REMARKS ON COMPLETENESS OF CONTINUOUS FUNCTION LATTICE

By

Takashi ITÔ

Let E be an arbitrary topological space and $C(E)$ be a vector lattice of all real valued continuous functions on E. In general the lattice $C(E)$ is neither conditionally complete[1] nor conditionally σ-complete[2]. H. Nakano shows in [1] that a sufficient condition for $C(E)$ to be conditionally σ-complete (conditionally complete) is that E is *σ-universal* (*universal*), that is, every open F_σ-set has an open closure (every open set has an open closure) (cf. [2] Chap. VII, Theorem 41.1, Theorem 41.4). Under the assumption that E is normal (completely regular) σ-universality (universality) of E is a necessary condition for $C(E)$ to be conditionally σ-complete (conditionally complete). L. Gillman and M. Jerison in their book [3] show that for a completely regular space E the necessary and sufficient condition for $C(E)$ to be conditionally σ-complete is that E is *basically disconnected*, that is, every cozero-set[3] has an open closure ([3] p. 51, 3N). In this note we shall remark the necessary and sufficient topological condition for $C(E)$ on an arbitrary topological space E to be conditionally σ-complete or conditionally complete.

In the sequel a cozero-set P of $f \in C(E)$ will be denoted by $P(f)$; $P(f) = \{x \mid f(x) \neq 0\} = \{x \mid |f|(x) > 0\}$.

Theorem 1. *$C(E)$ is a conditionally σ-complete lattice if and only if the following two conditions are satisfied*

a) *there exists the smallest open-closed set $U(P)$ containing P for any cozero-set P.*

b) *if $P_1 \frown P_2 = \phi$ for two cozero-sets P_1 and P_2, then $U(P_1) \frown U(P_2) = \phi$.*

Proof. Suppose $C(E)$ is conditionally σ-complete and P is a cozero-set of some $f \in C(E)$, $P = P(f)$, then by the conditional σ-completeness of $C(E)$ f gives the orthogonal decomposition of the constant function $\mathbf{1}$ as follows

$$\mathbf{1} = [f]\mathbf{1} + [f]^{\perp}\mathbf{1}$$

1) every family with an upper bound in $C(E)$ has a supremum in $C(E)$.
2) every countable family with an upper bound in $C(E)$ has a supremum in $C(E)$.
3) $\{x \mid f(x) = 0\}$ is a zero-set of $f \in C(E)$, cozero-set is a complement of a zero-set.

where $[f]\mathbf{1}=\bigcup_{n=1}^{\infty}(\mathbf{1}_\frown n|f|)$ and $[f]^\perp\mathbf{1}=\mathbf{1}-[f]\mathbf{1}$. $[f]\mathbf{1}_\frown[f]^\perp\mathbf{1}=0$ implies that $[f]\mathbf{1}$ is a characteristic function χ_U for some open-closed set U, and $|f|_\frown[f]^\perp\mathbf{1}=0$ implies $U\supset P(f)$. The fact that $[f]\mathbf{1}_\frown|g|=0$ for all $g\in C(E)$ such that $|f|_\frown |g|=0$ shows that U is the smallest open-closed set containing $P(f)$. If $P_1{}_\frown P_2=\phi$ for cozero-sets $P_1=P(f_1)$ and $P_2=P(f_2)$, then by $|f_1|_\frown|f_2|=0$ we have $[f_1]\mathbf{1}_\frown[f_2]\mathbf{1}=0$, namely $U(P_1)$ and $U(P_2)$ are disjoint from the above argument $\chi_{U(P_1)}=[f_1]\mathbf{1}$ and $\chi_{U(P_2)}=[f_2]\mathbf{1}$.

Conversely, let a) and b) satisfied. To prove the conditional σ-completeness of $C(E)$ it is sufficient to show the existence of an infimum $\bigcap_{n=1}^{\infty} f_n$ for any sequence $\{f_n\}$ of non-negative continuous functions. If we put $E_\alpha^{(n)}=\{x\,|f_n(x)<\alpha\}$ and $E_\alpha=\bigcup_{n=1}^{\infty}E_\alpha^{(n)}$ for all $\alpha>0$, then obviously E_α is a cozero-set of a continuous function $g_\alpha=\sum_{n=1}^{\infty}\frac{1}{2^n}(\alpha\mathbf{1}-f_n)^+$ [4]. Hence from a) we can find the smallest open-closed set U_α containing E_α $(\alpha>0)$. We have then

$$(1)\qquad U_\alpha\supset U_\beta\ (\alpha>\beta>0),\qquad\qquad (2)\qquad \bigcup_{\alpha>0}E_\alpha=E\,.$$

If we put $f_0(x)=\inf_{x\in U_\alpha}\alpha$ $(x\in E)$, then by (2) f_0 is a non-negative real valued function on E, and by (1) we see

$$\{x|f_0(x)<\alpha\}=\bigcup_{\alpha>\beta>0}U_\beta,\qquad\qquad \{x|f_0(x)\leqq\alpha\}=\bigcap_{\beta>\alpha}U_\beta\ (\alpha>0)\,.$$

This implies the continuity of f_0. Since $E_\alpha^{(n)}\subset E_\alpha\subset U_\alpha\subset\{x|f_0(x)\leqq\alpha\}$ $(n=1,2,\cdots;\ \alpha>0)$, we have $f_n\geqq f_0$ $(n=1,2,\cdots)$, that is, f_0 is a lower bound of $\{f_n\}$. And if $f_n\geqq g\geqq 0$ $(n=1,2,\cdots)$, for some $g\in C(E)$, then we have $E_\alpha{}_\frown\{x|g(x)>\alpha\}=\phi$ $(\alpha>0)$. Hence from the assumption b) we see $U_\alpha{}_\frown\{x|g(x)>\alpha\}=\phi$ $(\alpha>0)$, and so $\{x|f_0(x)<\alpha\}\subset U_\alpha\subset\{x|g(x)\leqq\alpha\}$ $(\alpha>0)$. hence $g\geqq f_0$. Therefore f_0 is an infimum of $\{f_n\}$.

Similarly it is easy to give a necessary and sufficient condition for $C(E)$ to be a conditionally complete lattice.

Let $C(E)$ be conditionally complete, then E satisfies the following condition c) in addition to a) and b) in Theorem 1;

c) *all of open-closed sets of E constitutes a complete lattice.*

In fact, let U_λ $(\lambda\in\Lambda)$ be any system of open-closed sets in E, then since χ_{U_λ} $(\lambda\in\Lambda)$ has an infimum $f=\bigcap_{\lambda\in\Lambda}\chi_{U_\lambda}$ and a supremum $g=\bigcup_{\lambda\in\Lambda}\chi_{U_\lambda}$ in $C(E)$, easily it is shown that $U(P(f))$ and $U(P(g))$ are respectively an infimum and a supremum of U_λ $(\lambda\in\Lambda)$ in all of open-closed sets of E. Conversely, suppose E

4) the positive part of $\alpha\mathbf{1}-f_n$; $(\alpha\mathbf{1}-f_n)\cup 0$.

satisfies a), b) and c), to show the conditional completeness of $C(E)$ only a slight modification of the definition of U_α in the proof of Theorem 1 is necessary. Namely, for any system $f_\lambda \geqq 0$ $(\lambda \in \Lambda)$ of non-negative continuous functions, putting $E_\alpha^{(\lambda)} = \{x \mid f_\lambda(x) < \alpha\}$ and $E_\alpha = \bigcup_{\lambda \in \Lambda} E_\alpha^{(\lambda)}$ $(\alpha > 0)$, by the condition c) we can define U_α as the smallest open-closed set containing E_α; U_α is the supremum of $U_\alpha^{(\lambda)}$ $(\lambda \in \Lambda)$ in all of open-closed sets, where $U_\alpha^{(\lambda)}$ is the smallest open-closed set containing $E_\alpha^{(\lambda)}$.

Theorem 2. *$C(E)$ is a conditionally complete lattice if and only if E satisfies the condions* a), b) *and* c).

Finally we shall remark an extension theorem. If we replace cozero-sets in a) and b) by open F_σ-sets;

a′) *there exists the smallest open-closed set $U(F)$ containing F for any open F_σ-set F,*

b′) *if $F_1 \frown F_2 = \phi$ for two open F_σ-sets F_1 and F_2, then $U(F_1) \frown U(F_2) = \phi$,* then we have a purely topological sufficient condition for $C(E)$ to be conditionally σ-complete. Obviously it is weaker than σ-universality in [1].

Under the assumptions a′) and b′) we have a following extension theorem which is a slight generalization of Theorem 41.2 of [2].

Suppose E satisfies a′) *and* b′), *then a continuous function φ defined on an open F_σ-set F has a continuous extension ψ over E, provided ψ may take values $+\infty$ and $-\infty$.*

To prove this it is sufficient to show that φ has a continuous extension over $U(F)$. Putting $F_\alpha = \{x \mid \varphi(x) < \alpha\}$ $(+\infty > \alpha > -\infty)$ since F is an open F_σ-set in E, F_α is also an open F_σ-set in E. Hence by a′) we can find the smallest open-closed set U_α containing F_α for all α, and $\{U_\alpha\}$ has the properties $U_\alpha \supset U_\beta$ $(\alpha > \beta)$ and $U(F) \supset \bigcup_{+\infty > \alpha > -\infty} U_\alpha \supset F$. Similarly to the latter part of the proof of Theorem 1 we define $\psi(x) = \inf_{x \in U_\alpha} \alpha$ $\left(x \in \bigcup_{+\infty > \alpha > -\infty} U_\alpha\right)$, $\psi(x) = +\infty$ $\left(x \in U(F) - \bigcup_{+\infty > \alpha > -\infty} U_\alpha\right)$, then we see easily ψ is a continuous function on $U(F)$ and $\varphi \geqq \psi$ on F. Since by b′) $\{x \mid \varphi(x) > \alpha\} \frown F_\alpha = \phi$ implies $\{x \mid \varphi(x) > \alpha\} \frown U_\alpha = \phi$, we have $\psi \geqq \varphi$ on F.

References

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(Received July 16, 1963)