A NOTE ON HOCHSCHILD COHOMOLOGY GROUPS

By

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Let K be a commutative ring with identity element, and let Λ be a K-algebra, that is, an algebra over K with identity element 1. We denote by Λ^* the opposite algebra of Λ, which is in an opposite-isomorphism $\lambda \rightarrow \lambda^*$ with Λ. Any right Λ-module is converted into a left Λ^*-module (and conversely) by setting the left multiplication of λ^* to be the right multiplication of λ. Furthermore, every two-sided Λ-module is, and in particular Λ is, regarded as a left module for the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^*$[1] in the natural manner.

Let A be a two-sided Λ-module. Hochschild defined in [2] the cohomology groups of Λ for A as the homology groups $H^n(\Lambda, A)$ of the complex whose components are the cochain groups $C^n(\Lambda, A)$, i.e., the module of all K-multilinear mappings $f = f(\lambda_1, \cdots, \lambda_n)$ of Λ into A, and whose differentiation operators δ^n: $C^n(\Lambda, A) \rightarrow C^{n+1}(\Lambda, A)$ are defined by

$$(\delta^n f)(\lambda_1, \cdots, \lambda_{n+1}) = \lambda_1 f(\lambda_2, \cdots, \lambda_{n+1}) +$$

$$\sum_{i=1}^{n} (-1)^i f(\lambda_1, \cdots, \lambda_i \lambda_{i+1}, \cdots, \lambda_{n+1}) + (-1)^{n+1} f(\lambda, \cdots, \lambda_n) \lambda_{n+1}.$$

Here, the two-sided Λ-module A needs not be assumed to be unital (i.e., the identity element 1 of Λ does not necessarily act on A as the identity-operator on both left and right hands), but we may replace A by the unital module $1A1$ to obtain the same cohomology groups according to Hchschild [3], Th. 1: $H^n(\Lambda, A) \cong H^n(\Lambda, 1A1)$. On the other hand, Cartan and Eilenberg gave in [1] another definition of cohomology groups for a unital two-sided Λ-module A; namely, they called $\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$ the n-th cohomology group of Λ for A. The defined two groups $H^n(\Lambda, A)$ and $\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$, for unital A, coincide (that is, are naturally equivalent as functors of A) always for $n=0$, 1 ([1], Chap. IX, Prop. 4.1.). But this is not the case in general for $n>1$. It is shown in [1], Chap. IX, §6 that the both groups coincide if Λ is K-projective. In this note, we shall however generalize this to prove that the both groups coincide whenever Λ^e is projective as a right Λ-module. In this connection, it may be of some interest to give in Proposition 2 below a homological significance of the Hochschild two-sided Λ-module $\mathrm{Hom}_K(\Lambda, A)$ introduced in [2].

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1) We shall mean by the mere $\otimes$ the tensor product over K; thus, $\Lambda \otimes \Lambda^* = \Lambda \otimes_K \Lambda^*$.

Now, consider three K-algebras Λ, Γ, Σ. Let A, B, C be respectively a unital two-sided Λ-Γ-, Γ-Σ-, Λ-Σ-module; they are also regarded respectively as a left $\Lambda\otimes\Gamma^*$-, $\Gamma\otimes\Sigma^*$- $\Lambda\otimes\Sigma^*$-module in the natural manner. Then there is a well-known natural isomorphism ([1], Chap. IX, Prop. 2.2)

$$(1)\qquad \mathrm{Hom}_{\Lambda\otimes\Gamma^*}(A, \mathrm{Hom}_{\Sigma}(B, C)) \cong \mathrm{Hom}_{\Lambda\otimes\Sigma^*}(A\otimes_{\Gamma}B, C).$$

On the basis of this isomorphism, we have the following analogy of [1], Chap. IX, Th. 2.8a:

Proposition 1. *In the situation* $({}_{\Lambda}A_{\Gamma}, {}_{\Gamma}B_{\Sigma}, {}_{\Lambda}C_{\Sigma})$ *assume that* A *is* Γ-*projective,* B *is* Σ-*projective,* and $\Lambda\otimes\Gamma^*$ *is (left)* Γ^*-*projective*[2]. *Then there is a natural isomorphism*

$$\mathrm{Ext}^n_{\Lambda\otimes\Gamma^*}(A, \mathrm{Hom}_{\Sigma}(B, C)) \cong \mathrm{Ext}^n_{\Lambda\otimes\Sigma^*}(A\otimes_{\Gamma}B, C).$$

Proof. Let X be a $\Lambda\otimes\Gamma^*$-projective resolution of A. Then, since $\Lambda\otimes\Gamma^*$ is Γ^*-projective, X is also a (Γ^*- or) Γ-projective resolution of A. Hence the homology group of $X\otimes_{\Gamma}B$ is $\mathrm{Tor}^{\Gamma}(A, B)$. But, since A is Γ-projective, $\mathrm{Tor}^{\Gamma}_n(A, B)=0$ for $n>0$. Further, since B is Σ-projective, $X\otimes_{\Gamma}B$ is $\Lambda\otimes\Sigma^*$-projective and so a $\Lambda\otimes\Sigma^*$-projective resolution of $A\otimes_{\Gamma}B$ by [1], Chap. IX, Prop. 2.3. Thus, replacing in (1) A by X and taking the homology group, we have the desired isomorphism.

As the particular case where $\Lambda=\Gamma=\Sigma$ and $A=\Lambda$, we obtain.

Corollary. *Let* B *and* C *be unital two-sided* Λ-*modules, and assume that* B *is right* Λ- *(or left* Λ^*-*) projective and* Λ^e *is left* Λ^*-*projective. Then there is a natural isomorphism*

$$\mathrm{Ext}^n_{\Lambda^e}(\Lambda, \mathrm{Hom}_{\Lambda^*}(B, C)) \cong \mathrm{Ext}^n_{\Lambda^e}(B, C).$$

We next consider a Λ^e-epimorphism $\rho: \Lambda^e\to\Lambda$ which is defined by $\rho(\lambda\otimes\mu^*)$ $(\lambda\otimes\mu^*)1=\lambda\mu$. Let J be the kernel of ρ. Then J is a left ideal of Λ^e, and we have an exact sequence of left Λ^e-modules

$$(2)\qquad 0\longrightarrow J\xrightarrow{\iota}\Lambda^e\xrightarrow{\rho}\Lambda\longrightarrow 0,$$

where ι means the imbedding mapping. Hence, to every unital two-sided Λ-module A, there corresponds, for each $n>1$, an exact sequence

$$\mathrm{Ext}^{n-1}_{\Lambda^e}(\Lambda^e, A)\longrightarrow\mathrm{Ext}^{n-1}_{\Lambda^e}(J, A)\longrightarrow\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)\longrightarrow\mathrm{Ext}^n_{\Lambda^e}(\Lambda^e, A).$$

But, since Λ^e is Λ^e-projective, the first and the last terms $=0$, and thus we have a natural isomorphism

(2) We regard $\Lambda\otimes\Gamma^*$ as a left Γ^*-module by setting the left multiplication of γ^* as the left multiplication of $1\otimes\gamma^*$.

(3) $$\mathrm{Ext}^{n-1}_{\Lambda^e}(J, A) \cong \mathrm{Ext}^n_{\Lambda^e}(\Lambda, A), \qquad n>1.$$

On the other hand, since (2) is exact as left Λ^*-modules too and since Λ is Λ^*-projective, (2) must splits, i.e., $\Lambda^e = J \oplus \Lambda^*$. It follows therefore that if Λ^e is Λ^*-projective then J is also Λ^*-projective. Thus, there is, by the above corollary, a natural isomorphism

$$\mathrm{Ext}^n_{\Lambda^e}(\Lambda, \mathrm{Hom}_{\Lambda^*}(J, A)) \cong \mathrm{Ext}^n_{\Lambda^e}(J, A),$$

if Λ^e is Λ^*-projective. Now, replacing here n by $n-1$ and comparing with (3), we obtain the following reduction theorem:

Theorem 1. *Let A be a unital two-sided Λ-module, and assume that Λ^e is left Λ^*-projective. Then there is a natural isomorphism*

$$\mathrm{Ext}^{n-1}_{\Lambda^e}(\Lambda, \mathrm{Hom}_{\Lambda^*}(J, A)) \cong \mathrm{Ext}^n_{\Lambda^e}(\Lambda, A), \qquad n>1.$$

Now, let A be any two-sided Λ-module. Hochschild converted $\mathrm{Hom}_K(\Lambda, A)$ into a two-sided Λ-module by setting, for any $f \in \mathrm{Hom}_K(\Lambda, A)$ and $\lambda \in \Lambda$, λf and $f\lambda$ as the mappings $\mu \to \lambda f(\eta)$ and $\mu \to f(\lambda\mu) - f(\lambda)\mu$, $\mu \in \Lambda$, respectively ([2], § **1**). Since Λ is K-unital we have $1f(\mu) = f(\mu) = f(\mu)1$ for every $f \in \mathrm{Hom}_K(\Lambda, A)$ and $\mu \in \Lambda$, and this implies first that $\mathrm{Hom}_K(\Lambda, A)$ is unital as a left Λ-module. However, it is not necessarily unital as a right Λ-module (even if A is unital), and in fact $\mathrm{Hom}_K(\Lambda, A)\cdot 1$ consists of those f in $\mathrm{Hom}_K(\Lambda, A)$ which satisfy $f(1)=0$, because if $f = g1$ with some $g \in \mathrm{Hom}_K(\Lambda, A)$ then $f(1) = g(1) - g(1)1 = 0$ and conversely if $f(1)=0$ then $f(\mu) - f(1)\mu = f(\mu)$ for all $\mu \in \Lambda$, showing $f1 = f$.

Proposition 2. *Let A be a two-sided Λ-module and* $\mathrm{Hom}_K(\Lambda, A)$ *the Hochschild two-sided Λ-module. Let $\varphi: \Lambda \to J$ be the K-homomorphism defined by $\varphi(\lambda) = \lambda \otimes 1 - 1 \otimes \lambda^*$. Then, by associating with each $h \in \mathrm{Hom}_{\Lambda^*}(J, A)$ the product mapping $h \cdot \varphi \in \mathrm{Hom}_K(\Lambda, A)$, we obtain a natural isomorphism $\mathrm{Hom}_{\Lambda^*}(J, A) \cong \mathrm{Hom}_K(\Lambda, A) \cdot 1$ as two-sided Λ-modules.*

Proof. Let $h \in \mathrm{Hom}_{\Lambda^*}(J, A)$ and put $f = h \cdot \varphi$. Then, since $\varphi(1) = 0$, it follows that $f(1) = h(\varphi(1)) = h(0) = 0$. Further, for any λ, μ in Λ, we have $\lambda f(\mu) = \lambda h(\mu \otimes 1 - 1 \otimes \mu^*)$, i.e., $\lambda f = (\lambda h) \cdot \varphi$, while $f(\lambda\mu) - f(\lambda)\mu = f(\lambda\mu) - \mu^* f(\lambda) = h((\lambda\mu) \otimes 1 - 1 \otimes (\lambda\mu)^*) - h((1 \otimes \mu^*)(\lambda \otimes 1 - 1 \otimes \lambda^*)) = h((\lambda\mu) \otimes 1 - \lambda \otimes \mu^*) = h((\lambda \otimes 1)(\mu \otimes 1 - 1 \otimes \mu^*))$, i.e., $f\lambda = (h\lambda) \cdot \varphi$. These together show that the mapping $h \to h \cdot \varphi$ gives a homomorphism $\mathrm{Hom}_{\Lambda^*}(J, A) \to \mathrm{Hom}_K(\Lambda, A) \cdot 1$ as two-sided Λ-modules. Now J is, as a left Λ^*-module, generated by the elements $\varphi(\lambda)$, $\lambda \in \Lambda$, because $\sum \lambda_i \otimes \mu_i^* \in J$ means $\sum \lambda_i \mu_i = 0$ whence $\sum \lambda_i \otimes \mu_i^* = \sum \lambda_i \otimes \mu_i^* - 1 \otimes (\sum \lambda_i \mu_i)^* = \sum (\lambda_i \otimes \mu_i^* - 1 \otimes (\mu_i^* \lambda_i^*)) = \sum (1 \otimes \mu_i^*)(\lambda_i \otimes 1 - 1 \otimes \lambda_i^*)$ (cf. [1], Chap. IX, Prop. 3.1), which implies that the mapping $h \to h \cdot \varphi$ is one-to-one. In order to see that the mapping is moreover an epimorphism, take any f from $\mathrm{Hom}_K(\Lambda, A) \cdot 1$. As is easily seen, there exists a K-homomorphism $\bar{h}: \Lambda^e \to A$ such that $\bar{h}(\lambda\otimes\mu^*)=(f(\lambda)\mu=)\mu^*f(\lambda)$, and $\bar{h}$ is necessarily a Λ^*-homomorphism because $\bar{h}((1\otimes\nu^*)(\lambda\otimes\mu^*))=\bar{h}(\lambda\otimes(\mu\nu)^*)=(\mu\nu)^*f(\lambda)=\nu^*\mu^*f(\lambda)=\nu^*\bar{h}(\lambda\otimes\mu^*)$. Let h be the restriction of $\bar{h}$ to J. Then $h\in\mathrm{Hom}_{\Lambda^*}(J, A)$ and $h(\lambda\otimes 1-1\otimes\lambda^*)=\bar{h}(\lambda\otimes 1)-\bar{h}(1\otimes\lambda^*)=f(\lambda)1-\lambda^*f(1)=f(\lambda)$, i.e., $h\cdot\varphi=f$. This completes our proof.

We can now prove

Theorem 2. *Let A be a unital two-sided Λ-module and $H^n(\Lambda, A)$ the n-th Hochschild cohomology group of Λ for A. Suppose that $\Lambda^e=\Lambda\otimes\Lambda^*$ is left Λ^*-projective (or equivalently, right Λ-projective)[3]. Then there is a natural isomorphism $H^n(\Lambda, A)\cong\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$.*

Proof. The theorem is true for $n=0, 1$ by [1], Chap. IX, Prop. 4.1 (even without assuming the Λ^*-projectivity of Λ^e). So we may assume $n>1$, and we suppose the theorem is true for $n-1$ instead of n and for all (unital) A. Then in particular

$$(4)\qquad H^{n-1}(\Lambda, \mathrm{Hom}_{\Lambda^*}(J, A))\cong\mathrm{Ext}^{n-1}_{\Lambda^e}(\Lambda, \mathrm{Hom}_{\Lambda^*}(J, A)).$$

By Theorem 1 the right side of (4) is naturally isomorphic to $\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$. On the other hand, the left side of (4) is by Proposition 2, whence, by [3], Th. 1, naturally isomorphic to $H^{n-1}(\Lambda, \mathrm{Hom}_K(\Lambda, A))$, and moreover this is isomorphic to $H^n(\Lambda, A)$ according to [2], Th. 3.1. Thus, $H^n(\Lambda, A)\cong\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$, and by induction this proves our theorem completely.

References

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2) G. HOCHSCHILD: On the cohomology groups of an associative algebra, Ann. of Math. vol. 46 (1945) pp. 58–67.
3) G. HOCHSCHILD: On the cohomology theory for associative algebra, Ann. of Math. vol. 47 (1946) pp. 568–579.

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3) The mapping $\lambda\otimes\mu^*\to\mu\otimes\lambda^*$ defines an involution of Λ^e, and by this mapping $\nu^*(\lambda\otimes\mu^*)$ is mapped on $(\mu\otimes\lambda^*)\nu=(\mu\nu)\otimes\lambda^*$. This shows that Λ^e is left Λ^*-projective if and only if it is right Λ-projective.