AN INVERSE THEOREM OF GROSS'S STAR THEOREM

Dedicated to Prof. Kinjiro Kunugi on his 60th birthday

By

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Let $w=w(z)$ be an analytic function of z in a Riemann surface R whose values fall on the w-sphere. Let $z=z^{-1}(w)$ be its inverse. Let $e(w, w_0)$ be an arbitrary regular element of $z^{-1}(w)$. We continue analytically $e(w, w_0)$, using only regular element (without any algebraic element) along every ray: $\arg(w-w_0)=\theta$ $(0\leqq\theta<2\pi)$ toward infinity. Then, there arise two cases whether the continuation defines a singularity ω_θ in a finite distance or not, in the former case, we call the ray a singular ray. For each singular ray: $\arg(w-w_0)=\theta$, we exclude the segment between the singularity ω_θ and $w=\infty$ from the w-plane. The remaining domain Ω is clearly a (single valued) regular branch of $z=z^{-1}(w)$. Let $\rho=\rho(\theta)$ the polar coordinate of the singularity ω_θ or ∞ according as the singular ray exists or not. Then $\rho(\theta)$ is clearly lower semicontinuous and $S_n=E[\theta:\rho(\theta)\leqq n]$ is closed. We call the set $E[\theta:\rho(\theta)<\infty]$ the singular set S of Ω. Then by $S=\sum_{n=1}^{\infty} S_n$ S is an F_σ set. Then the famous Gross's Star Theorem is as follows:

Theorem. *Let R be a domain such that $R=E[z:|z|<\infty]$ in the z-plane and let $f(z)$ be an analytic function of $z\in R$ whose values fall on the w-plane. Let Ω be a star domain. Then S is a set of linear measure zero.*

This theorem was extended by M. Tsuji[1] to the case: R is a domain in the z-plane such that the boundary of R is a set of capacity zero and also extended by Z. Yûjôbo[2] to the case: R is a Riemann surface with null-boundary. The method used by them is essentially the same as used by W. Gross. On the other hand, T. Yoshida[3] showed that the Gross's theorem holds for not only conformal mappings but also for quasiconformal mappings

1) M. TSUJI: Theory of meromorphic functions in a neighbourhood of a closed set of capacity zero, Jap. Journ. Math., 19 (1944–1948).

2) Z. YUJÔBÔ: On the Riemann surfaces, no Green function of which exists, Math. Japonicae, 2 (1951).

3) T. YOSHIDA: On the behaviour of a pseudo-regular functions in a neighbourhood of a closed set of capacity zero, Proc. Japan Acad., 26 (1950).

and M. Ohtsuka[4] extended the class of conformal mappings to a little wider class than quasiconformal mappings in which the Gross's theorem holds. Further we proved that there exists a Riemann surface $R \in O_{HP}$[5] such that the covering surface over the w-plane (mapped by an analytic function $w=w(z)$: $z \in R$) has not Gross's property (singular set of Ω is $|w|=1$) and also there exists a domain $D \in O_{AB}$[6] in the z-plane such that ∂D is a set of linear measure zero on a straight and its covering surface (mapped by an analytic function in D) has not Gross's property. Above two examples show that the validity of the Gross's theorem depends on the size of the boundary (boundary of R must be so small that R has null-boundary) but on the complexity of the boundary. In the present paper we consider an inverse of Gross's theorem i. e. to consider "how to construct a covering surface for given singular set ?".

Let $F_i(i=1,2,\cdots)$ be a closed set on $|w|=1$. If $\text{dist}(F_i, \sum_{j\neq i} F_j)>0$, we call $\sum F_i$ a *discrete* F_σ set. We shall prove

Theorem. *Let S be an arbitrary discrete F_σ set of linear measure zero on $|w|=1$. Then we can construct a covering surface $\mathfrak{R}$ which is conformally equivalent to a planer domain with null-boundary such that $\mathfrak{R}$ has a star domain Ω whose singular set is S.*

At present we cannot prove the above theorem under the condition that the connectivity of $\mathfrak{R}$ is one. We suppose that the above theorem is valid for arbitrary F_σ but it is complicated too much to construct a covering surface for any F_σ. Now by this theorem we know that *Gross's theorem cannot be improved for Riemann surface of connectivity* ∞ but it remains the problem: *Is the singular set S of a star domain of a covering surface (which is conformally equivalent to $|z|<\infty$) smaller than sets of measure zero*?

I. *Extension of $\mathcal{L}$ through D.* Let $\mathcal{L}$ be a leaf identical to the whole w-plane. Let D be a circular echelon

$$D: \qquad R e^{-\alpha} < |w| < R, \quad -\frac{\theta}{2} < \arg w < \frac{\theta}{2}, \quad \theta < \pi, \quad \alpha > 0.$$

Let $U(w)$ be a C_1-function[7] in D such that $U(w)=2(\alpha+\log R-\log|w|)/\alpha$ in

4) M. OHTSUKA: Thérèmes ètoilées de Gross et leurs applications, Annales de L'institut Fourier, V (1953–1954).

5) Z. KURAMOCHI: On covering properties of abstract Riemann surfaces, Osaka Math. Journ., 6 (1954).

6) Z. KURAMOCHI: On the behaviour of analytic functions on abstract Riemann surfaces, Osaka Math. Journ., 7 (1955).

7) If a continuous function has partial derivatives almost everywhere, we call it a C_1 function.

the part $D': Re^{-\alpha} < |w| < Re^{-\frac{\alpha}{2}}$ of D and $U(w)=1$ in the part $D'': R > |w| \geqq Re^{-\frac{\alpha}{2}}$ of D. Let $V(w)$ be a harmonic function in the complementary set of $D+C: C=E[w: |w| \leqq 1]$ such that $V(w)=0$ on ∂C and $V(w)=U(w)$ on ∂D. Then $D(V(w))$ depends on $U(w)$ and D. Let $\tilde{U}(w)$ be a C_1-function in the complementary set of C such that $\tilde{U}(w)=U(w)$ in D and $U(w)=0$ on ∂C. Then by the Dirichlet principle

$$D(\tilde{U}(w)) \geqq \underset{D}{D}(U(w)) + \underset{CD}{D}(V(w)) .$$

As $\theta \to 0$, i. e. D becomes narrow, $D(U(w)) \to 0$ but $D(V(w))$ does not tend to zero. Now we shall prove the following

Lemma. *Let C and D and $U(w)$ be as above. We can construct a closed Riemann surface $\mathfrak{R}_D$, covering surface of a finite number of sheets $\mathcal{L}, \mathcal{L}_1, \cdots, \mathcal{L}_{n+1}$ over the w-plane of genus 0 satisfying the following conditions:*

1). *Every branch point lies on J:* $\arg w = -\frac{\theta}{2}$, $Re^{-\alpha} < |w| < R$ *and* $J': \arg w = \frac{\theta}{2}$, $Re^{-\alpha} < |w| < R$, where $Re^{-\alpha} > 1$.

2). *D connects $\mathcal{L}, \mathcal{L}_1, \cdots, \mathcal{L}_{n+1}$ so that every $\mathcal{L}_i$ $(i=1, 2, \cdots, n)$ contains a part D_i of D, $\mathcal{L}$ and $\mathcal{L}_{n+1}$ do not contain any part of D.*

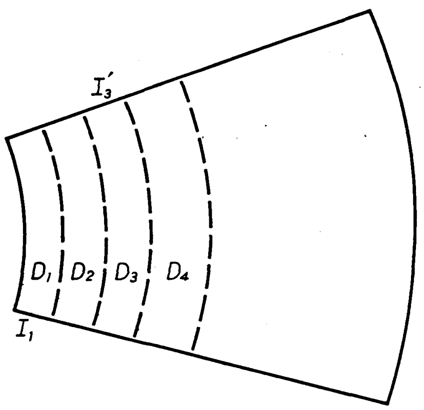

Fig. 1.

3). *There exists a C_1-function $\tilde{U}(w)$ in $\mathfrak{R}_D$ such that $\tilde{U}(w)=U(w)$ in D, $\tilde{U}(w)=0$ in $\mathcal{L}$, $\tilde{U}(w)=1$ in $\mathcal{L}_{n+1}$ and*

$$D(\tilde{U}(w)) \leqq 3D(U(w)) = \frac{6\theta}{\alpha} .$$

Such operation (to construct $\mathfrak{R}_D$ through D) will be called *extension of $\mathcal{L}$ though D.*

Proof. Let D_i and I_i and I_i' $(i=1, 2, \cdots, n)$ be an echelon and a segment as follows:

$$D_i: \quad Re^{-\alpha+(i-1)\gamma} < |w| < Re^{-\alpha+i\gamma}, \qquad -\frac{\theta}{2} < \arg w < \frac{\theta}{2},$$

$$I_i: \quad Re^{-\alpha+(i-1)\gamma} < |w| < Re^{-\alpha+i\gamma}, \qquad \arg w = -\frac{\theta}{2},$$

$$I_i': \quad Re^{-\alpha+(i-1)\gamma} < |w| < Re^{-\alpha+i\gamma}, \qquad \arg w = \frac{\theta}{2},$$

$$\sum_{i=1}^{n} D_i = D', \quad \text{where } \gamma = \frac{\alpha}{2n} .$$

Then D_i's are conformally equivalent. Map D_i by $\xi=\log w$ onto a rectangle K_i such that K_i: $\log R-\alpha+(i-1)\gamma<\eta<\log R-\alpha+i\gamma$, $-\frac{\theta}{2}<\zeta<\frac{\theta}{2}$, where $\xi=\eta+i\zeta$. Then $U(w)\rightarrow U_i(\xi)=2(\alpha-\log R+\eta)/\alpha$, $U_i(\xi)=(i-1)/n$ on $\eta=\log R-\alpha+(i-1)\gamma$ and $U_i(\xi)=i/n$ on $\eta=\log R-\alpha+i\gamma$. We shall define a function $\widetilde{U}_i(\xi)$ corresponding to K_i from $U_i(\xi)$. Let K_i' be the symmetric image of K_i with respect to $\eta=\log R+i\gamma-\alpha$:

$$K_i':\ \log R-\alpha+i\gamma<\eta<\log R-\alpha+(i+1)\gamma,\quad -\frac{\theta}{2}<\zeta<\frac{\theta}{2}.$$

Let Γ_U and Γ_L be semicircles as follows:

$$\Gamma_U:\ |\xi-p_U|<\gamma,\ \pi\geqq\arg(\xi-p_U)\geqq 0,$$
$$\Gamma_L:\ |\xi-p_L|<\gamma,\ 2\pi\geqq\arg(\xi-p_L)\geqq\pi,$$

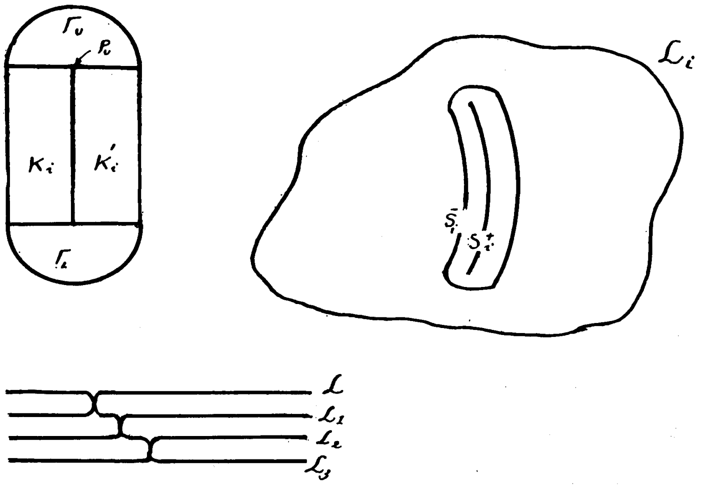

Fig. 2.

where p_U: $\xi=(\log R-\alpha+i\gamma)+\frac{i\theta}{2}$ and p_L: $\xi=(\log R-\alpha+i\gamma)-\frac{i\theta}{2}$. Now the function $U_i(\xi)$ is defined only in K_i. We continue it into $K_i+K_i'+\Gamma_U+\Gamma_L$ so that $\tilde{U}_i(\xi)=U_i(\xi)$ in K_i, $\tilde{U}(\xi)=-2(\alpha+\eta-\log R)/\alpha+\frac{2i}{n}$ in K_i', $\tilde{U}_i(\xi)=2(\gamma-\rho)/\alpha+\frac{i-1}{n}$ in $(\Gamma_U+\Gamma_L)$, where $\rho=|\xi-p_U|$ in Γ_U and $\rho=|\xi-p_L|$ in Γ_L respectively. Then $\tilde{U}_i(\xi)$ is a C_1-function and $\tilde{U}_i(\xi)=\frac{i-1}{n}$ on $\partial(K_i+K_i'+\Gamma_U+\Gamma_L)$. Consider the Dirichlet integral of $\tilde{U}_i(\xi)$. Then $D_{K_i+K_i'}(\tilde{U}_i(\xi))=\frac{2}{n}D(U(w))=\frac{2}{n}\times\frac{2\theta}{\alpha}=\frac{4\theta}{n\alpha}$ and $D_{\Gamma_U}(\tilde{U}_i(\xi))=D_{\Gamma_L}(\tilde{U}_i(\xi))=\int_{\rho=0}^{\frac{\alpha}{2n}}\int_0^{\pi}\left\{\left(\frac{\partial}{\partial\rho}\tilde{U}_i(\xi)\right)^2+\frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi}\tilde{U}_i(\xi)\right)^2\right\}\rho d\rho d\varphi=\left(\frac{\alpha}{2n}\right)^2\frac{4\pi}{\alpha^2}=\frac{\pi}{n^2}$, where $(\xi-p_U)=\rho e^{i\varphi}$ in Γ_U and

$(\xi-p_L)=\rho\ e^{i\varphi}$ in Γ_L

Hence

$$\underset{K_i+K_i'+\Gamma_U+\Gamma_L}{D}(\tilde{U}_i(\xi))=\frac{4\theta}{n\alpha}+\frac{2\pi}{n^2}.$$

Map $K_i+K_i'+\Gamma_U+\Gamma_L$ in the ξ-plane to $D_i+D_i'+\Gamma_U^W+\Gamma_L^W$ in the w-plane by $w=e^\xi$ and consider the function $\tilde{U}_i(w)$ such that $\tilde{U}_i(w)=\tilde{U}_i(\log w)=\tilde{U}_i(\xi)$ in $D_i+D_i'+\Gamma_L^W+\Gamma_U^W$ and $\tilde{U}_i(w)=\frac{i-1}{n}$ outside of $D_i+D_i'+\Gamma_U^W+\Gamma_L^W$. This w-plane is denoted by $\mathcal{L}_i$ in which $\tilde{U}_i(w)$ is defined. Clearly

$$D_{\mathcal{L}_i}(\tilde{U}_i(w))=\frac{4\theta}{n\alpha}+\frac{2\pi}{n^2}. \qquad (1)$$

Structure of $\mathfrak{R}_D$. Let S_i^+ and S_i^- be slits as follows:

$$S_0^-:\quad |w|=Re^{-\alpha},\quad -\frac{\theta}{2}<\arg w<\frac{\theta}{2}\ \text{ in } \mathcal{L}.$$

$$\left.\begin{aligned}S_i^+:&\quad |w|=Re^{-\alpha+(i-1)r},\quad -\frac{\theta}{2}<\arg w<\frac{\theta}{2}\ \text{ in } \mathcal{L}_i,\\ S_i^-:&\quad |w|=Re^{-\alpha+ir},\quad -\frac{\theta}{2}<\arg w<\frac{\theta}{2}\ \text{ in } \mathcal{L}_i,\end{aligned}\right\}\quad i=1,2,\cdots,n$$

$$S_{n+1}^+:\quad |w|=Re^{-\frac{\alpha}{2}},\quad -\frac{\theta}{2}<\arg w<\frac{\theta}{2}\ \text{ in } \mathcal{L}_{n+1},$$

where $\mathcal{L}_{n+1}$ is a leaf identical to the w-plane.
Connect $\mathcal{L}$ with $\mathcal{L}_1$ crosswise on $S_0^-(=S_1^+)$, connect $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ crosswise on $S_i^-(=S_{i+1}^+)$ $(i=1,2,\cdots,n)$. Then we have an $n+2$ sheeted covering surface over the w-plane. Clearly $\mathfrak{R}_D$ is closed and of genus zero. We define a new C_1-function $\hat{U}(w)$ in $\mathfrak{R}_D$ as follows: Put $\hat{U}(w)=\tilde{U}_0(w)=0$ in $\mathcal{L}$, $\hat{U}(w)=\tilde{U}_i(w)$ in $\mathcal{L}_i$ $(i=1,2,\cdots,n)$ and $\hat{U}(w)=1$ in $\mathcal{L}_{n+1}$. Then since $\hat{U}(w)=\tilde{U}_i(w)=U(w)$ on $S_i^{-1}(=S_{i+1}^+)$ $(i=0,1,2,\cdots,n)$ through which $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ are connected and since $\tilde{U}_i(w)$ is a C_1-function, $\hat{U}(w)$ is a C_1-function in $\mathfrak{R}_D$, where $\mathcal{L}_0$ means $\mathcal{L}$. Then the Dirichlet integral of $\hat{U}(w)$ is given as $D(\hat{U}(w))=n\left(\frac{4\theta}{n\alpha}\right)+n\left(\frac{4\pi}{n^2}\right)$. Choose a number n such that $\frac{4\pi}{n}<\frac{2\theta}{\alpha}$. Then $D(\hat{U}(w))<3D(U(w))=\frac{6\theta}{\alpha}$, hence we have the lemma.

II. *Extension of* $\mathcal{L}$ *through* $\sum_m D_m$ $(m=1,2,\cdots,m_0)$. Let

$$D_m:\quad Re^{-\alpha}<|w|<R,\ \theta_m<\arg w<\theta_m',\ \theta_m'<\theta_{m+1}.$$

In every D_m let $U(w)=2(\alpha-\log R+\log w)/\alpha$ for $Re^{-\alpha}<|w|<Re^{-\frac{\alpha}{2}}$ and $U(w)$

$=1$ for $Re^{-\frac{\alpha}{2}} \leqq |w| < R$. We define $\mathcal{L}_{m,1}, \mathcal{L}_{m,2}, \cdots, \mathcal{L}_{m,n(m)+1}$ and connect them on slits contained in D_m as mentioned in I) such that there exists a C_1-function $\hat{U}_m(w)$ in $\mathcal{L}_{m,1}+\mathcal{L}_{m,2}\cdots+\mathcal{L}_{m,n(m)+1}$), $\hat{U}_m(w)=0$ in $\mathcal{L}$, $\hat{U}_m(w)=1$ in $\mathcal{L}_{m,n(m)+1}$ and $D(\hat{U}_m(w)) \leqq 3D_{D_m}(U(w))$. Then we have a covering surface $\mathfrak{R}_{\Sigma D}$ of $1+(n(1)+1)+(n(2)+1)\cdots+(n(m_0)+1)$ number of sheets and of genus zero. Put $\hat{U}(w)=\hat{U}_m(w)$ in $\mathcal{L}+\sum_{m=1}^{m=n_0}\sum_{i=1}^{n(m)+1}\mathcal{L}_{m,i}$ Then since $\hat{U}_m(w)=0$ in $\mathcal{L}$, $\hat{U}(w)$ is a C_1-function in $\mathfrak{R}_{\Sigma D}$ and

$$D(\hat{U}(w)) = \sum_{m=1}^{m_0} D(\hat{U}_m(w)) \leqq 3\sum_{m=1}^{m_0} D_{D_m}(U(w)) = \frac{6}{\alpha}\sum_{m=1}^{m_0}(\theta'_m-\theta_m)\,.$$

Now the projection of every branchpoint of $\mathfrak{R}_{\Sigma D}$ lies on $\arg w=\theta_m$ and $\arg w=\theta'_m (m=1, 2, \cdots, m_0)$. We consider the star domain Ω of $\mathfrak{R}_{\Sigma D}$ with centre at $w=0$ of $\mathcal{L}$. Then $\partial\Omega$ consists of segments $|w|>Re^{-\alpha}$, $\arg w=\theta_m$ and $|w|>Re^{-\alpha}$, $\arg w=\theta'_m (m=1, 2, \cdots, m_0)$ and Ω is composed of the following parts:

$$\Omega=\hat{\mathcal{L}}+\sum_{m=1}^{m_0}\sum_{n=1}^{n(m)+1}\hat{\mathcal{L}}_{m,n} \quad \text{and} \quad D_m \subset \sum_{n=1}^{n(m)}\mathcal{L}_{m,n}$$

where

$\hat{\mathcal{L}}$: $E[w: |w|<Re^{-\alpha}]+\sum_{m=1}^{m_0} E[w: |w|>Re^{-\alpha},\ \theta'_m<\arg w<\theta_{m+1}]$
$+E[w: |w|>Re^{-\alpha},\ \theta'_{m_0}<\arg w<\theta_1]$ of $\mathcal{L}$.

$\hat{\mathcal{L}}_{m,1}$: $E[w: Re^{-\alpha+\gamma_m}>|w|>Re^{-\alpha},\ \theta_m<\arg w<\theta'_m]$ of $\mathcal{L}_{m,1}$

....

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$\hat{\mathcal{L}}_{m,n}$: $E[w: Re^{-\alpha+n\gamma_m}>|w|>Re^{-\alpha+(n-1)\gamma_m},\ \theta_m<\arg w<\theta'_m]$ of $\mathcal{L}_{m,n}$
$(n=2, 3, \cdots, n(m))$

....

....

$\hat{\mathcal{L}}_{m,n(m)+1}$: $E[w: |w|>Re^{-\frac{\alpha}{2}},\ \theta_m<\arg w<\theta'_m$, where $\gamma_m=\alpha/2n(m)$ and $m=1, 2, \cdots, m_0$ and $n=1, 2, \cdots, n(m)+1$.

The function $\hat{U}(w)$ in Ω is as follows: $\hat{U}(w)=0$ in $\hat{\mathcal{L}}$, $\hat{U}(w)=2(\alpha-\log R+\log|w|)/\alpha$ in $\sum_{m=1}^{m_0}\sum_{n=1}^{n(m)}\hat{\mathcal{L}}_{m,n}$ and $\hat{U}(w)=1$ in $\sum_{m=1}^{m_0}\mathcal{L}_{m,n(m)+1}$.

III. *Extension of $\mathcal{L}$ through a closed set F of linear measure zero on* $|w|=1$. The complementary set of $F=CF=\sum_{l=1}^{\infty} I_l$, where I_l is an open interval. Put $F_l=\Gamma-\sum_{i=1}^{l} I_i(\Gamma: |w|=1)$. Then $F_l=J_{l,1}+J_{l,2}+\cdots+J_{l,m(l)}+p_{l,1}$

$+p_{l,2}+\cdots+p_{l,m'(l)}$, where $J_{l,i}$ is a closed interval and $p_{l,i}$ is an isolated point of F_l. Clearly $F=\bigcap_{l=1}^{\infty} F_l$. Let $l'(l)$ be the smallest number such that mes $F_{l'(l)}<\frac{1}{4^l}$. For simplicity we denote $F_{l'(l)}$ by F_l. Then $F=\bigcap_{l=1}^{\infty} F_l$ and mes $F_l<\frac{1}{4^l}$. Put $A_l=J_{l,1}+J_{l,2}+\cdots+J_{l,m(l)}$ and $B_l=p_{l,1}+\cdots+p_{l,m'(l)}$. Then $F_l=A_l+B_l$ and $F_{l+1}=A_{l+1}+B_{l+1}$. Since every $p_{l,m}$ is isolated in F_l and is contained in F, $B_l\subset B_{l+1}$. Hence by $F_l\supset F_{l+1}$

$$B_{l+1}-B_l\subset A_l \quad\text{and}\quad A_{l+1}\subset A_l\,. \tag{2}$$

And

$$F_l=A_l+\sum_{i=1}^{l}(B_i-B_{i-1}),\ F=\bigcap_{l=1}^{\infty}A_l+\sum_{l=1}^{\infty}(B_l-B_{l-1}),\ \text{where } B_0=0\,.$$

Since mes $F=0$, F does not contain any closed interval and

$F=\overline{\sum_{l=1}^{\infty}\sum_{m=1}^{m(l)}(q_{l,m}+q'_{l,m})+\lim_{l=\infty}B_l}$, where $q_{l,m}$ and $q'_{l,m}$ are endpoints of $J_{l,m}$.

We define echelons $D_{l,m}$ $\hat{D}_{l,m}$, $\hat{\hat{D}}_{l,m}$ $(D_{l,m}=\hat{D}_{l,m}+\hat{\hat{D}}_{l,m})$ and a slit $t_{l,m'}$ from F_l as follows:

$$D_{l,m}:\quad Re^{-\frac{\alpha}{2^{l-1}}}<|w|<R,\ \theta_{l,m}<\arg w<\theta'_{l,m},$$

$$\hat{D}_{l,m}:\quad Re^{-\frac{\alpha}{2^{l-1}}}<|w|<Re^{-\frac{\alpha}{2^l}},\ \theta_{l,m}<\arg w<\theta'_{l,m}.$$

$$\hat{\hat{D}}_{l,m}:\quad Re^{-\frac{\alpha}{2^l}}<|w|<R,\ \theta_{l,m}<\arg w<\theta'_{l,m},\quad m=1,2,\cdots,m(l)$$

$$t_{l,m}:\quad Re^{-\frac{\alpha}{2^{l-1}}}<|w|<Re^{-\frac{\alpha}{2^l}},\ \arg w=\arg p_{l,m}\in(B_l-B_{l-1}),$$

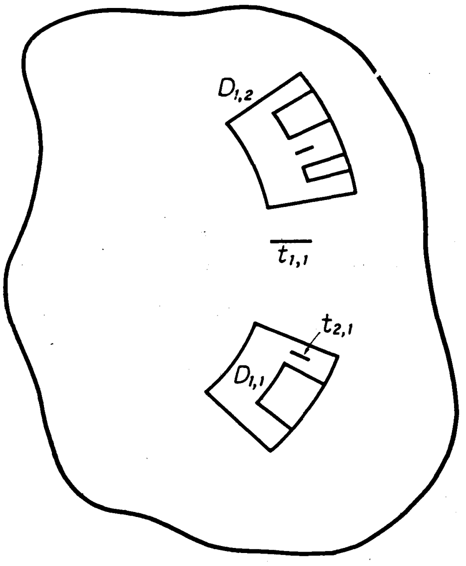

Fig. 3.

where $\theta_{l,m}=\min_{W\in J_{l,m}}\arg w$ and $\theta'_{l,m}=\max_{W\in J_{l,m}}\arg w$. Then

$$\sum_{m=1}^{m(l)} D_{l,m}\supset \sum_{m=1}^{m(l+1)} D_{l+1,m},\ \sum_{m=1}^{m(l)} \hat{D}_{l,m}\cap \sum_{m=1}^{m(l+1)} \hat{D}_{l+1,m}=0$$

$$\text{and } \sum_{m=1}^{m(l)} D_{l,m}\supset \sum_{m=1}^{m'(l+1)} t_{l+1,m},\ \sum_{m=1}^{m(l)} D_{l,m}\cap \sum_{m=1}^{m'(l)} t_{l,m}=0.$$

IV. *Extnsion of* 1*st step of* $\mathcal{L}$ *through* F. Let $\mathcal{L}$ be a leaf. Let $U_1(w)=2(\alpha-\log R+\log|w|)/\alpha$ in $\sum_{m=1}^{m(1)}\hat{D}_{1,m}$ and $U_1(w)=1$ in $\sum_{m=1}^{m(1)}\hat{\hat{D}}_{1,m}$. We extend $\mathcal{L}$ though $\sum_{m=1}^{m(1)} D_{1,m}$ (see II) to $\mathfrak{R}'_1=\mathcal{L}+\sum_{m=1}^{m(1)}\sum_{n=1}^{n(1,m)+1}\mathcal{L}_{1,m,n}$ such that there exists a C_1-function $\hat{U}_1(w)$ in $\mathfrak{R}'_1$ such that $\hat{U}_1(w)=0$ in $\mathcal{L}$, $\hat{U}_1(w)=1$ in $\sum_{m=1}^{m(1)}\mathcal{L}_{1,m,n(m)+1}$ and $D(\hat{U}_1(w))\leqq 3D(U_1(w))=\frac{6\theta_1}{\alpha}$, where $\theta_1=\sum_{m=1}^{m(1)}(\theta'_{1,m}-\theta_{1,m})\leqq\frac{1}{4}$.
Next we connect a leaf $\mathcal{L}'_{1,m}$ with $\mathcal{L}$ crosswise on $t_{1,m}$ ($m=1,2,\cdots,m'(1)$). Put $\mathfrak{R}'(F,1)=\mathfrak{R}'_1+\sum_{m=1}^{m'(1)}\mathcal{L}'_{1,m}$ and put $U(w,F,1)=0$ in $\sum_{m=1}^{m'(1)}\mathcal{L}'_{1,m}$ and $U(w,F,1)=\hat{U}_1(w)$ in $\mathfrak{R}'_1$. Then since $\hat{U}_1(w)=0$ in $\mathcal{L}$, $U(w,F,1)$ is also a C_1-function in $\mathfrak{R}'(F,1)$ and $D(U(w,F,1))=D(\hat{U}_1(w))$. Put

$$\mathfrak{R}(F,1)=\mathfrak{R}'(F,1)-\sum_{m=1}^{m(1)}\mathcal{L}_{1,m,n(m)+1}\ .$$

Then $\partial\mathfrak{R}(F,1)$ is composed of $m(1)$ number of compact relative boundary components $B(F,1)$ such that each component lies on the slits on which $\mathcal{L}_{1,m,n(m)+1}$ is connected. Such operation is called the *extension of first step of* $\mathcal{L}$ through F.

Extension of 2nd step of $\mathcal{L}$ through F. We extend every $\mathcal{L}_{1,m,n(m)+1}$ ($m=1,2,\cdots,m(1)$) through $\sum_{m'}' D_{2,m'}$ ($\sum'$ means the sum over $D_{2,m}$ contained in $D_{1,m}$) by defining $\mathcal{L}_{2,m,n}$ ($n=1,2,\cdots,n(2,m)+1$) and connect $\mathcal{L}'_{2,m}$ on $t_{2,m}$ ($m=1,2,\cdots,m'(2)$) crosswise to obtain $\mathfrak{R}'(F,2)=\mathfrak{R}'(F,1)+\sum_{m=1}^{m(2)}\mathcal{L}'_{2,m}+\sum_{m=1}^{m(2)}\sum_{n=1}^{n(2,m)+1}\mathcal{L}_{2,m,n}$. Put $\mathfrak{R}(F,2)=\mathfrak{R}'(F,2)-\sum_{m=1}^{m(2)}\mathcal{L}_{2,m,n(2,m)+1}$. Then there exists a C_1-function $U(F,w,2)$ in $\mathfrak{R}(F,2)$ such that $U(F,w,2)=0$ in $\mathfrak{R}'(F,1)$, $U(F,w,2)=1$ on $B(F,2)=\partial\mathfrak{R}(F,2)$ and

$$D(U(F,w,2))=3D(U_2(w))=\frac{6\theta_2}{\frac{\alpha}{2}}\ ,$$

where $U_2(w)=2\left(\frac{\alpha}{2}-\log R+\log|w|\right)\Big/\frac{\alpha}{2}$ in $\sum_{m=1}^{m(2)}\hat{D}_{2,m}$ and $U_2(w)=1$ in $\sum_{m=1}^{m(2)}\hat{\hat{D}}_{2,m}$ and $\theta_2=\sum_{m=1}^{m(2)}(\theta'_{2,m}-\theta_{2,m})\leqq\frac{1}{4^2}$.

Suppose $\mathfrak{R}(F, l)$ is defined, we define $(l+1)$-th step and $\mathfrak{R}(F, l+1)$ as follows: we extend $\mathcal{L}_{l,m,n(l,m)+1}$ $(m=1, 2, \cdots, m(l))$ through $\sum_m' D_{l+1,m}$ ($\sum'$ means over $D_{l+1,m}$ contained in $D_{l,m}$) by defining $\mathcal{L}_{l+1,m,n}$ $(n=1, 2, \cdots, n(l+1, m)$, $n(l+1, m)+1))$ and connecting $\sum_{m=1}^{m'(l+1)} \mathcal{L}'_{l+1,m}$ on $\sum_{m=1}^{m'(l+1)} t_{l+1,m}$ $(\subset \sum_{m=1}^{m(l)} D_{l,m})$. Put

$$\mathfrak{R}'(F, l+1) = \mathfrak{R}'(F, l) + \sum_{m=1}^{m(l+1)} \sum_{n=1}^{n(l+1,m)+1} \mathcal{L}_{l+1,m,n} + \sum_{m=1}^{m(l+1)} \mathcal{L}'_{l+1,m} \text{ and}$$

$$\mathfrak{R}(F, l+1) = \mathfrak{R}'(F, l+1) - \sum_{m=1}^{m(l+1)} \mathcal{L}_{l+1,m,n(l+1,m)+1} .$$

There exists a C_1-function $U(F, w, l+1)$ in $\mathfrak{R}(F, l+1)$ such that $U(F, w, l+1)=0$ in $\mathfrak{R}'(F, l)$, $U(F, w, l+1)=1$ on $B(F, l+1)$ and

$$D(U(F, w, l+1)) \leqq \frac{6\theta_{l+1}}{\frac{1}{2^{l+1}}} : \theta_{l+1} = \sum_{m=1}^{m(l+1)} (\theta'_{l+1,m} - \theta_{l+1,m}) \leqq \frac{1}{4^{l+1}} .$$

Such extension in called *the extension of* $l+1$-*th step of* $\mathcal{L}$ *through F.* Put $\mathfrak{R}_F = \lim_l \mathfrak{R}(F, l)$. Then $\mathfrak{R}_F$ has the following properties:

1). *$\mathfrak{R}_F$ is a Riemann surface of planer character of connectivity $\leqq \infty$ and has null-boundary.*

2). *Let Ω_F be a star domain of $\mathfrak{R}_F$ with centre at $w=0$ of $\mathcal{L}$. Then Ω_F contains the part of $\mathcal{L}$ outside of $\sum_{m=1}^{m(1)} K_{1,m} + \sum_{m=1}^{m'(1)} K'_{1,m}$, where*

$$K_{1,m}: \quad Re^{-\alpha} < |w| < \infty, \ \theta_{1,m} < \arg w < \theta'_{1,m}, \qquad m = 1, 2, \cdots, m(1)$$

$$K'_{1,m}: \quad Re^{-\alpha} < |w| < \infty, \ \arg w = \theta''_{1,m} = \arg t_{1,m}, \qquad m = 1, 2, \cdots, m'(1)$$

3). *Let Ω_F be as above. Then the singular set of Ω_F is F.*

1). Clearly every $\mathfrak{R}(F, l)$ is of planer character and $B(F, l)$ consists of $n(l)$ number of components. Hence $\mathfrak{R}_F$ is of planer character and its connectivity $\leqq \infty$. Now $\mathfrak{R}(F, l)$ $(l=1, 2, \cdots, \quad)$ is an exhaustion of $\mathfrak{R}_F$, let $\omega_l(w)$ be a harmonic function in $\mathfrak{R}(F, l) - C$ such that $\omega_l(w)=0$ on ∂C and $\omega_l(w)=1$ on $B(F, l)$, where $C = E[w: |w| < 1]$ of $\mathcal{L}$. Then by the Dirichlet principle $D(\omega_l(w)) \leqq D(U(F, w, l)) \leqq \dfrac{6}{4^l \times \frac{\alpha}{2^{l+1}}} = \dfrac{6}{2^{l-1}\alpha}$. Whence $\lim_{l=\infty} \omega_l(w)=0$ and $\mathfrak{R}_F$ has null-boundary.

2). is clear from the structure of $\mathfrak{R}(F, 1)$.

3). If p is an accumulating point of $\lim_{l=\infty} B_l$, $p \in A_l$ for any l and $p \in \bigcap_{l=1}^{\infty} A_l$, by $F_l = A_l + (B_l - B_{l-1}) \supset F$. Now mes $F=0$ and F does not contain any arc. Whence if $p \in \bigcap_{l=1}^{\infty} A_l$, $p \in \sum_{l=1}^{\infty} \sum_{m=1}^{m(l)} (q_{l,m} + q'_{l,m})$. Corresponding fact occcurs for singular raies. In fact for $p \in \sum_{l}^{\infty} \sum_{m=1}^{m(l)} (q_{l,m}+q'_{l,m})+\lim_{l=\infty} B_l$, there exists a ray: $r(p)$ such that $r(p): Re^{-\frac{\alpha}{2^l}} \leqq |w| < \infty$, $\arg w = \arg p$. Suppose $p \in \overline{\sum_l \sum_m (q_{l,m}+q'_{l,m})} - \sum_l \sum_m (q_{l,m}+q'_{l,m}) - \lim_l B_l$. Then since there exist only a finite number of points $\{q_{l,m}\}$ and $\{q'_{l,m}\}$ for given l, there exists a sequence $q_{l_1,i_1}, q_{l_2,i_2} \cdots \to p$, $l_1 < l_2 \cdots$. Hence there exists a sequence of raies $r_{l_i,i}: Re^{-\frac{\alpha}{2^{l_i}}} \leqq |w| < \infty$, $\arg w = \arg q_{l,i}$ tending to the ray $r(p)$. Thus to every $p \in F$ a singular ray corresponds and the singular set S of Ω_F is F.

V. *Extension of $\mathcal{L}$ through a discrete F_σ set of measure zero.*

Let $F_\sigma = \sum_{i=1}^{\infty} F_i$, $\delta_i = \operatorname{dist}(F_i, \sum_{j \neq i} F_j) > 0$ and $\tilde{F}_i = E\left[p : \operatorname{dist}(F_i, p) \leqq \frac{\delta_i}{2}\right]$. Then $\tilde{F}_i \supset F_i$ and $\tilde{F}_i \cap \tilde{F}_j = 0$ for $i \neq j$. Every F_i is expressed by

$$F_i = \bigcap_{l=1}^{\infty} A_{i,l} + \sum_{l=1}^{\infty} (B_{i,l} - B_{i,l-1}), \text{ where } \quad B_{i,0} = 0, \text{ and} \tag{3}$$

$A_{i,l} = J_{i,l,1} + J_{i,l,2} + \cdots + J_{i,l,m(i,l)}$ and $B_{i,l} = p_{i,l,1} + p_{i,l,2} + \cdots + p_{i,l,m'(i,l)}$, where $J_{i,l,m}$ is a closed interval and $p_{i,l,m}$ is an isolated point of F_i. Since mes $F_i = 0$, there exists a number $l(i)$ such that mes $A_{i,l} < \frac{\delta_i}{4}$ and $A_{i,l} + B_{i,l} \subset \tilde{F}_i$ for $l > l(i)$. On the other hand, by (3) we suppose without loss of generality that

$$A_{i,l} + B_{i,l} \subset \tilde{F}_i, \quad i = 1, 2, \cdots. \tag{4}$$

Also we can suppose

$$\text{mes } A_{i,l} \leqq 1/2^l, \quad l = 1, 2, \cdots. \tag{5}$$

We define $D_{i,l,m}$ $(m=1, 2, \cdots, m(i,l))$ and $t_{i,l,m}$ $(m=1, 2, \cdots, m'(i,l))$ from $F_{i,l} = A_{i,l} + \sum_{j=1}^{l} (B_{i,j} - B_{i,j-1})$ $(F_i = \bigcap_l A_{i,l} + \lim_l B_{i,l}$ for every $i)$. Let

$$R_i = e^{\beta^i} \quad \text{and} \quad \alpha_i = \alpha^i, \text{ where } \beta > \alpha > 2. \tag{6}$$

$$D_{i,l,m}: \quad R_i e^{-\frac{\alpha_i}{2^{l-1}}} < |w| < R_i, \ \min_{\theta \in J_{i,l,m}} \theta < \arg w < \max_{\theta \in J_{i,l,m}} \theta, \ l = 1, 2, \cdots \text{ and } m = 1, 2, \cdots, m(i,l)$$

$$t_{i,l,m}: \quad R_i e^{-\frac{\alpha_i}{2^{l-1}}} < |w| < Re^{-\frac{\alpha_i}{2^l}}, \ \arg w = \arg p_{i,l,m}, \ l = 1, 2, \cdots \text{ and } m = 1, 2, \cdots, m'(i,l).$$

By (6) we have

$$\log(R_{i+1} e^{-\frac{\alpha_{i+1}}{2}} / R_i) = \beta^{i+1} - \beta^i - \frac{\alpha^{i+1}}{2} > \beta^i \left(\frac{\beta}{2} - 1\right) \to \infty \text{ as } i \to \infty. \tag{7}$$

At first we extend $\mathcal{L}$ through F_1 by $\{D_{1,l,m}\}+\{t_{1,l,m}\}$ and we obtain $\mathfrak{R}_{F_1}$. Then

1°). By (2) of III Ω_{F_1} (star domain of $\mathfrak{R}_{F_1}$ with centre at $w=0$ of $\mathcal{L}$ contains the part of $\mathcal{L}$ outsider of K_1: $R_1 e^{-\frac{\alpha_1}{2}} \leqq |w| \leqq \infty$, $\arg w=\theta \in \widetilde{F}_1$.

2°). There exists a sequence of bordered Riemann surface $\{\mathfrak{R}(F_1, l)\}$ $(l=1,2,\cdots)$, where $\mathfrak{R}(F_1, l) \rightarrow \mathfrak{R}(F_1)$ as $l \rightarrow \infty$ and $\mathfrak{R}(F_1, l)$ has compact relative boundary $B(F_1, l.)$. This $\mathfrak{R}(F_1, l)$ is extended through $(\sum_{m=1}^{m(1,1)} D_{1,1,m}+\sum_{m=1}^{m'(1,1)} t_{1,1,m})+(\sum_{m=1}^{m(1,2)} D_{1,2,m}+\sum_{m=1}^{m'(1,2)} t_{1,2,m})+\cdots+(\sum_{m=1}^{m(1,l)} D_{1,l,m}+\sum_{m=1}^{m(1,l)} t_{1,l,m})$. We call $\mathfrak{R}(F_1 l)$, a surface of l-th step through F_1.

3°). There exists a C_1-function $U(F_1, w, l)$ in $\mathfrak{R}(F_1, l)$ such that $U(F_1, w, l)=0$ in $\mathcal{L}$, $U(F_1, w, l)=1$ on $B(F_1, l)$ and $D(U(F_1, w, l)) \leqq \frac{6}{\alpha_1}\text{mes}(A_{1,l}) \leqq \frac{6}{2^l \alpha_1}$.

By (1) Ω_{F_1} contains the part of $\mathcal{L}$ over $\sum_{m=1}^{m(2,1)} D_{2,1,m}+\sum_{m=1}^{m'(2,1)} t_{2,1,m}$, because $A_{2,1}+B_{2,1} \subset \widetilde{F}_2 \subset$ complementary set of $\widetilde{F}_1$. We extend $\mathcal{L}$ through $\sum_{m=1}^{m(2,l)} D_{2,l,m}+\sum_{m=1}^{m'(2,l)} t_{2,l,m}$: $l=1, 2, \cdots$. Then Ω_{F_1} is extended through F_2 and we have $\mathfrak{R}_{F_1+F_2}$.

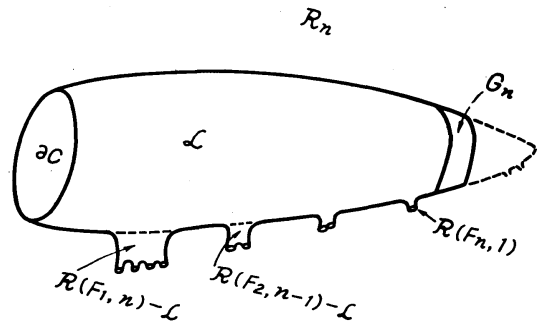

Fig. 4.

Suppose we have extended $\mathcal{L}$ through $F_1+F_2+\cdots+F_i$ and denote it by $\mathfrak{R}_{\sum^i F_l}$. Then the star domain $\Omega_{\sum^i F_l}$ of $\mathfrak{R}_{\sum^i F_l}$ with centre at $w=0$ of $\mathcal{L}$ contains the part of $\mathcal{L}$ not lying on $(K_1+K_2+\cdots+K_i)$ where K_i: $R_i e^{-\frac{\alpha_i}{2}} \leqq |w| \leqq \infty$ $\arg w=\theta \in \widetilde{F}_i$. Hence $\sum_{m=1}^{m(i+1,1)} D_{i+1,1,m}+\sum_{m=1}^{m'(i+1,1)} t_{i+1,1,m}$ is contained in $\Omega_{\sum^i F_l}$. Whence the extension of $\Omega_{\sum^i F_l}$ through F_{i+1} can be performed. This is the extension of $\mathcal{L}$ through F_{i+1}. Thus we can define the extension of $\Omega_{\sum^i F_l}$ through F_{i+1} to obtain $\mathfrak{R}_{\sum^{i+1} F_l}$. Also there exists a C_1-function $U(F_{i+1}, w, l)$ in $\mathfrak{R}(F_{i+1}, l)$ such that $U(F_{i+1}, w, l)=0$ in $\mathcal{L}$, $U(F_{i+1}, w, l)=1$ on $B(F_{i+1}, l)$ and

$$D(U(F_{i+1}, w, l)) \leqq \frac{6}{2^l \alpha_{i+1}}. \tag{7}$$

Put $\mathfrak{R}_{F_\sigma}=\lim_i \mathfrak{R}_{\sum^i F_l}$. *Then* $\mathfrak{R}_{F_\sigma}$ *has the following properties*:

1°). $\mathfrak{R}_{F_\sigma}$ *is the surface of planer character and* $\mathfrak{R}_{F_\sigma}$ *has null-boundary.*

2°). *The singular set* S *of the star domain* Ω*of* $\mathfrak{R}_{F_\sigma}$ *with centre at* $w=0$ *on* $\mathcal{L}$ *is* F_σ

Proof of 1°). We must define an exhaustion $\Re_n$ $(n=1,2,\cdots)$ of $\Re_{F_\sigma}$ with compact relative boundary $\partial\Re_n$. Let C be the circle $|w|<1$ in $\mathcal{L}$. We extend $\mathcal{L}$ through F_1 till n-th step, $\mathcal{L}$ through F_2 till (n-1)-th step$\cdots$ and through F_n till first step. Then we have the surface composed of $\mathcal{L}+(\Re(F_1,n)-\mathcal{L})+(\Re(F_2,n-1)-\mathcal{L})+\cdots+(\Re(F_n,1)-\mathcal{L})$. This surface has compact relative boundary consisting of $B(F_1,n)+B(F_2,n-1)+\cdots+B(F_n,1)$. We extract the part $\Gamma_n: R_{n+1}e^{-\alpha_{n+1}}<|w|<\infty$ (in which $\mathcal{L}$ will be extended through F_{n+1}, F_{n+2} $\cdots$) from $\mathcal{L}$ of this surface. The remaining surface $\Re_n$ has relative boundary $\sum_{j=1}^{n} B(F_1,\ n+1-j)+\partial\Gamma_n$. Let G_n be a ring in $\mathcal{L}$ such that $R_n<|w|<R_{n+1}e^{-\alpha_{n+1}}$ and let $V_n(w)$ be a C_1-function in $\Re_n$ such that $V_n(w)$ is harmonic in G_n, $V_n(w)=0$ in $\Re_n-G_n$, $V_n(w)=1$ on ∂G_n lying on $|w|=R_{n+1}e^{-\alpha_{n+1}}$. Then by (6) $D(V_n(w))=2\pi/\log\dfrac{R_{n+1}e^{-\alpha_{n+1}}}{R_n}\leqq\dfrac{2\pi}{\beta^n\left(\dfrac{\beta}{2}-1\right)}$. Let $\hat{U}_n(w)=V_n(w)$ in G_n and $\hat{U}_n(w)=U(F_1,w,n)$ $(=U(F_2,w,n-1)=\cdots,=U(F_n,w,1)=V_n(w)=0)$ in $\mathcal{L}-G_n$, $\hat{U}_n(w)=U(F_1,w,n)$ in $\Re(F_1,n)-\mathcal{L}$, $\hat{U}_n(w)=U(F_2,w,n-1)$ in $\Re(F_2,n-1)-\mathcal{L}\cdots$, $\hat{U}_n(w)=U(F_n,w,1)$ in $\Re(F_n,1)-\mathcal{L}$ and $\hat{U}_n(w)=V(w)$ in G_n. Then $\hat{U}_n(w)$ is a C_1-function in $\Re_n$ such that $\hat{U}_n(w)=0$ on ∂C and $\hat{U}_n(w)=1$ on $\partial\Re_n$. Then by the Dirichlet principle and by (6) and (7)

$$D(\omega_n(w))\leqq D(\hat{U}_n(w))\leqq\sum_{i=1}^{n}D(U(F_i,w,n+1-i))+D(V_n(w))\leqq$$

$$\sum_{i=1}^{n}\frac{6}{2^{n+1-i}\alpha_i}+\frac{2\pi}{\beta^n\left(\frac{\beta}{2}-1\right)}\leqq\frac{6n}{2^{n+1}}+\frac{2\pi}{\alpha^n\left(\frac{\beta}{2}-1\right)},$$

where $\omega_n(w)$ is a harmonic function in $\Re_n-C$ such that $\omega_n(w)=1$ on $\partial\Re_n$ and $\omega_n(w)=0$ on ∂C. It is evident that $\lim_n \omega_n(w)=0$ and $\{\Re_n\}$ is an exhaustion of $\Re_{F_\sigma}$. Hence $\Re_{F_\sigma}$ has null-boundary. Next since every $\Re_n$ is a surface of planer character, $\Re_{F_\sigma}$ is of planer character.

2°). It can be proved that the singular set S of Ω satisfies $S=\sum F_i=F_\sigma$ as III. Hence $\Re_{F_\sigma}$ is the Riemann surface required and the Theorem is proved.

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