ON THE DISTRIBUTION OF INTEGERS REPRESENTABLE AS A SUM OF TWO *h*-TH POWERS

By

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Our aim in this note is to present some elementary results concerning the distribution of integers which can be expressed as a sum of two h-th powers, where $h \geqq 2$ is a fixed integer.

1. According to P. Erdös [2], R. P. Bambah and S. Chowla [1] have proved that for some *sufficiently large* constant C the interval $(n, n+Cn^{\frac{1}{4}})$ always contains an integer of the form x^2+y^2, n, x and y being integral, and Erdös [2] coniectures (among others) that this holds for every C if $n \geqq n_0(C)$. We cannot, at present, prove this conjecture of Erdös, but it is possible to refine the result of Bambah and Chowla in the following form:

Theorem 1. *For every $n \geqq 1$ there are integers x, y with $xy \neq 0$ satisfying*

$$n < x^2 + y^2 < n + 2^{\frac{3}{2}} n^{\frac{1}{4}} .$$

Proof. For $n=1$ and $n=2$ the result is obvious. Assume now that $n \geqq 3$.

Let δ, $0<\delta<1$, be a fixed real number: the exact value of δ (which may depend on n) will be determined in a moment later.

Write

$$[n^{\frac{1}{2}}] = n^{\frac{1}{2}} - (1-\varepsilon) \qquad (0<\varepsilon \leqq 1) .$$

Here, and in what follows, $[t]$ denotes, as usual, the greatest integer not exceeding t.

We distinguish two cases.

Case 1: $0<\varepsilon\leqq\delta$. We take

$$x = [n^{\frac{1}{2}}]+1 , \qquad y = 1 .$$

Then we have

$$n < x^2 + y^2 = n + 2\varepsilon n^{\frac{1}{2}} + \varepsilon^2 + 1 < n + 2^{\frac{3}{2}} n^{\frac{1}{4}} ,$$

if

$$2\varepsilon n^{\frac{1}{2}}+\varepsilon^2+1<2^{\frac{3}{2}}n^{\frac{1}{4}}$$

or

$$(1) \qquad \delta^2+2\delta n^{\frac{1}{2}}-(2^{\frac{3}{2}}n^{\frac{1}{4}}-1)<0\,.$$

Case 2: $\delta<\varepsilon\leqq 1$. We put

$$x=[n]\,,\qquad y=\left[(n-[n^{\frac{1}{2}}]^2)^{\frac{1}{2}}\right]+1\,.$$

Then we have

$$n<x^2+y^2\leqq n+2\left(2(1-\varepsilon)n^{\frac{1}{2}}-(1-\varepsilon)^2\right)^{\frac{1}{2}}+1<n+2^{\frac{3}{2}}n^{\frac{1}{4}}\,,$$

if

$$2\left(2(1-\varepsilon)n^{\frac{1}{2}}-(1-\varepsilon)^2\right)^{\frac{1}{2}}+1<2^{\frac{3}{2}}n^{\frac{1}{4}}$$

or

$$(2) \qquad \delta^2+2\delta(n^{\frac{1}{2}}-1)-\left(2^{\frac{1}{2}}n^{\frac{1}{4}}-\frac{5}{4}\right)>0\,.$$

Now, let δ_0 be the (unique) positive zero of the quadratic equation

$$\delta_0^2+2\delta_0(n^{\frac{1}{2}}-1)-\left(2^{\frac{1}{2}}n^{\frac{1}{4}}-\frac{5}{4}\right)=0\,.$$

It is easy to see that $\delta_0<1$ and that

$$\delta_0^2+2\delta_0 n^{\frac{1}{2}}-(2^{\frac{3}{2}}n^{\frac{1}{4}}-1)=2\delta_0-\left(2^{\frac{1}{2}}n^{\frac{1}{4}}+\frac{1}{4}\right)<0$$

for $n\geqq 3$. Thus, we may take any δ less than 1 and slightly greater than δ_0, so that the inequalities (1) and (2) hold true simultaneously. This proves the theorem.

Corollary 1. *For every $\varepsilon>0$ the set of integers n for which the interval $(n, n+\varepsilon n^{\frac{1}{4}})$ contains an integer of the form x^2+y^2 has a positive density.*

Proof. For every δ, $0<\delta\leqq 1$, the set of integers n satisfying $n^{\frac{1}{2}}-\delta<[n^{\frac{1}{2}}]\leqq n^{\frac{1}{2}}$ is of positive density. It suffices to take $\delta=\delta(\varepsilon)$ small enough.

2. Here we wish to state two conjectures related to Theorem 1. They are:

Conjecture 1. *Let C_1 be a constant $>2^{-\frac{1}{4}}\cdot 3$. Then for all $n\geqq 1$ there are integers x, y with $xy\neq 0$ satisfying*

$$n<x^2+y^2<n+C_1n^{\frac{1}{4}}\ ;$$

and

Conjecture 2. *Let C_2 be a constant $>2^{-\frac{1}{2}}\cdot 5^{\frac{3}{4}}$. Then for all $n\geqq 1$ there are integers x, y satisfying*

$$n<x^2+y^2<n+C_2n^{\frac{1}{4}}\ .$$

Either of these conjectures, if true, is the best possible in the sense that if $C_1=2^{-\frac{1}{4}}\cdot 3$ or $C_2=2^{-\frac{1}{2}}\cdot 5^{\frac{3}{4}}$ then the corresponding result cannot be correct any longer (to see this we put, for instance, $n=2$ or $n=20$). We note also that our Conjectures 1 and 2 have been verified by M. Uchiyama up to $n=1000$.

3. For a general $h\geqq 2$ we shall mention the following rather trivial

Theorem 2. *Let $g(n)$ be a function of n satisfying the inequality*

$$g(n)>\sum_{j=1}^{h}\binom{h}{j}(n-[n^{1/h}]^h)^{(h-j)/h}$$

for $n\geqq n_0$. Then there exist integers x, y with $xy\neq 0$ such that

$$n<x^h+y^h<n+g(n)$$

for all $n\geqq n_0$.

Proof. Put

$$x=[n^{1/h}],\qquad y=\left[(n-[n^{1/h}]^h)^{1/h}\right]+1\ .$$

Corollary 2. *For any $\varepsilon>0$ there is an $n_0=n_0(\varepsilon)$ such that for all $n\geqq n_0$ there exist integers x, y with $xy\neq 0$ satisfying*

$$n<x^h+y^h<n+(c+\varepsilon)n^a\ ,$$

where

$$a=\left(1-\frac{1}{h}\right)^2,\qquad c=h^{(2h-1)/h}\ .$$

For $h=2$ this is of course weaker than Theorem 1.

Corollary 3. *For every $\varepsilon>0$ the set of integers n for which the interval $(n, n+\varepsilon n^a)$, with $a=\left(1-\frac{1}{h}\right)^2$, contains an integer of the form x^h+y^h has a positive density.*

Proof is similar to that of Corollary 1.

References

[1]* R. P. BAMBAH and S. CHOWLA: On numbers which can be expressed as a sum of two squares, Proc. Nat. Inst. Sci. India, vol. 13 (1947), pp. 101–103.

[2] P. ERDÖS: Some unsolved problems, Publ. Math. Inst. Hungar. Acad. Sci., vol. 6, ser. A (1961), pp. 221–254.

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* The writer has been unable to consult this paper.