{i\neq i_0}C_i)^-\supsetneqq A^-$ by assumption, we have $V_{i_0}=(\cap_{i\neq i_0}C_i)\cap A^c\neq 0$ for all i_0. And then, as $V_i\cap C_i=0$, each V_i is uniform by Prop. 2.5, and $\sum V_i=\sum\oplus V_i\subseteq A^c$. Hence, $n\leq\dim A^c=$ codim A. Each M/C_i is uniform by Prop. 2.5, and the direct sum $M^*=\sum\oplus M/C_i$ is locally uniform by Prop. 1.3, so that M^* is finite-dimensional by Cor. to Th. 2.8. Since M/A is a subdirect sum of M/C_i's and $(C_1+\cap_{i\neq 1}C_i)/C_1\oplus\cdots\oplus(C_n+\cap_{i\neq n}C_i)/C_n$ is dense in M^* by Cor. to Prop. 1.1, M/A is dense in M^*, whence it follows $\dim M/A=\dim M^*=n$ and that M/A is finite-dimensional. Now, A^c being embedded in M/A, $\dim M/A=n\leq\dim A^c$ yields the equality between them and the density of A^c in M/A, which means that A is complemented (Prop. 1.13).

Corollary. *M is finite-dimensional if and only if M has a finite set of maximal complemented submodules which has zero intersection. If it is the case, a complemented submodule is nothing but an s-irredundant (finite) meet of maximal complemented submodules.* (Cf. [2; Lemma (3.7)].)

Proof. Let M be finite-dimensional. If C is a proper complemented submodule, C^c being dense in M/C, M/C is finite-dimensional (so that locally uniform). C is therefore an s-irredundant finite meet of maximal complemented submodules by Th. 2.9 (i). The first assertion and the converse of the second one are evident by Th. 2.9.

A finite chain of submodules of M: $0=C_0\subsetneqq C_1\subsetneqq\cdots\subsetneqq C_n=M$ is called a c-composition series of M if each C_i is a maximal complemented submodule of C_{i+1}. If $0=C_0\subsetneqq C_1\subsetneqq\cdots\subsetneqq C_n=M$ is a c-composition series of M, then each C_i

is complemented in M by Th. 2.6 and C_i/C_{i-1} is uniform by Prop 2.5. If V_i is a complement of C_{i-1} in C_i, V_i is uniform by Prop. 2.5 and dense in C_i/C_{i-1}. Since each $C_{i-1}\oplus V_i$ is dense in C_i, so is $V_1\oplus\cdots\oplus V_n$ in M (cf. Cor. to Prop. 1.1). Hence, M is finite dimensional and $n=\dim M$. Conversely, if M is finite-dimensional then Th. 2.8 secures the existence of a c-compoistion series. Combining the above with Th. 1.10, we readily obtain the following:

Proposition 2.10. *M is finite-dimensional if and only if M has a c-composition series. If it is the case, the length of any c-composition series of M is equal to* $\dim M$ *and for any two c-composition series $0=C_0\subsetneqq C_1\subsetneqq\cdots\subsetneqq C_n=M$ and $0=C_0'\subsetneqq C_1'\subsetneqq\cdots\subsetneqq C_n'=M$ there exists a* 1–1 *mapping f of $\{C_i/C_{i-1};\ i=1,2,\cdots,n\}$ onto $\{C_i'/C_{i-1}';\ i=1,2,\cdots,n\}$ such that $C_i/C_{i-1}\sim f(C_i/C_{i-1})$ for all i.*

§3. Throughout this section we assume that the d.c-correspondence is a closure operation, or what is the same, $X^{cc}(X\in\mathfrak{M})$ is uniquely determined by X (Cor. 1. to Th. 2. 3).

Proposition 3.1. *If the d.c-correspondence is a closure operation, then there hold the following*:

(i) *A finite or infinite meet of complemented submodules is complemented.*

(ii) *For any $A, B\in\mathfrak{M}$, $A\cap B^{cc}$ is the unique double complement of $A\cap B$ in A.*

(iii) For $A, B\in\mathfrak{M}$, $A^-\subseteq B^-$ if and only if $A^{cc}\subseteq B^{cc}$. Consequently, $A^-=B^-$ if and only if $A^{cc}=B^{cc}$.

Proof. (i). Let $C_\lambda(\lambda\in\Lambda)$ be complemented submodules. Then, as $\cap C_\lambda\subseteq C_{\lambda_0}$ for all λ_0, $(\cap C_\lambda)^{cc}\subseteq C_{\lambda_0}$ for all λ_0 by Prop. 1.7, and hence $(\cap C_\lambda)^{cc}\subseteq\cap C_\lambda$, that is, $(\cap C_\lambda)^{cc}=\cap C_\lambda$. (ii). Let $0\neq(A\cap B)'$ be a double complement of $A\cap B$ in A, then $(A\cap B)'\subseteq(A\cap B)^{cc}\subseteq B^{cc}$, and so $(A\cap B)'\subseteq A\cap B^{cc}$. Since B is dense in B^{cc}, so is $A\cap B$ in $A\cap B^{cc}$ by Prop. 1.4. Hence $(A\cap B)'=A\cap B^{cc}$. (iii). By Prop. 1.4, if $A^-\subseteq B^-$ then $A^-=(A\cap B)^-$. Hence $A^{cc}=(A\cap B)^{cc}\subseteq B^{cc}$. Conversely, if $A^{cc}\subseteq B^{cc}$ then $A^-=(A^{cc})^-\subseteq(B^{cc})^-=B^-$.

Under the same notations as in Th. 1.11, we obtain the following.

Theorem 3.2. *If the d.c-correspondence is a closure operation, then there hold the following*:

(i) *$(\sum_{\lambda\in\Lambda_{P_0}}V_\lambda)^{cc}$ depends on P_0 only (and is independent of the choice of $\{V_\lambda\}$). We set then $C(P_0)=(\sum_{\lambda\in\Lambda_{P_0}}V_\lambda)^{cc}$.*

(ii) *$(\sum_{\lambda\in\Lambda_{P_0}}V_\lambda)^{c}$ depends on P_0 only (and is independent of the choice of $\{V_\lambda\}$ and complements).*

(iii) $\mathrm{C}(P_0)$ *and* $\mathrm{C}(P_0)^c$ *are the unique complements of each other.* $\mathrm{C}(P)$ *is zero or the unique maximal locally uniform submodule, and* $\mathrm{C}(P)^c$ *is the unique maximal submodule containing no uniform submodules, and is the meet of all maximal complemented submodules.*

Proof. (i) and (ii) are immediate consequences of Th. 1.11 and Prop. 3.1 (iii). (iii). By (i) and (ii), $\mathrm{C}(P_0)^c$ is evidently the unique complement of $\mathrm{C}(P_0)$. Let C be a complement of $\mathrm{C}(P_0)^c$. Then $C\sim\mathrm{C}(P_0)$ by Prop. 1.13. If $C^-\not\subseteq\mathrm{C}(P_0)^-$, then $X\subseteq C$ and $X\cap\mathrm{C}(P_0)=0$ for some $X\in\mathfrak{M}$. As $C\sim\mathrm{C}(P_0)\sim\sum_{\lambda\in\Lambda_{P_0}}V_\lambda$, we may assume that $X(\subsetneqq C)$ is isomorphically mapped in $\sum_{\lambda\in\Lambda_{P_0}}V_\lambda$. Then, by Prop. 1.3, X contains a uniform submodule U such that $\tilde{U}\in P_0$. As $X\cap\mathrm{C}(P_0)=0$, $U\cap\sum_{\lambda\in\Lambda_{P_0}}V_\lambda=0$. But this contradicts that $\{V_\lambda;\ \lambda\in\Lambda_{P_0}\}$ is a maximal independent subset of $\cup_{\rho\in P_0}\rho$. Hence $C^-\subseteq\mathrm{C}(P_0)^-$, and so, by Prop. 3.1(iii), $C=C^{cc}\subseteq\mathrm{C}(P_0)^{cc}=\mathrm{C}(P_0)$. Since C is a complement of $\mathrm{C}(P_0)^c$ and $\mathrm{C}(P_0)^c\cap\mathrm{C}(P_0)^{cc}=0$, we have $C=\mathrm{C}(P_0)$. Hence $\mathrm{C}(P_0)$ is the unique complement of $\mathrm{C}(P_0)^c$. Evidently $\mathrm{C}(P)^c$ does not contain a uniform submodule, and $\mathrm{C}(P)$ is locally uniform, because locally uniform $\sum V_\lambda$ is dense in $(\sum V_\lambda)^{cc}=\mathrm{C}(P)$. If A is a locally uniform submodule, then $A\cap\mathrm{C}(P)^c=0$, and so $A\subseteq\mathrm{C}(P)^{cc}=\mathrm{C}(P)$. If B is a submodue containing no uniform submodules, then $B\cap\mathrm{C}(P)=0$, and so $B\subseteq\mathrm{C}(P)^c$. Next, if C_1 is a maximal complemented submodule, then C_1^c is uniform, and C_1^c is contained in the unique maximal locally uniform submodule $\mathrm{C}(P)$. Hence, by Th. 2.6, the unique complement $\mathrm{C}(P)^c$ of $\mathrm{C}(P)$ is contained in C_1. By Prop. 2.5, the meet of all maximal complemented submodules does not contain a uniform submodule, and hence it is contained in $\mathrm{C}(P)^c$. Hence we conclude that $\mathrm{C}(P)^c$ coincides with the meet of all maximal complemented submodules.

Theorem 3.3. *If the d.c-correspondence is a closure operation, then the following conditions are equivalent to one another*:

(i) *M is locally uniform.*

(ii) *The meet of all maximal complemented submodules is zero.*

(iii) *M is an irredundant subdirect sum of uniform modules.*

Proof. Since $M/\mathrm{C}(P)^c(\sim\mathrm{C}(P))$ is locally uniform, $\mathrm{C}(P)^c$ is an irredundant meet of maximal complemented submodules by Th. 2.9. Since $\mathrm{C}(P)$ is the meet of all maximal complemented submodules and the unique maximal submodule containing no uniform submodules, our equivalences will be obvious.

Theorem 3.4. *If the d.c-correspondence is a closure operation then, for $A, B\in\mathfrak{M}$, there hold* $\dim A+\dim B=\dim(A\cap B)+\dim(A+B)$ *and* $\rho\text{-}\dim A+\rho\text{-}\dim B=\rho\text{-}\dim(A\cap B)+\rho\text{-}\dim(A+B)\,(\rho\in P)$.

Proof. Since B^{cc} and $A^{cc}\cap B^{cc}=(A^{cc}\cap B^{cc})^{cc}$ are complemented in $A^{cc}+B^{cc}$

and A^{cc} respectively, we have ρ-dim $(A^{cc}+B^{cc})=\rho$-dim $(A^{cc}+B^{cc})/B^{cc}$[3] $+\rho$-dim B^{cc} and ρ-dim $A^{cc}=\rho$-dim $A^{cc}/(A^{cc}\cap B^{cc})+\rho$-dim$(A^{cc}\cap B^{cc})$ (Prop. 1.12 and Prop. 1.13). Now, $(A^{cc}+B^{cc})/B^{cc}$ is isomorphic to $A^{cc}/(A^{cc}\cap B^{cc})$, and so ρ-dim $(A^{cc}+B^{cc})/B^{cc}=\rho$-dim $A^{cc}/(A^{cc}\cap B^{cc})$. Hence ρ-dim $(A^{cc}\cap B^{cc})+\rho$-dim $(A^{cc}+B^{cc})=\rho$-dim $A^{cc}+\rho$-dim B^{cc}. By Prop. 1.4, $A\cap B$ is dense in $A^{cc}\cap B^{cc}$, and so ρ-dim $(A\cap B)=\rho$-dim $(A^{cc}\cap B^{cc})$. Since A and B are dense in A^{cc} and B^{cc} respectively, ρ-dim $A=\rho$-dim A^{cc} and ρ-dim $B=\rho$-dim B^{cc}. Hence, as ρ-dim $(A+B)\leq\rho$-dim $(A^{cc}+B^{cc})$, we have ρ-dim $(A\cap B)+\rho$-dim $(A+B)\leq\rho$-dim$A+\rho$-dim B. Next, we take a maximal independent set $\{U_\lambda\}$ of $\rho(A\cap B)=\{X\in\rho;\ X\subseteq A\cap B\}$, which can be extended to maximal independent sets $\{U_\lambda\}\cup\{A_\mu\}$, $\{U_\lambda\}\cup\{B_\nu\}$ of $\rho(A)$ and $\rho(B)$, respectively. Then $\{U_\lambda\}\cup\{A_\mu\}\cup\{B_\nu\}$ is an independent set of $\rho(A+B)$. Because, if $(\sum A_\mu+\sum U_\lambda)\cap\sum B_\nu\neq 0$, then by Prop. 1.3, this contains a member of $\rho(A\cap B)$, and hence $0\neq\sum U_\lambda\cap((\sum A_\mu+\sum U_\lambda)\cap\sum B_\nu)=\sum U_\lambda\cap\sum B_\nu$, a contradiction. Thus we have ρ-dim $(A\cap B)+\rho$-dim$(A+B)\geq\rho$-dim $A+\rho$-dim B. Hence ρ-dim $(A\cap B)+\rho$-dim $(A+B)=\rho$-dim $A+\rho$-dim$B$ for every $\rho\in P$, and dim $(A\cap B)+$dim $(A+B)=$dim $A+$dim B.

The d.c-closure operation (in M) is called *continuous* if for each endomorphism φ of M the inverse image $C\varphi^{-1}$ of any complemented submodule C of M is complemented in M.

Proposition 3.5. *If the d.c-correspondence is a closure operation, then the following conditions are equivalent*:

(i) *The d.c-closure operation is continuous.*

(ii) $X^{cc}\varphi\subseteq(X\varphi)^{cc}$ *for any* $X\in\mathfrak{M}$ *and any endomorphism* φ *of* M.

(iii) *For any endomorphism* φ *of* M, Ker φ *is a complemented submodule of* M.

Proof. (i)$\Rightarrow$(ii). As $X\varphi\subseteq(X\varphi)^{cc}$, $X\subseteq(X\varphi)^{cc}\varphi^{-1}$. Since $(X\varphi)^{cc}\varphi^{-1}$ is complemented, $X^{cc}\subseteq(X\varphi)^{cc}\varphi^{-1}$ and so $X^{cc}\varphi\subseteq(X\varphi)^{cc}$. (ii)$\Rightarrow$(iii). $(\text{Ker }\varphi)^{cc}\varphi\subseteq((\text{Kea }\varphi)\varphi)^{cc}=0^{cc}=0$, and hence Ker $\varphi=(\text{Ker }\varphi)^{cc}$, as desired. (iii)$\Rightarrow$(i). We may assume $\varphi\neq 0$. If C is a complemented submodule of M, then $C\cap M\varphi$ is complemented in Mφ, by Prop. 3.1 (ii). Now, $M/\text{Ker }\varphi\cong M\varphi$, and Ker φ is complemented in M by assumption. Since $C\varphi^{-1}=(C\cap M\varphi)\varphi^{-1}$, $C\varphi^{-1}/\text{Ker }\varphi$ is complemented in $M/\text{Ker }\varphi$. Hence $C\varphi^{-1}$ is complemented in M, by Th. 2.2.

Let K be the (R-) endomorphism ring of M acting on the right. If the d.c-closure operation is continuous then, by Prop. 3.5. (ii), the (R-) double complement of any R-K-submodule is also an R-K-submodule. We set $\mathrm{H}(\rho)=\sum_{V\in\rho}V$, and $\mathrm{H}(P_0)=\sum_{\rho\in P_0}\mathrm{H}(\rho)$. Each $\mathrm{H}(\rho)$ is called an (R-) homogeneous component of M. $\mathrm{H}(P_0)\subseteq\mathrm{C}(P_0)$ by Th. 3.2 (i), and evidently $\mathrm{H}(P_0)$ is dense in

3) For $A\in\mathfrak{M}$, $A\sim M/A^c$ by Prop. 1.13.

$C(P_0)$, that is, $H(P_0)^{cc}=C(P_0)$.

Theorem 3.6. *If the d.c-correspondence is a continuous closure operation, then there hold the following*:

(i) *The contraction of an endomorphism of M to a uniform submodule is zero or* 1–1.

(ii) *For any non-empty subset P_0 of P, $H(P_0)$, $C(P_0)$ $(=H(P_0)^{cc})$ and $C(P_0)^c$ are all R-K-submodules of M.*

(iii) *For any direct summand $C\in\mathfrak{M}$ of M, the d.c-correspondence in C is a continuous closure operation.*

Proof. (i). Let $V\in\mathfrak{M}$ be uniform, and let φ be any endomorphism of M. If $V\cap\operatorname{Ker}\varphi\neq 0$, then $V\subseteq V^{cc}=(V\cap\operatorname{Ker}\varphi)^{cc}\subseteq(\operatorname{Ker}\varphi)^{cc}=\operatorname{Ker}\varphi$, and hence $V\varphi=0$. (ii). $H(P_0)$ is R-K-admissible by (i), so that $H(P_0)^{cc}=C(P_0)$ is. Next, if $C(P)^c$ is not K-admissble then $C(P)^c\varphi$ contains a uniform submodule for some endomorphism φ of M by Th. 3.2 (iii). Then, for some non-zero submodule A contained in $C(P)^c$, $A\varphi$ is uniform, so that $(A\varphi)^{cc}$ is uniform and $A^{cc}\varphi\subseteq(A\varphi)^{cc}$. Since $\operatorname{Ker}\varphi$ is a complemented submodule, $\operatorname{Ker}\varphi\cap A^{cc}$ is a complemented submodule of M properly contained in A^{cc}. Hence there exists some $X\in\mathfrak{M}$ with $(\operatorname{Ker}\varphi\cap A^{cc})\oplus X\subseteq A^{cc}$. Then, φ maps X isomorphically into the uniform submodule $(A\varphi)^{cc}$, and hence X is uniform. Accordingly, A being dense in A^{cc}, A contains a uniform submodule $A\cap X$. This contradiction proves that $C(P)^c$ is R-K-admissible. We set $P_1=P-P_0$. Then, since $C(P)^c+C(P_1)+C(P_0)=C(P)^c\oplus C(P_1)\oplus C(P_0)$, $C(P)^c+C(P_1)$ is contained in $C(P_0)^c$, so that dense in $C(P_0)^c$. Hence, as $C(P)^c+C(P_1)$ is R-K-admissible, so is $(C(P)^c+C(P_1))^{cc}=C(P_0)^c$. (iii). By Prop. 3.1 (ii), the d.c-correspondence in C is a closure operation. Any endomorphism φ of C can be extended to an endomorphism $\bar{\varphi}$ of M. Let C_0 be a complemented submodule of C. As C_0 is complemented in M by Th. 2.6 (i), $C_0\bar{\varphi}^{-1}$ is complemented in M, and therefore $C\cap C_0\bar{\varphi}^{-1}=C_0\varphi^{-1}$ is complemented (in M, and so) in C.

§4. Quasi-injective modules. A unital R-left module M is said to be *R-quasi-injective*, if every R-homomorphism of any R-submodule into M can be extended to an R-endomorphism of M (cf. [6]). Throughout this section, "quasi-injective" implies always "R-quasi-injective".

Proposition 4.1. *M is (R-) quasi-injective if and only if $M\cdot\operatorname{Hom}_R(\hat{M},\hat{M})\subseteq M$, where $\hat{M}$ is the R-injective envelope of M.* (See [6; Theorem 1.1].)

Corollary. *Let M be quasi-injective, and let $\{A_\lambda;\ \lambda\in\Lambda\}$ be an independent set of submodules* of M. *Then* $M\cap\sum A_\lambda=\sum(M\cap A_\lambda)$.

Proof. Let φ_λ be the projection to A_λ. Then, each φ_λ can be extended to an endomorphism $\bar{\varphi}_\lambda$ of $\hat{M}$. Let $u=u_{\lambda_1}+\cdots+u_{\lambda_n}$ be any element of $M\cap\sum A_\lambda$, where $u_{\lambda_i}\in A_{\lambda_i}$ $(i=1,2,\cdots,n)$. Then, since M is quasi-injective, $u_{\lambda_i}=u\varphi_{\lambda_i}=u\bar{\varphi}_{\lambda_i}\in M$ by Prop. 4.1. Hence $M\cap\sum A_\lambda\subseteq\sum(M\cap A_\lambda)$. As $M\cap\sum A_\lambda\supseteq\sum(M\cap A_\lambda)$ is obvious, we have $M\cap\sum A_\lambda=\sum(M\cap A_\lambda)$.

Porosition 4.2. (i) *Let M_i $(i=1,2)$ be non-zero R-left modules, and let φ be an R-left homomorphism of M_1 into M_2. If a contraction of φ to a dense R-submodule M_{10} of M_1 is* 1–1, *then so is φ.*

(ii) *Let M_i $(i=1,2)$ be non-zero R-left modules. Then, $M_1\sim M_2$ (similar) if and only if $\hat{M}_1\cong\hat{M}_2$, where $\hat{M}_i$ means the R-injective envelope of M_i.*

(iii) *Every complemented R-submodule of an injective R-left module I is an R-direct summand of I.*

Proof. (i). Since M_{10} is dense in M_1, $\operatorname{Ker}\varphi\cap M_{10}=0$ yields $\operatorname{Ker}\varphi=0$. (ii). If a dense R-submodule M_{10} of M_1 is isomorphic to a dense R-submodule M_{20} of M_2, then $\hat{M}_1=\hat{M}_{10}\cong\hat{M}_{20}=\hat{M}_2$. Hence $\hat{M}_1\cong\hat{M}_2$. Conversely, assume $\hat{M}_1\cong\hat{M}_2$, then $M_1\sim\hat{M}_1\cong\hat{M}_2\sim M_2$. Hence $M_1\sim M_2$. (iii). For any R-submodule of I, its R-injective envelope is embedded isomorphically in I. Hence, by Prop. 1.7, every R-complemented submodule of I is R-injective, and is an R-direct summand of I.

Theorem 4.3. *Let M be quasi-injective, and let C be a complemented submodule of M. Then C is (R-) quasi-injective, and $M=C\oplus C^c$ for every complement C^c of C.*

Proof. $C\oplus C^c$ is dense (in M, and so) in $\hat{M}$. Let C^{dd} and $(C^c)^{dd}$ be double complements of C and C^c in $\hat{M}$ respectively. Then $C^{dd}\oplus(C^c)^{dd}$ is injective by Prop. 4.2 (iii), and dense in $\hat{M}$, and hence $\hat{M}=C^{dd}\oplus(C^c)^{dd}$. By Cor. to Prop. 4.1, $M=(M\cap C^{dd})\oplus(M\cap(C^c)^{dd})$. As $M\cap C^{dd}=C$ and $M\cap(C^c)^{dd}=C^c$ by Th. 2.1, we have $M=C\oplus C^c$. Next, let A be any submodule of C, and φ any homomorphism of A into C. Then φ can be extended to an endomorphism φ_1 of C^{dd}, because C^{dd} is injective. Furthermore, φ_1 can be extended to an endomorphism φ_2 of $\hat{M}$, and, as $C=M\cap C^{dd}$, $C^{dd}\varphi_2=C^{dd}\varphi_1\subseteq C^{dd}$ and $M\varphi_2\subseteq M$ yield $C\varphi_2\subseteq M\cap C^{dd}=C$. The contraction of φ_2 to C is an extension of φ to an endomorphism of C.

Proposition 4.4. *Let M be quasi-injective. Then there hold the following*:

(i) *Every extension of an isomorphism between dense submodules of M is always an automorphism of M.*

(ii) *If $A\sim B$ $(A,B\in\mathfrak{M})$ then $A^{cc}\cong B^{cc}$.*

Proof. (i). By Prop. 4.2 (i), this is evident. (ii). If $A\sim B$, then $(A^{cc})^{dd}\cong$

$(B^{cc})^{dd}$ by some isomorphism φ, because $(A^{cc})^{dd}$ and $(B^{cc})^{dd}$ are injective envelopes of A and B, respectively (Prop. 4.2 (iii)). And, φ is given by some endomorphism φ_1 of $\hat{M}$. Since M is quasi-injective, $A^{cc}\varphi = A^{cc}\varphi_1 = (M \cap (A^{cc})^{dd})\varphi \subseteq M \cap (B^{cc})^{dd} = B^{cc}$ (Th. 2.1), and symmetrically $B^{cc}\varphi^{-1} \subseteq A^{cc}$, and hence $A^{cc}\varphi_1 = B^{cc}$.

Now, for quasi-injective modules, Th. 1.10 can be sharpened as follows.

Theorem 4.5. *Let M be quasi-injective, and let $\{V_\lambda; \lambda \in \Lambda\}$ and $\{W_\gamma; \gamma \in \Gamma\}$ be maximal independent sets of complemented uniform submodules. Then there exists a 1–1 mapping f of Λ onto Γ such that $V_\lambda \cong W_{f(\lambda)}$ for all $\lambda \in \Lambda$. Furthermore there exists an automorphism φ of M such that $V_\lambda \varphi = W_{f(\lambda)}$ for all $\lambda \in \Lambda$.*

Proof. The first half is a direct consequence of Th. 1.10 and Prop. 4.4 (ii), and then there exists an isomrophism φ_1 of $\sum V_\lambda$ onto $\sum W_\gamma$ such that $V_\lambda \varphi_1 = W_{f(\lambda)}$ for all λ. By Th. 1.11, an arbitrary complement C of $\sum V_\lambda$ is a complement of $\sum W_\gamma$ as well. Hence, $x + y \to x + y\varphi_1$ $(x \in C,\ y \in \sum V_\lambda)$ is an isomorphism φ_2 between the dense submodules $C \oplus \sum V_\lambda$ and $C \oplus \sum W_\gamma$, and then φ_2 can be extended to an automorphism φ of M by Prop. 4.4 (i).

Corollary 1. *If M is quasi-injective and finite-dimensional then M is a direct sum of a finite number of quasi-injective uniform submodules, and such a representation of M is unique up to isomorphism.*

Proof. By the validity of Th. 4.5, it suffices to prove that M is a direct sum of a finite number of uniform submodules. Let $\{V_i; i = 1, \cdots, n\}$ be a maximal independent set of complemented uniform submodules of M. Then $\{V_i^{dd}; i = 1, \cdots, n\}$ is independent, where V_i^{dd} is a double complement of V_i in $\hat{M}$. Since each V_i^{dd} is injective by Prop. 4.2 (iii), so is the sum $\sum V_i^{dd}$, and hence $\sum V_i^{dd}$ is a direct summand of $\hat{M}$. On the other hand, M being locally uniform, we readily see that $\sum V_i^{dd}$ is dense in $\hat{M}$, whence it follows $\hat{M} = \sum V_i^{dd}$. Since $M = M \cap \sum V_i^{dd} = \sum (M \cap V_i^{dd})$ and $V_i = M \cap V_i^{dd}$ by Cor. to Prop. 4.1 and Th. 2.1, we obtain eventually $M = \sum V_i$, as desired.

Corollary 2. *Let M be quasi-injective.*

(i) *Every isomorphism of a finite-dimensional submodule A of M into M can be extended to an automorphism of M.*

(ii) *If finite-dimensional submodules A, B of M are similar, then $A^{cc} \cong B^{cc}$ and $A^c \cong B^c$.*

Proof. (i). Let φ be an isomorphism of A into M. As A^{cc} is quasi-injective (Th. 4.3) and finite-dimensional, A^{cc} is a direct sum of a finite number of uniform submodules (Cor. 1 to Th. 4.5). Hence, by the proof of Th. 4.5, we can extend φ to an automorphism ϕ of M. (ii). By Th. 4.3 and Prop. 4.4 (ii), $M = A^{cc} \oplus A^c = B^{cc} \oplus B^c$ and $A^{cc} \cong B^{cc}$. We have seen in (i) that the isomorphism $A^{cc}\cong B^{cc}$ can be extended to an automorphism ϕ of M. And, $B^{cc}\oplus B^{c}=M=M\phi=A^{cc}\phi\oplus A^{c}\phi=B^{cc}\oplus A^{c}\phi$, whence it follows $B^{c}\cong A^{c}\phi\cong A^{c}$.

Theorem 4.6. *If R is a left Noetherian ring with 1, and M is quasi-injective, then M is a direct sum of uniform submodules, and such a representation of M is unique up to isomorphism.*

Proof. For any non-zero element u of M, Ru is an R-module with the ascending chain condition for its submodules. Hence, Ru is locally uniform by Th. 2.7, so that M is locally uniform. Now, let $\{V_\lambda\}$ be a maximal independent set of complemented uniform submodules of M. Each double complement V_λ^{dd} of V_λ in $\hat{M}$ is injective by Prop. 4.2 (iii), and so $\sum\oplus V_\lambda^{dd}$ is an injective[4] dense submodule of $\hat{M}$. Hence, $\hat{M}=\sum\oplus V_\lambda^{dd}$. Recalling here that $M\cap V_\lambda^{dd}=V_\lambda$ by Th. 2.1, Cor. to Prop. 4.1 yields $M=M\cap\sum\oplus V_\lambda^{dd}=\sum\oplus(M\cap V_\lambda^{dd})=\sum\oplus V_\lambda$. The final assertion is a consequence of Th. 4.5.

The proof of the following lemma proceeds just like in [1; Th. 22.3].

Lemma 4.7. *Let $M=A\oplus B$. In order that B is R-K-admissible, it is necessary and sufficient that $M=A\oplus B'$ implies $B=B'$.*

Under the same notations as in Th. 1.11, there holds the following:

Theorem 4.8. *Let the d.c-correspondence in a quasi-injective module M be a closure operation. If P_0 is a non-empty subset of P then M is the direct sum of R-K-submodules $(\sum_{\lambda\in\Lambda_{P_0}}\oplus V_\lambda)^c$ and $\mathrm{C}(P_0)=(\sum_{\lambda\in\Lambda_{P_0}}\oplus V_\lambda)^{cc}$.*

Proof. By Th. 4.3, $M=(\sum_{\lambda\in\Lambda_{P_0}}\oplus V_\lambda)^c\oplus\mathrm{C}(P_0)=\mathrm{C}(P_0)^c\oplus\mathrm{C}(P_0)$. And, by Th. 3.2 (iii) and Lemma 4.7, $\mathrm{C}(P_0)^c$ and $\mathrm{C}(P_0)$ are R-K-submodules.

Proposition 4.9. *Let M' be a unital R'-K'-module, where R', K' are rings with 1. And, assume that each R'-homomorphism of any finitely generated R'-submodule of M' into M' is induced by an element of K'.*

(i) *Let u be a non-zero element of M'. If $R'u$ is a uniform R'-submodule and each $\alpha\in K'$ with $u\alpha\neq 0$ induces an R'-isomorphism of $R'u$ onto $R'u\alpha$, then uK' is a minimal K'-submodule of M', and conversely.*

(ii) *Let uK' and vK' ($u, v\in M'$) be minimal K'-submodules of M'. If $R'u$ is similar to $R'v$ then uK' is K'-isomorphic to vK'.*

Proof. (i). Assume first that uK' is minimal. If $R'u$ is not uniform, there exist two non-zero elements au, bu ($a, b\in R'$) with $R'au\cap R'bu=0$. $x+y\to x$ ($x\in R'au$, $y\in R'biu$) defines evidently an R'-homomorphism φ of $R'au\oplus R'bu$ into M', which is induced by an element γ of K'. Since uK' is minimal, $uK'\cong auK'$ and $uK'\cong buK'$ naturally, and hence $auK'\cong buK'$ where

4) Since R is a left Noetherian ring, every left ideal of R is finitely generated. Therefore, every homorphic image of any left ideal of R is finitely generated.

$au \longleftrightarrow bu$. Therefore, as $0 \neq au = (au)\varphi = au\gamma$, $(bu)\varphi = bu\gamma \neq 0$. This contradiction proves the uniformity of $R'u$. For any $\alpha \in K'$ with non-zero $u\alpha$, we have $u\alpha K' = uK'$, and hence $au\alpha = 0$ $(a \in R')$ implies $au = 0$. Conversely, assume that $R'u$ is a uniform R'-submodule and each $\alpha \in K'$ with non-zero $u\alpha$ induces an R-isomorphism $R'u \cong R'u\alpha$. Then, for any α with non-zero $u\alpha$, there exists an element δ of K' such that $(u\alpha)\delta = u$. Hence, $u = u\alpha\delta \in u\alpha K'$. This implies that uK' is minimal. (ii). Let $R'au \cong R'bv$, $0 \neq au \longleftrightarrow bv$ $(a, b \in R')$. Then, there exists an element $\gamma \in K'$ such that $au\gamma = bv$. Accordingly, $uK' \cong auK' = au\gamma K' = bvK' \cong vK'$, and hence $uK' \cong vK'$.

§ 5. A unital R-left module M is called an *R-c. q. i-module* if M is R-quasi-injective and the R-d. c-correspondence in M is a continuous closure operation. We set $K = \mathrm{Hom}_R(M, M)$, which acts on the right.

Noting that the kernel of any R-endomorphism of an R-c. q.i-module is an R-direct summand (Th. 4. 3), the next proposition will be proved as in [7; 3.3 Theorem].

Proposition 5. 1. *If M is an R-c. q-i-module, then K is a regular ring.*

Corollary. *Let M be an R-c. q.i-module. If C and C' are R-direct summands (or equivalently, R-complemented submodules) of M then so is* $C + C'$ (cf. [6; 1.4 Theorem]).

Proof. As is well known, $C = M\sigma$ and $C' = M\sigma'$ with some idempotent elements $\sigma, \sigma' \in K$. Then, K being a regular ring by Prop. 5.1, $K\sigma + K\sigma' = K\varepsilon$ with an idempotent element $\varepsilon \in K$, and so $M\sigma + M\sigma' = M \cdot K\sigma + M \cdot K\sigma' = M \cdot (K\sigma + K\sigma') = M \cdot K\varepsilon = M\varepsilon$. $C + C'$ is therefore an R-direct summand of M.

Theorem 5. 2. *Let M be an R-c .q. i-module.*

(i) *Let u be a non-zero element of M. Ru is uniform if and only if uK is minimal.*

(ii) *Every K-uniform submodule of M is isomorphic to a minimal (or equivalently, uniform) right ideal of K.*

(iii) *Let Ru, Rv $(u, v \in M)$ be uniform. $Ru \sim Rv$ (similar) if and only if $uK \cong vK$ (or equivalently, $uK \sim vK$).*

(iv) *The sum* $\mathrm{H}(P)$ *of all R-uniform submodules of M coincides with the K-socle (i. e. the sum of all minimal K-submodules) of M. The set* $\{\mathrm{H}(\rho); \rho \in P\}$ *of all R-homogeneous components of M coincides with the set of all K-homogeneous components of (the K-socle of) M, and each* $\mathrm{H}(\rho)$ *is a direct sum of R-uniform submodules (as well as of minimal K-submodules).*

(v) *If Ru $(u \in M)$ contains an R-uniform submodule then uK contains a minimal K-submodule,and conversely.* (Cf. [4; pp. 60–64 and pp. 124–126].)

Proof. (i). Combining Prop. 3.6 (i) and Prop. 4.9 (i), it will be evident. (ii). Let uK $(u\in M)$ be uniform, and set $r(u)=\{\alpha\in K;\ u\alpha=0\}$. Then, $(Ru)^{cc}=M\varepsilon$ with an idempotent $\varepsilon\in K$, and $M\varepsilon\cdot r(u)=(Ru)^{cc}\cdot r(u)\subseteq(Ru\cdot r(u))^{cc}=0$ by Prop. 3.5, whence it follows $r(u)=r(M\varepsilon)=(1-\varepsilon)K$. Hence, we have $uK\cong K/r(u)=K/(1-\varepsilon)K\cong\varepsilon K$. Since K is a regular ring, a uniform right ideal of K is minimal. Hence uK $(\cong\varepsilon K)$ is minimal. (iii) and (iv). Each R-homogeneous component $\mathrm{H}(\rho)$ is R-K-admissible by Th. 3.6 (ii), and is contained in a K-homogeneous component of M by (i) and Prop. 4.9 (ii). And, by (i), the sum $\sum_\rho\oplus\mathrm{H}(\rho)$ of all R-uniform submodules coincides with the K-socle. Now, let $\{V_\lambda;\ \lambda\in\Lambda\}$ be a maximal independent set of complemented R-uniform submodules of M, and let V be arbitrary R-uniform submodule of M. Then, $V\cap(V_{\lambda_1}\oplus\cdots\oplus V_{\lambda_n})\neq0$ for some finite subset $\{V_{\lambda_i}\}$ of $\{V_\lambda\}$, and so $V\subseteq V^{cc}=(V\cap(V_{\lambda_1}\oplus\cdots\oplus V_{\lambda_n}))^{cc}\subseteq(V_{\lambda_1}\oplus\cdots\oplus V_{\lambda_n})^{cc}=V_{\lambda_1}\oplus\cdots\oplus V_{\lambda_n}$ by Cor. to Prop. 5.1, whence it follows that $\sum_\rho\oplus\mathrm{H}(\rho)=\sum_\lambda\oplus V_\lambda$. Further, noting that $\mathrm{H}(\rho)\supseteq\sum_{\lambda\in\Lambda_\rho}\oplus V_\lambda$ $(\Lambda_\rho=\{\lambda\in\Lambda;\ V_\lambda\in\rho\})$, we obtain $\mathrm{H}(\rho)=\sum_{\lambda\in\Lambda_\rho}\oplus V_\lambda$. Choose a K-homogeneous component N containing $\mathrm{H}(\rho)$. If we set $S=\mathrm{Hom}_K(N,N)$ acting on the left, then it is well known that N is S-K-minimal (cf. [4]). For any $V_\lambda(\lambda\in\Lambda_\rho)$, $M=V_\lambda\oplus V_\lambda^c$ and the projection π of M onto V_λ is contained in K, so that for each $a\in S$ and $v\in V_\lambda$ we have $av=a(v\pi)=(av)\pi\in V_\lambda$. Hence each $V_\lambda(\lambda\in\Lambda_\rho)$ and so $\mathrm{H}(\rho)$ is an S- submodule of N, which implies $\mathrm{H}(\rho)=N$. And, at the same time, we obtain (iii). (v). Let Rau $(a\in R)$ be an R-uniform submodule of Ru, and set $(Rau)^{cc}=M\varepsilon$ with an idempotent $\varepsilon\in K$. As $M\varepsilon$ is still uniform, $K\varepsilon$ is directly indecomposable, whence so is εK. Further, recalling that K is a regular ring, εK is minimal. Since $M\varepsilon=M\varepsilon\cdot\varepsilon=(Rau)^{cc}\varepsilon\subseteq(Rau\varepsilon)^{cc}$ yields $u\varepsilon\neq0$, we have then $\varepsilon K\cong u\varepsilon K\subseteq uK$. Conversely, let $u\delta K$ $(\delta\in K)$ be a minimal K-submodule of uK. Then $Ru\delta$ is uniform by (i). Since the unique maximal R-submodule $\mathrm{H}(P)^c$ containing no R-uniform submodules is K-admissible (Th. 3.6 (ii)), Ru have to contain an R-uniform submodule.

In particular, Th. 5.2 (iv) and (v) yield at once.

Corollary. *M is R-locally uniform if and only if it is K-locally uniform, and M is a (direct) sum of R-uniform submodules if and only if it is a (direct) sum of K-uniform (or equivalently, K-minimal) submodules (i.e. M is K-completely reducible).* (Cf. Th. 4.5.)

Combining Prop. 3.6 (iii) and Th. 4.3, we obtain

Theorem 5.3. *If M is an R-c.q.i-module, then every complemented R-submodule of M is an R-direct summand of M and an R-c.q.i-module.*

We set $Q=\mathrm{Hom}_K(M,M)$, which acts on the left. We note here that every R-direct summand of M is Q-admissible, and so a Q-direct summand of M.

Now, let Q_0 be an arbitrary intermediate ring of Q and the ring R_0 of all the (additive group) endomorphisms induced by R. For any Q_0-submodule A, one will easily see that a complement A^c in the R-module M is a complement A^d in the Q_0-module M, and conversely. And then, we see also that any double complement A^{dd} in the Q_0-module M coincides with A^{cc} uniquely determined. Noting here that $\mathrm{Hom}_{Q_0}(M, M)=K$ and for each $\alpha\in K$ there holds $A^{dd}\alpha=A^{cc}\alpha\subseteq(A\alpha)^{cc}=(A\alpha)^{dd}$, the d.c-correspondence in the Q_0-module M is seen to be a continuous closure operation. If φ is a Q_0-homomorphism of a Q_0-submodule A into M, then φ is given by an element of K, because φ is an R-homomorphism. Since $\mathrm{Hom}_{Q_0}(M, M)=K$, this implies that M is Q_0-quasi-injective. We have proved thus

Theorem 5.4. *Let M be an R-c.q.i-module. Then, for any intermediate ring Q_0 between Q and R_0, M is a Q_0-c.q.i-module, and for any Q_0-submodule A, $\{A^c\}=\{A^d\}$ and $A^{cc}=A^{dd}$.*

Lemma 5.5. *Let T be a ring with* 1, *which has no nilpotent* (*one-sided*) *ideals, and let e be an idempotent element of T such that Te is a* (*two-sided*) *ideal. Then e belongs to the center of T.*

Proof. Since Te is an ideal, $Te\cdot T(1-e)=0$. As $(T(1-e)\cdot Te)^2=0$, $T(1-e)\cdot Te=0$ and hence $T(1-e)\subseteq l(Te)=\{a\in T;\, aTe=0\}$. As $(l(Te)\cap Te)^2=0$, $l(Te)\cap Te=0$. Hence $T(1-e)=l(Te)$ is an ideal of T. Let $1=f+g$, where $f\in Te$, $g\in T(1-e)$. Then, as is easily seen, f and g are idempotent elements belonging to the center of T. As $Te=Tf$, we have $f=e$.

Let M be a unital R-left, K_1-right module, where K_1 is a non-zero ring with 1. Let K_1^0 be the opposite ring of K_1. We can consider M as a unital $R\otimes_J K_1^0$-left module by means of $(a\otimes\beta^0)u=au\beta$ $(a\in R,\ \beta\in K_1,\ u\in M)$, where J means the ring of rational integers. If M is an $R\otimes K_1^0$-c.q.i-module, M is called an *R-K_1-c.q.i-module.*

Let M be an R-c.q.i-module. If B is an R-K-submodule, then $B^{cc}\alpha\subseteq(B\alpha)^{cc}\subseteq B^{cc}$ for every $\alpha\in K$, so that B^{cc} is also an R-K-submodule. If we set $B^{cc}=M\varepsilon$ with an idempotent ε in K, then $M\varepsilon\cdot K\subseteq M\varepsilon$, and hence $\varepsilon K\subseteq K\varepsilon$, that is, $K\varepsilon$ is an ideal of K. And, K being a regular ring, ε is a central idempotent of K by the preceding lemma. As $M=B^c\oplus B^{cc}$, B^c is also an R-K-submodule of M, and is the unique complement of B in the R-module M (Lemma 4.7). Hence, to be easily seen, the complement B^e of B in the R-K- (or $R\otimes K_1^0$-) module M coincides with the one of B in the R-module M, which implies also $B^{cc}=B^{ee}$. Since $\mathrm{Hom}_{R,K}(M, M)$ is the center of K, for each γ of the center of K, we have $B^{ee}\gamma=B^{cc}\gamma\subseteq(B\gamma)^{cc}=(B\gamma)^{ee}$. Hence, the R-K- (or $R\otimes K_1^0$-) d.c-correspondence in M is a continuous closure operation. Let N be a dense R-K-submodule of M and let φ be an R-K-homomorphism of N into M. Extending φ as an R-homomorphism to an element δ of K, we have $N(\alpha\delta-\delta\alpha)=0$ for all $\alpha\in K$. By the continuity, $M(\alpha\delta-\delta\alpha)=N^{cc}(\alpha\delta-\delta\alpha)\subseteq(N(\alpha\delta-\delta\alpha))^{cc}=0$, whence $\alpha\delta-\delta\alpha=0$ for all $\alpha\in K$. Thus we have proved the following theorem:

Theorem 5.6. *If M is an R-c.q.i-module, then M is an R-K-c.q.i-module, and $B^c=B^e$ (uniquely determined), $B^{cc}=B^{ee}$ for every R-K-submodule B of M.*

Theorem 5.7. *Let M be an R-c.q.i-module. If N is any R-K-submodule of M then N is an R-c.q.i and R-K-c.q.i-module.* (See Prop. 3.1 (ii).)

Proof. The R-quasi-injectivity of N is evident. And, by Prop. 3.1 (ii), the R-d.c-correspondence in N is a closure operation. In fact, if A is an R-submodule of N, then $A^{cc}\cap N$ is the unique R-double complement of A in N. Now, $K'=\mathrm{Hom}_R(N,N)$ is the contraction of K to N. For any $\gamma\in K$, $(A^{cc}\cap N)\gamma\subseteq A^{cc}\gamma\cap N\subseteq(A\gamma)^{cc}\cap N=$ the R-double complement of $A\gamma$ in N. Hence, N is an R-c.q.i-module. Moreover, by Th. 5.6, N is an R-K'-c.q.i-module, or what is the same, N is an R-K-c.q.i-module.

The following lemma is well known.

Lemma 5.8. *If $r_M(\mathfrak{l})=\{u\in M;\ \mathfrak{l}u=0\}=0$ for every dense left ideal $\mathfrak{l}$ of R, then the R-d.c-correspondence is a continuous closure operation. In fact, if A is an R-submodule of M then $A^{cc}=\{u\in M;\ \mathfrak{l}u\subseteq A$ for some dense left ideal $\mathfrak{l}\}$.*

Proof. Let A be a non-zero R-submodule of M. For any $u\in A^{cc}$, $R\ni a\to au\in A^{cc}$ is an R-homomorphism of R into a double complement A^{cc} of A. Since A is dense in A^{cc}, $\{a\in R;\ au\in A\}$ is a dense left ideal of R by Prop. 1.5. Hence A^{cc} is contained in $A^+=\{u\in M;\ \mathfrak{l}u\subseteq A$ for some dense left ideal $\mathfrak{l}\}$. If $\mathfrak{l}_1u_1,\ \mathfrak{l}_2u_2\subseteq A$ for dense left ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ ($u_1,u_2\in M$), then $\mathfrak{l}_1\cap\mathfrak{l}_2$ is a dense left ideal and $(\mathfrak{l}_1\cap\mathfrak{l}_2)(u_1\pm u_2)\subseteq A$. Further for any $a\in R$, $(\mathfrak{l}_1:a)_l=\{b\in R;\ ba\in\mathfrak{l}_1\}$ is a dense left ideal by Prop. 1.5, and $(\mathfrak{l}_1:a)_l au_1\subseteq\mathfrak{l}_1u_1\subseteq A$. Hence, A^+ is an R-submodule of M. Next, if $A^{cc}\subsetneqq A^+$, A^{cc} being non-dense in A^+, there exists a non-zero submodule X of A^+ with $A^{cc}\cap X=0$. Choosing an arbitrary non-zero element $u\in X$, there exists a dense left ideal $\mathfrak{l}$ with $\mathfrak{l}u\subseteq A$. On the other hand, as $u\in X$, $\mathfrak{l}u\subseteq X$, and hence $\mathfrak{l}u=0$, contradicting our assumption. Hence, we have $A^{cc}=A^+$. The continuity of R-d.c-correspondence will be evident by Prop. 3.5 (ii).

As is seen from the above proof, $0^+=\{u\in M;\ \mathfrak{l}u=0$ for some dense left ideal $\mathfrak{l}\}$ is an R-K-submodule of M, and is called the *R-singular part* (or singular submodule) of M. (And the K-singular part is defined in the similar way.) Lemma 5.8 is now restated as follows: If the R-singular part of M is zero, then the R-d.c-correspondence is a continuous closure operation.

Proposition 5.9. *If M is an R-c.q.i-module, then the K-singular part of M and the left singular part of K (i.e. the singular part of K as a K-left module) are* 0.

Proof. Let $\mathfrak{r}$ be an arbitrary dense right ideal of K. If $u\mathfrak{r}=0$ $(u\in M)$ then $Ru\mathfrak{r}=0$. Setting $(Ru)^{cc}=M\varepsilon$ with an idempotent $\varepsilon\in K$, $M\varepsilon\mathfrak{r}=(Ru)^{cc}\mathfrak{r}\subseteq(Ru\mathfrak{r})^{cc}=0$, that is, $\varepsilon\mathfrak{r}=0$, whence $\mathfrak{r}\cap\varepsilon K=\varepsilon(\mathfrak{r}\cap\varepsilon K)=0$. Since $\mathfrak{r}$ is dense in K, εK has to be 0, and so we have $u=0$. Next, let $\mathfrak{l}$ be an arbitrary dense left ideal of K. If $r(\mathfrak{l})=\{\alpha\in K;\mathfrak{l}\alpha=0\}$ is non-zero, $r(\mathfrak{l})$ contains a non-zero idempotent τ, and so $\mathfrak{l}\subseteq K(1-\tau)$, which contradicts the density of $\mathfrak{l}$.

The next theorem has been stated in [7] without proof.

Theorem 5.10. *If M is an R-c.q.i-module, then K is an injective K-left module in which the K-d.c-correspondence is a continuous closure operation (or equivalently, K is a maximal left quotient ring with zero left singular part.* (Cf. [8].))

Proof. Since the left singular part of K is $\{0\}$ by Prop. 5.9, the K-d.c-correspondence in the K-left module K is a continuous closure operation by Lemma 5.8. Accordingly, it is left only to prove the injectivity of K. Let $\mathfrak{l}$ be a left ideal, and φ a K-homomorphism of $\mathfrak{l}$ into K. For given $u_i\in M$ and $\alpha_i\in\mathfrak{l}$ $(i=1,\cdots,n)$, choose an element $\gamma\in K$ with $\sum K\alpha_i=K\gamma$, and set $\alpha_i=\alpha_i'\gamma$ $(\alpha_i'\in K)$, $u=\sum u_i\alpha_i'$. If we set $r_K(u)=\{\alpha\in K;\ u\alpha=0\}$ then, by Prop. 3.5 (ii), $r_K(u)=r_K(Ru)=r_K((Ru)^{cc})=\varepsilon K$ with an idempotent $\varepsilon\in K$. Hence, if $\sum u_i\alpha_i=0$ then $\gamma\in\varepsilon K\cap\mathfrak{l}$, and so $\gamma\varphi=(\varepsilon\gamma)\varphi=\varepsilon(\gamma\varphi)\in\varepsilon K$. And then, $\sum u_i(\alpha_i\varphi)=u(\gamma\varphi)=0$, which enables us to see that $\sum v_j\beta_j\rightarrow\sum v_j(\beta_j\varphi)$ $(v_j\in M,\ \beta_j\in\mathfrak{l})$ defines an R-homomorphism ψ of $M\mathfrak{l}$ into M. Since M is R-quasi-injective, ψ can be extended to some $\delta\in K$. And, we have then $\beta\varphi=\beta\delta$ for all $\beta\in\mathfrak{l}$, which proves that K is injective.

Let A be an R-submodule of an R-c.q.i-module M, and set $A^{cc}=M\varepsilon$ with an idempotent $\varepsilon\in K$. Then, $r_K(A)=r_K(A^{cc})=(1-\varepsilon)K$ and $l_M((1-\varepsilon)K)=\{u\in M;\ u(1-\varepsilon)K=0\}=M\varepsilon=A^{cc}$. In particular, by Th. 5.10, $l(r(\mathfrak{l}))$ coincides with the double complement $\mathfrak{l}''$ of $\mathfrak{l}$ in K for any left ideal $\mathfrak{l}$ of K, where $r(*)$, $l(*)$ denote the right annihilator and left annihilator of $*$ in K. As $l(r_K(A))=l((1-\varepsilon)K)=K\varepsilon$, $A^{cc}=M\cdot l(r_K(A))$ and, in particular, $(M\cdot\mathfrak{l})^{cc}=M\cdot l(r_K(M\mathfrak{l}))=M\cdot l(r(\mathfrak{l}))=M\mathfrak{l}''$. We have proved therefore the following:

Proposition 5.11. *Let M be an R-c.q.i-module. If A is an R-submodule of M, and $\mathfrak{l}$ a left ideal of K with the double complement $\mathfrak{l}''$ in K, then $A^{cc}=l_M(r_K(A))=M\cdot l(r_K(A))$, $\mathfrak{l}''=l(r(\mathfrak{l}))$ and $(M\mathfrak{l})^{cc}=M\mathfrak{l}''$.*

Let A be an R-submodule of M. For $\alpha \in K$, $A\alpha \cdot r_K(A\alpha)=0$, and so $\alpha \cdot r_K(A\alpha) \subseteq r_K(A)$. Hence $l_M(r_K(A)) \cdot \alpha \cdot r_K(A\alpha)=0$, that is, $l_M(r_K(A)) \cdot \alpha \subseteq l_M(r_K(A\alpha))$. Thus we have the following:

Corollary. *M is an R-c.q.i-module if and only if M is R-quasi-injective and $A^{cc}=l_M(r_K(A))$ for any R-submodule A.*

Let M be an R-c.q.i-module. Every R-complemented submodule of M is an R-direct summand of M (Th. 4.3). Every R-K-complemented submodule of M is an R-K-direct summand of M, by Th. 4.3 and Th. 5.6. Consequently, by Th. 5.10, every complemented left ideal of K is a left direct summand of K, and every complemented ideal of K is a two-sided direct summand of K. For any R-direct summand $M\varepsilon$ ($\varepsilon^2=\varepsilon \in K$), we correspond a left direct summand $K\varepsilon=l(r_K(M\varepsilon))$ of K. Then this is an order-isomorphism of the R-direct summands of M onto the left direct summands of K. From this fact and Th. 2.7, M is R-locally uniform if and only if K is (K-) left locally uniform. And, $M\varepsilon$ is uniform if and only if $K\varepsilon$ is uniform (Cor. to Th. 2.3). Therefore, M contains an R-uniform submodule if and only if K contains a uniform (or equivalently, minimal) left ideal. To be easily seen, $M\varepsilon$ is K-admissible if and only if $K\varepsilon$ is an ideal, that is $\varepsilon K \subseteq K\varepsilon$. In this case ε is a central idempotent (Lemma 5.5). Hence M is R-K-locally uniform if and only if K is ideal- (i.e. K-K-) locally uniform (Th. 2.7). And, $M\varepsilon$ is R-K-uniform if and only if $K\varepsilon$ is a uniform ideal (Cor. 2 to Th. 2.3). Therefore M contains an R-K-uniform submodule if and only if K contains a uniform ideal. Let $\{\varepsilon_\lambda;\ \lambda \in \Lambda\}$ be a set of idempotent elements of K. Then, $\sum M\varepsilon_\lambda=\sum \oplus M\varepsilon_\lambda$ if and only if $\sum K\varepsilon_\lambda=\sum \oplus K\varepsilon_\lambda$. To prove this fact, let $M\varepsilon_1+\cdots+M\varepsilon_n=M\varepsilon_1 \oplus \cdots \oplus M\varepsilon_n$, where $\varepsilon_i^2=\varepsilon_i \in K$. If $K\varepsilon_1 \cap (K\varepsilon_2+\cdots+K\varepsilon_n)$ contains $0 \neq \alpha \in K$, then $0 \neq u\alpha$ for some $u \in M$. Then $0 \neq u\alpha \in M\varepsilon_1 \cap (M\varepsilon_2+\cdots+M\varepsilon_n)$ a contradiction. Conversely, we assume that $K\varepsilon_1+\cdots+K\varepsilon_n=K\varepsilon_1 \oplus \cdots \oplus K\varepsilon_n$. Let $K\varepsilon_1+\cdots+K\varepsilon_n=K\varepsilon$, $\varepsilon^2=\varepsilon \in K$ and $\varepsilon=\varepsilon_1'+\cdots+\varepsilon_n'$, $\varepsilon_i' \in K\varepsilon_i$. Then $K\varepsilon_i'=K\varepsilon_i$, $\varepsilon_i'^2=\varepsilon_i'$ and $\varepsilon_i'\varepsilon_j'=0$, if $i \neq j$. Hence, as $M\varepsilon_i'=M\varepsilon_i$, $M\varepsilon_1+\cdots+M\varepsilon_n=M\varepsilon_1 \oplus \cdots \oplus M\varepsilon_n$. From this fact, R-dim M is equal to the left dimension of K, and R-K-dim M is equal to the ideal-dimension of K.

Let W, W' be two maximal R-K-uniform submodules such that $W \sim W'$ as R-K-submodules. Then, by the R-quasi-injectivity, there exists some $\gamma \in K$ such that $W\gamma \cap W' \neq 0$. As $W\gamma \subseteq W$, $W \cap W' \neq 0$, and so $W=(W \cap W')^{cc}=W'$ (Th. 5.6). This shows that M has the unique maximal independent set of maximal R-K-uniform submodules (Th. 1.10). Let V and V' be R-uniform submodules such tnat $V \sim V'$. Then, by the R-quasi-injectivity, $V\delta \cap V' \neq 0$ for some $\delta \in K$. Hence each R-homogeneous component $\mathrm{H}(\rho)$ is R-K-uniform, and each ($\mathrm{H}(\rho)^{cc}=$) $\mathrm{C}(\rho)$ is an R-K-homogeneous component of M. Hence the unique maximal R-locally uniform submodule $C(P)=(\sum \oplus H(\rho))^{cc}$ is R-K-locally uniform. Hence the unique maxmal R-locally uniform submodule is contained in the unique maximal R-K-locally uniform submodule. Consequently, by Th. 3.2 (iii) and Th. 4.8 (and Th. 5.6), M has the following representation: $M=M_1\oplus M_2\oplus M_3$. Each M_i is R-K-admissible. The first component M_1 is R-locally uniform. The second component M_2 is R-K-locally uniform, but does not contain an R-uniform submodule. The third component M_3 contains neither an R-uniform submodule nor an R-K-uniform submodule. In this meaning, such a representation of M is unique. Because M_1 is the unique maximal R-locally uniform submodule, and $M_1\oplus M_2$ is the unique maximal R-K-locally uniform submodule (see Lemma 4.7). Let each τ_i $(i=1,2,3)$ be the projection to M_i. Then each τ_i is a central idempotent. And, $K=K\tau_1\oplus K\tau_2\oplus K\tau_3$ and, to be easily seen, $\mathrm{Hom}_R(M_i, M_i)=\tau_i K\tau_i=K\tau_i$. And further $\mathrm{Hom}_R(M\tau_1\oplus M\tau_2, M\tau_1\oplus M\tau_2)=K(\tau_1+\tau_2)=K\tau_1+K\tau_2$ and $\mathrm{Hom}_R(M\tau_2\oplus M\tau_3, M\tau_2\oplus M\tau_3)=K(\tau_2+\tau_3)=K\tau_2+K\tau_3$. As M_1 is R-locally uniform, $K\tau_1$ is left locally uniform. As $M\tau_2+M\tau_3$ does not contain an R-uniform submodule, $K\tau_2+K\tau_3$ does not contain a uniform left ideal. Hence $K\tau_1$ is the unique maximal locally uniform left ideal of K. Similarly we can see that $K\tau_1+K\tau_2$ is the unique maximal locally uniform ideal of K. Hence $K\tau_i$ is the i-th component of a left injective ring K with zero (left) singular part.

Let A be an R-submodule. By the R-quasi-injectivity, $\mathrm{Hom}_R(A, M)=K/r_K(A)$. Since $r_K(A)=r_K(A^{cc})$ (Prop. 3.5 (ii)), $\mathrm{Hom}_R(A, M)=\mathrm{Hom}_R(A^{cc}, M)$. If B is an R-K-submodule, then B^{cc} is also an R-K-submodule and ($\mathrm{Hom}_R(B, M)=$) $\mathrm{Hom}_R(B, B)=\mathrm{Hom}_R(B^{cc}, B^{cc})$. Let $M\tau=B^{cc}$, where $\tau^2=\tau\in K$. Then $\mathrm{Hom}_R(B^{cc}, B^{cc})=\tau K\tau$, and $\tau K\tau=K\tau$ is a two-sided direct summand of K. Let $M\varepsilon$ be R-uniform, where $\varepsilon^2=\varepsilon\in K$. Then $K\varepsilon$ is a uniform left ideal, and further, as K is a regular ring, $K\varepsilon$ is a minimal left ideal. Hence $\mathrm{Hom}_R(M\varepsilon, M\varepsilon)=\varepsilon K\varepsilon$ is a division ring.

Theorem 5.12. (i) *K is a direct sum of three rings $\{K_i; i=1,2,3\}$. The ring K_1 is left locally uniform. The ring K_2 is ideal-locally uniform, but does not contain a uniform left ideal. The ring K_3 contains neither a uniform left ideal nor a uniform ideal. Such a representatiin of K is unique. And, the first component K_1 is uniquely represented as a complete direct sum of right*[5] *endomorphism rings of vector spaces (over division rings). The second component K_2 is uniquely represented as a complete direct sum of prime rings containing no uniform left ideals.*

(ii) *The center of K is also an injective ring with zero singular part.*

5) The "right" implies "acting on the right".

For any idempotent ν of K, $\nu K\nu$ is also a left injective ring with zero left singular part.

Proof. The first half was already proved. In fact $K_i = K\tau_i$ $(i=1,2,3)$. Since $\sum \oplus \mathrm{H}(\rho)$ is R-K-admissible and R-dense in M_1, $K_1 = \mathrm{Hom}_R(M_1, M_1) = \mathrm{Hom}_R(\sum \mathrm{H}(\rho), \sum \mathrm{H}(\rho))$, and further, since each $\mathrm{H}(\rho)$ is R-K-admissibie, $K_1 = \sum^c \oplus \mathrm{Hom}_R(\mathrm{H}(\rho), \mathrm{H}(\rho))$ (complete direct sum). Each $\mathrm{H}(\rho)$ is a direct sum of R-uniform submodules which are isomorphic to one another. Let V_ρ be a complemented uniform submodule belonging to ρ. Then $\mathrm{Hom}_R(\mathrm{H}(\rho), \mathrm{H}(\rho))$ is isomorphic to the ring of row-finite (ρ-dim M)-dimensional matrixes over the division ring $\mathrm{Hom}_R(V_\rho, V_\rho)$, that is, the right endomorphism ring of a (ρ-dim M)-dimensional $\mathrm{Hom}_R(V_\rho, V_\rho)$-left vector space. Let $\{W_\gamma : \gamma \in \Gamma\}$ be the maximal independent set of complemented R-K-uniform submodules of M_2. Since $\sum \oplus W_\gamma$ is R-K-admissible and (R-K-dense in M_2, and so) R-dense in M_2, $\mathrm{Hom}_R(M_2, M_2) = \mathrm{Hom}_R(\sum W_\gamma, \sum W_\gamma)$, and further $\mathrm{Hom}_R(M_2, M_2) = \sum^c \oplus \mathrm{Hom}_R(W_\gamma, W_\gamma)$. Since each W_γ is an R-c.q.i.-module (Th. 5.7), $\mathrm{Hom}_R(W_\gamma, W_\gamma)$ is a regular ring (Prop. 5.1). And, since W_γ is R-K-uniform and an R-K-direct summand of M, $\mathrm{Hom}_R(W_\gamma, W_\gamma)$ is an ideal-direct summand of K and an ideal-uniform (and regular) ring. Hence each $\mathrm{Hom}_R(W_\gamma, W_\gamma)$ is a prime ring. Let $K_1 \cong \sum^c_{\lambda\in\Lambda} \oplus K^*_{1\lambda}$ and $K_{1\lambda} \longleftrightarrow (0, \cdots, 0, K^*_{1\lambda}\ 0, \cdots, 0)$ (finite or infinite), where each $K^*_{1\lambda}$ is a left locally uniform ring. Then, to be easily seen, $\sum K_{1\lambda} = \sum \oplus K_{1\lambda}$ is a dense ideal (i.e. K_1-K_1-dense submodule) of K_1, and each $K_{1\lambda}$ is a complemented uniform ideal of K_1, because each $(0, \cdots, 0, K^*_{1\lambda}, 0, \cdots, 0)$ is a two-sided direct summand of $\sum^c \oplus K^*_{1\lambda}$. Hence $\{K_{1\lambda} ; \lambda \in \Lambda\}$ is a maximal independent set of complemented uniform ideals of K_1, which is uniquely determined. The uniqueness of the representation of K_2 is similarly proved. (ii) follows from Th. 5.3, Th. 5.6 and Th. 5.10.

Remark. Let $\{D_\gamma ; \gamma \in \Gamma\}$ be a collection of division rings, and let $\{\Lambda_\gamma ; \gamma \in \Gamma\}$ be a collection of sets. We denote by $D_\gamma^{(\Lambda_\gamma)}$ the direct sum of Λ_γ copies of the D_γ-left vector space D_γ. Then $D_\gamma^{(\Lambda_\gamma)}$ is a $\sharp\Lambda_\gamma$-dimensional D_γ-left vector space. And, $\mathrm{Hom}_{D_\gamma}(D_\gamma^{(\Lambda_\gamma)}, D_\gamma^{(\Lambda_\gamma)}) = \mathrm{End}\,({}_{D_\gamma}D_\gamma^{(\Lambda_\gamma)})$ acting on the right is (isomorphic to) the ring of row-finite $\sharp\Lambda_\gamma$-dimensional matrixes over D_γ. Next, we consider the $\sum \oplus D_\gamma$-left module $\sum \oplus D_\gamma^{(\Lambda_\gamma)}$, where $D_{\gamma_1} \cdot \sum \oplus D_{\gamma_2}^{(\Lambda_{\gamma_2})} = 0$, if $\gamma_1 \neq \gamma_2$. Then $\sum \oplus D_\gamma^{(\Lambda_\gamma)}$ is $\sum \oplus D_\gamma$-left completely reducible. Hence $\sum \oplus D_\gamma^{(\Lambda_\gamma)}$ is a $\sum \oplus D_\gamma$-c.q.i-module. Therefore the $\sum \oplus D_\gamma$-endomorphism ring acting on the right of $\sum \oplus D_\gamma^{(\Lambda_\gamma)}$ is a left injective ring with zero (left) singular part. This ring is a complete direct sum of right endomorphism rings of vector spaces $\{D_\gamma^{(\Lambda_\gamma)} ; \gamma \in \Gamma\}$, because for any $\sum \oplus D_\gamma$-endomorphism φ of $\sum \oplus D_\gamma^{(\Lambda_\gamma)}$, $D_\gamma^{(\Lambda_\gamma)}\varphi = (1_\gamma \cdot D_\gamma^{(\Lambda_\gamma)})\varphi = 1_\gamma \cdot (D_\gamma^{(\Lambda_\gamma)}\varphi) \subseteq D_\gamma^{(\Lambda_\gamma)}$, where 1_γ is the identity of D_γ.

Theorem 5.13. *If an R-c.q.i-module M is R-locally uniform and R-faithful, then the following conditions are equivalent to each other*:

(i) *M is K-quasi-injective and* $\mathrm{Hom}_K(M, M)=R$.

(ii) *The R-singular part of M is zero, and every R-uniform submodule is minimal. And further R is a complete direct sum of left endomorphism rings of vector spaces (over division rings).*

Proof. (i)$\Rightarrow$(ii). This part follows from Th. 5.2 (ii), Prop. 5.9 and Th. 5.12. (ii)$\Rightarrow$(i). By Cor. to Th. 5.2, M is K-locally uniform. Let $\dot{M}$ be the K-injective envelope of M. Then, as M is K-dense in $\dot{M}$, the K-socle M_0 of M coincides with the K-socle of $\dot{M}$, and further, by assumption, coincides with the R-socle of M (Th. 5.2 (iv)). Since M_0 is R-K-admissible and R-dense in M, $\mathrm{Hom}_R(M_0, M_0)=K$. We set $R'=\mathrm{Hom}_K(\dot{M}, \dot{M})$ acting on the left. Then, since M_0 is R'-K-admissible and K-dense in $\dot{M}$ and the K-singular part (of M is zero, and so) of $\dot{M}$ is zero, $R'=\mathrm{Hom}_K(\dot{M}, \dot{M})=\mathrm{Hom}_K(M_0, M_0)$ by Lemma 5.8 and the K-injectivity of $\dot{M}$. We shall prove that $R'=R$. Let Ru be R-minimal. Then $R/l_R(u)\cong Ru$. Let $\mathfrak{l}a\subseteq l_R(u)$ $(a\in R)$, where $\mathfrak{l}$ is a dense left ideal of R. Then $\mathfrak{l}au=0$, and, since R-singular part of M is zero, $au=0$, that is, $a\in l_R(u)$. Hence $l_R(u)$ $(\neq R)$ is a complemented left ideal of R (Lemma 5.8). Since $l_R(u)^c$ is dense in $R/l_R(u)$, Ru is naturally isomorphic to a minimal left ideal of R. Conversely, let $\mathfrak{l}_0$ be a minimal left ideal. Then, since M is R-faithful, $\mathfrak{l}_0u\neq 0$ for some $u\in M$. Evidently $\mathfrak{l}_0\cong\mathfrak{l}_0u(\subseteq M_0)$. Hence $M_0=S\cdot M=\sum_{\rho\in P}S_\rho\cdot M$, where $S=\sum_{\rho\in P}\oplus S_\rho$ is the (left) socle of R and each S_ρ s a (left) homogeneous component of R such that $S_\rho\cdot M=\mathrm{H}(\rho)$ $(\rho\in P)$. Let $\mathfrak{l}_\rho$ be a minimal left ideal such that $\mathfrak{l}_\rho\subseteq S_\rho$. Then, S_ρ and $S_\rho\cdot M$ are direct sums of $\mathfrak{l}_\rho$'s (up to isomorphism). From this fact, $\mathrm{End}^2({}_RS_\rho)\cong\mathrm{End}^2({}_R\mathfrak{l}_\rho)$[6] and $\mathrm{End}^2({}_RS_\rho\cdot M)\cong\mathrm{End}^2({}_R\mathfrak{l}_\rho)$, where $\mathrm{End}^2({}_RS_\rho)$ means the $\mathrm{End}({}_RS_\rho)$-endomorphism ring of S_ρ acting on the left. Now, since R is a regular ring, $(S\cap r(S))^2=0$ implies $S\cap r(S)=0$, and symmetrically $S\cap l(S)=0$. Since R is a right locally uniform regular ring, S is a dense right ideal. Hence $r(S)=l(S)=0$. Since R is a right injective ring with zero (right) singular part, $\mathrm{End}(S_R)=\mathrm{End}(R_R)=R_l$ (the left multiplications of elements of R). As $\mathrm{End}({}_RS)\supseteq R_r$, $(R_l\subseteq)$ $\mathrm{End}^2({}_RS)\subseteq\mathrm{End}(S_R)=R_l$. Hence $R_l=\mathrm{End}^2({}_RS)$. Since S_ρ and $S_\rho\cdot M$ are an ideal and

6) Let N be an Ω-left module, where Ω is any operator domain, and let $N^{(\Lambda)}$ be a direct sum of Λ copies of N, where Λ is a non-empty set. Then $\mathrm{End}^2({}_\Omega N^{(\Lambda)})=\mathrm{End}^2({}_\Omega N)$ naturally. To see this, let $\varepsilon_{\lambda\lambda'}$ be the Ω-endomorphism such that $(0,\cdots,0,\overset{\lambda}{u},0,\cdots,0)\rightarrow(0,\cdots,0,\overset{\lambda'}{u},0,\cdots,0)$. Then any $\varphi\in\mathrm{End}^2({}_\Omega N^{(\Lambda)})$ is commutative with every $\varepsilon_{\lambda\lambda'}$. Set $N_\lambda=(0,\cdots,0,\overset{\lambda}{N},0,\cdots,0)$. Since $\varphi(N_\lambda\cdot\varepsilon_{\lambda\lambda})=(\varphi N_\lambda)\varepsilon_{\lambda\lambda}\subseteq N_\lambda$, we have $\varphi N_\lambda\subseteq N_\lambda$. We correspond $\varphi|N_\lambda$ (the contraction of φ to N_λ) to an element φ_λ in $\mathrm{End}^2({}_\Omega N)$, naturally. Then, since φ is commutative with every $\varepsilon_{\lambda\lambda'}$, $\varphi_\lambda=\varphi_{\lambda'}$ for all $\lambda, \lambda'\in\Lambda$.

an R-K-submodule respectively, $R_l=\mathrm{End}^2({}_RS)\cong\sum_\rho^c\oplus\mathrm{End}^2({}_RS_\rho)$ and $\mathrm{End}^2({}_RS\cdot M)\cong\sum_\rho^c\oplus\mathrm{End}^2({}_RS_\rho\cdot M)$ naturally. As $\mathrm{End}^2({}_RS_\rho)\cong\mathrm{End}^2({}_RI_\rho)\cong\mathrm{End}^2({}_RS_\rho\cdot M)$,$R_l=\mathrm{End}^2({}_RS)\cong\mathrm{End}^2({}_RS\cdot M)$. Hence we have $\mathrm{End}^2({}_RS\cdot M)=R$, as desired. Since $\mathrm{End}^2({}_RS\cdot M)=\mathrm{Hom}_K(\dot{M}.\dot{M})$, this implies that M is K-quasi-injective (Prop. 4.1) and $\mathrm{Hom}_K(M,M)=R$.

§ 6. Throughout this section, we assume that R is a ring with 1 such that for each non-zero left ideal I, R/I contains a minimal R-left submodule.

Theorem 6.1. *Let A be an R-submodule of M. Then, the following conditions are equivalent to each other*:

(i) *A is a complemented submodule of M.*

(ii) *Let I be a maximal left ideal of R, u an element of M. If $Iu\subseteq A$ then there exists an element $v\in A$ such that $au=av$ for all $a\in I$.*

Proof. (i)$\Rightarrow$(ii). We may assume that u is not contained in A (and so, A is a proper complemented submodule of M). Now, $(Ru+A)/A$ is a minimal submodule of M/A. Since A is complemented, $(A+A^c)/A$ is dense in M/A by Prop. 1.13, and hence $(Ru+A)/A\subseteq(A+A^c)/A$, that is, $u\in A+A^c$. Setting $u=v+v'$ with $v\in A$ and $u'\in A^c$, $Iu\subseteq A$ yields $Iv'=0$. Hence, $I(u-v)=0$, that is, $au=av$ for all $a\in I$. (ii)$\Rightarrow$(i). Suppose $A^{cc}\supsetneqq A$, and choose an arbitrary $x\in A^{cc}$ not contained in A. As A is dense in A^{cc}, there exists a non-zero $a\in R$ with $(0\neq)$ $ax\in A$. Then, $R/L\cong(Rx+A)/A\subseteq A^{cc}/A$, where $L=\{b\in R;\ bx\in A\}$ is a non-zero left ideal of R, so that, by the assumption for R, A^{cc}/A contains a minimal submodule $(Ru+A)/A$ $(u\notin A)$. As $I=\{b\in R;\ bu\in A\}$ is a maximal left ideal and $Iu\subseteq A$, there exists an element $v\in A$ such that $I(u-v)=0$. $R(u-v)$ is then a minimal submodule of A^{cc}, and so $R(u-v)\subseteq A$ by the density of A in A^{cc}. But, $u-v\in A$ and $v\in A$ yield a contradiction $v\in A$. We have proved therefore $A^{cc}=A$.

Theorem 6.2. *M is R-injective if and only if every R-homomorphism of any maximal left ideal of R into M can be extended to an R-homomorphism of R into M.*

Proof. It suffices to prove the "if" part. To this end, we consider the R-injective envelope $\hat{M}$ of M. Let I be a maximal left ideal of R, and let $Iu\subseteq M$ for an element $u\in\hat{M}$. Since $I\ni a\to au\in M$ *is an* R-homomorphism φ of I into M, φ can be extended to an R-homomorphism ψ of R into M. If $1\psi=v\in M$, then $a(u-v)=a\varphi-a\psi=0$ for all $a\in I$. Hence, M is complemented (and dense) in $\hat{M}$ by Th. 6.1, which proves $M=\hat{M}$.

Theorem 6.3. *Let R be further a left principal ideal ring.*

(i) *A is a complemented submodule of M if and only if $A\cap pM=pA$*

for each $p \in R$ generating a maximal left ideal of R.

(ii) *If $pM = M$ for each $p \in R$ generating a maximal left ideal of R, then M is R-injective.* (Cf. [1; p. 92].)

Proof. (i). Let Rp be an arbitrary maximal left ideal of R. Then, the condition that if $Rpu \subseteq A$ $(u \in M)$ then $Rp(u-v) = 0$ for some $v \in A$ is equivalent to $A \cap pM = pA$. Hence, (i) follows immediately from Th. 6.1. (ii). Let $\hat{M}$ be the R-injective envelope of M. Since $M \cap p\hat{M} \subseteq M = pM$, M is complemented (and dense) in $\hat{M}$ by (i). Hence $M = \hat{M}$, as desired.

Example. Let Ja and Jb be (additive) cyclic groups of orders 4 and 2 repectively, where J denotes the ring of rational integers. We consider $M = Ja \oplus Jb$. Now, $J(a+b)$ is a complemented submodule by Th. 6.3. In fact, $J(a+b) \cap 2M = \{0, a+b, 2a, 3a+b\} \cap \{0, 2a\} = \{0, 2a\} = 2(J(a+b))$ and $p(J(a+b)) = J(a+b)$ for all prime $p \neq 2$. Ja is a direct summand and $Ja \cap J(a+b) = \{0, 2a\} = J(2a)$. But $J(2a)$ is not complemented in M, for $J(2a) \cap 2M = \{0, 2a\} \neq 0 = 2(J(2a))$. This elementary example shows that the d. c-correspondence is not always a closure operation.

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