# ON THE SPECTRUM OF FUNCTION IN THE WEYL SPACE

By

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**1. Introduction.** Let $f$ be a bounded measurable function defined on the real line. By $\Lambda(f)$ we mean the set of real number $\lambda$ which satisfies the following property. For any integrable function $K(x)$ the condition

$$K*f=\int_{-\infty}^{\infty} K(x-y)f(y)dy=0 \qquad (-\infty<x<\infty) \tag{1.1}$$

implies

$$\int_{-\infty}^{\infty} e^{i\lambda y}K(y)dy=0. \tag{1.2}$$

(c. f. A. Beurling [1].

In the previous paper we introduced the analogous definitions. By $\Lambda_*(f)$ we mean the set of real number $\lambda$ which satisfies the following property. For any integrable function $K(x)$ the condition

$$K*f=\int_{-\infty}^{\infty} K(x-y)f(y)dy\sim 0 \tag{1.3}$$

implies

$$\int_{-\infty}^{\infty} e^{i\lambda y}K(y)dy=0 \tag{1.4}$$

where the notation $K*f\sim 0$ means

$$\overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} \frac{1}{l}\int_{-\infty}^{\infty} |K*f(x)|^2dx=0. \tag{1.5}$$

(c. f. S. Koizumi [4]. It is clear that

$$\Lambda_*(f)\subseteqq \Lambda(f). \tag{1.6}$$

The purpose of this paper is to investigate properties of the above defined set as for functions represented by the Fourier-Stieltjes transform and almost periodic functions in the sense of Weyl.

**2. General property of $\Lambda_*(f)$.**

**Theorem 1.** *The set $\Lambda_*(f)$ is closed.*

This is a trivial result.

**Theorem 2.** *Let $f$ be a bounded measurable function and $f\sim 0$. Then we have $\Lambda_*(f)=\emptyset$.*

*Proof of Theorem* 2. Using the same notation in our previous paper (S. Koizumi [4]) we get

$$\Lambda_*(f)=\Lambda_{Wy}(f) \tag{2.1}$$

From the assumption $f\sim 0$ we get

$$\overline{\lim_{l\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon,x)-s(u-\varepsilon,x)|^2du=0 \tag{2.2}$$

where $s(u)$ is the Fourier-Wiener transform of $f(x)$. Therefore we get $\Lambda_{Wy}(f)=\emptyset$ and thus we get $\Lambda_*(f)=\emptyset$.

## 3. Fourier-Stieltjes transform.

**Theorem 3.** *Let $f$ be represented by the Fourier-Stieltjes transform. That is*

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iux}d\sigma(u) \tag{3.1}$$

*where $\sigma(u)$ is a function of bounded variation on the real line. Then we have*

$$\overline{\lim_{l\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f(t)|^2dt=\frac{1}{2\pi}\sum|\sigma(\lambda+0)-\sigma(\lambda-0)|^2. \tag{3.2}$$

*proof of Theorem* 3. Let $\sigma_1(u)+\sigma_2(u)$ be the Lebesgue decomposition of $\sigma(u)$. The $\sigma_1(u)$ is its continuous part and the $\sigma_2(u)$ is its saltus part. Let $f_i(x)$ be the Fourier-Stieltjes transform of $\sigma_i(u)$ $(i=1,2)$ respectively. Then we have

$$\frac{1}{l}\int_x^{x+l}|f(t)|^2dt=\frac{1}{l}\int_x^{x+l}|f_1(t)|^2dt+2\Re\left(\frac{1}{l}\int_x^{x+l}f_1(t)\overline{f_2(t)}dt\right)$$
$$+\frac{1}{l}\int_x^{x+l}|f_2(t)|^2dt.$$

Firstly we shaly estimate the part of $f_1(t)$. We have

$$\frac{1}{l}\int_x^{x+l}|f_1(t)|^2dt=\frac{1}{2\pi l}\int_x^{x+l}dt\int_{-\infty}^{\infty}e^{iux}d\sigma_1(u)\int_{-\infty}^{\infty}e^{-ivx}\overline{d\sigma_1(v)}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1(u)\int_{-\infty}^{\infty}\overline{d\sigma_1(v)}\frac{e^{i(u-v)l}-1}{i(u-v)l}e^{i(u-v)x}$$

Here if we put $\sigma_*^1(u)=\int_{-\infty}^{u}|d\sigma_1(v)|$, then the $\sigma_1^*(u)$ is a continuous, nondecreasing and bounded function and we get

$$\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f_1(t)|^2dt\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1^*(u)\int_{-\infty}^{\infty}d\sigma_1^*(v)\left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^2$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1^*(u)\int_{|u-v|\geqq\delta}d\sigma_1^*(v)\left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^2$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1^*(u)\int_{|u-v|\geqq\delta}d\sigma_1^*(v)\left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^2=I_1+I_2,\ \text{say.}$$

The number $\delta$ will be determined in later. As for the $I_2$: for any given positive number $\eta$ there exist a number $N$ such that

$$|\sigma_1^*(\pm\infty)-\sigma_1^*(\pm N)|\leqq\eta/4V_1,$$

where $V_1=\sigma_1^*(\infty)=\int_{-\infty}^{\infty}|d\sigma(v)|$. Then if we fixe the number $N$, there exist a number $\delta$ such that

$$\sup_{-N\leqq u\leqq N}\sigma_1^*(u+\delta)-\sigma_1^*(u-\delta)<\eta/4V_1.$$

Hence we get

$$I_2\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1^*(u)\int_{u-\delta}^{u+\delta}d\sigma_1^*(v)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}(\sigma_1^*(u+\delta)-\sigma_1^*(u-\delta))d\sigma_1^*(u)$$

$$=\frac{1}{2\pi}\left(\int_{-\infty}^{-N}+\int_N^{\infty}\right)(\sigma_1^*(u+\delta)-\sigma_1^*(u-\delta))d\sigma_1^*(u)+\frac{1}{2\pi}\int_{-N}^{N}(\sigma_1^*(u+\delta)$$

$$-\sigma_1^*(u-\delta)d\sigma_1^*(u)\leqq\frac{1}{2\pi}2V_1[\{\sigma_1^*(-N)-\sigma_1^*(-\infty)\}+\{\sigma_1^*(\infty)$$

$$-\sigma_1^*(N)\}]+\frac{1}{2\pi}\int_{-N}^{N}\sup_{-N\leqq u\leqq N}\{\sigma_1^*(u+\delta)-\sigma_1^*(u-\delta)\}\,d\sigma_1^*(u)$$

$$\leqq(1/2\pi)(\eta/2+\eta/2)+(1/2\pi)(\eta/4)=5\eta/8\pi.$$

As for $I_1$: the condition $|u-v|\geqq\delta$ imply

$$\left(\frac{\sin(u-v)l/2}{(u-v)/2}\right)^2\to0 \qquad (\text{as } l\to\infty).$$

Therefore we get

$$\overline{\lim_{l\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f_1(t)|^2dt\leqq5\eta/8\pi.$$

and the number $\eta$ is an arbitrarily small. Thus we get

$$(3.3)\qquad \overline{\lim_{l\to\infty}}\ \sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f_1(t)|^2dt=0.$$

Secondly we shall estimate the part of $f_2(t)$. The $\sigma_2(u)$ has enumerable number of jump point $(\lambda_n)$ $(n=1,2\cdots)$. Here if we put

$$a_n=\sigma(\lambda_n+0)-\sigma(\lambda_n-0)$$

and

$$\sum|a_n|=\int_{-\infty}^{\infty}d\sigma_2(u)=V_2.$$

Then we have

$$f_2(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iux}d\sigma_2(u)=\frac{1}{\sqrt{2\pi}}\sum a_ne^{i\lambda_nx}.$$

Therefore we get

$$\frac{1}{l}\int_x^{x+l}|f_2(t)|^2dt=\lim_{N\to\infty}\frac{1}{2\pi l}\int_x^{x+l}\sum_{-N}^{N}a_ne^{i\lambda_nt}\overline{\sum_{-N}^{N}a_ne^{i\lambda_nt}}\,dt$$

$$=\frac{1}{2\pi}\sum_{-\infty}^{\infty}|a_n|^2+\frac{1}{2\pi}\sum_{m\neq n}a_m\bar{a}_ne^{i(\lambda_m-\lambda_n)x}\frac{e^{i(\lambda_m-\lambda_n)l}-1}{i(\lambda_m-\lambda_n)l}=J_1+J_2,\ \text{say}.$$

As for $J_2$: From $|\sum_{m\neq n}a_m\bar{a}_n|\leqq\sum|a_m|\sum|a_n|=V_2^2$, we get

$$\overline{\lim_{l\to\infty}}\ \sup_{-\infty<x<\infty}|J_2|\leqq\sum_{m,n}|a_m||\bar{a}_n|\overline{\lim_{l\to\infty}}\frac{e^{i(\lambda_m-\lambda_n)l}-1}{i(\lambda_m-\lambda_n)l}=0.$$

Therefore we get

$$(3.4)\qquad \overline{\lim_{l\to\infty}}\ \sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f_2(t)|^2dt=\frac{1}{2\pi}\sum|a_n|^2.$$

The estimations (3.3) and (3.4) read

$$(3.5)\qquad \overline{\lim_{l\to\infty}}\ \sup_{-\infty<x<\infty}\left|\frac{1}{l}\int_x^{x+l}f_1(t)\overline{f_2(t)}dt\right|$$

$$\leqq\overline{\lim_{l\to\infty}}\ \sup_{-\infty<x<\infty}\left(\frac{1}{l}\int_x^{x+l}|f_1(t)|^2dt\right)^{\frac{1}{2}}\left(\frac{1}{l}\int_x^{x+l}|f_2(t)|^2dt\right)^{\frac{1}{2}}=0.$$

Thus we obtain the (3.2).

The following results is well known: if $f(x)$ can be represented by the Fourier-Stieltjes transform (3.1), then we have

$$(3.6)\qquad \Lambda(f)=\{\lambda\,|\,\sigma(\lambda+\varepsilon)-\sigma(\lambda-\varepsilon)>0\ \text{for any}\ \varepsilon>0\}$$

(c.f. H. Pollard [5]). Now we shall prove the following theorem.

**Theorem 4.** *If $f(x)$ can be represented by the formula* (3.1), *then we get*

(3.7) $$\Lambda_*(f)=\overline{\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0\}}$$

*where the notation* $\overline{\{\ \}}$ *means the closure of the set* $\{\ \}$.

*Proof of Theorem* 4. The proof can be done by decomposing into three parts.

(i) Firstly we shall prove

(3.8) $$\Lambda^*(f)\supseteq\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0\}.$$

We have

$$g(x)=K*f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K(x-y)\,dy\int_{-\infty}^{\infty}e^{iuy}d\sigma(u)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\sigma(u)\int_{-\infty}^{\infty}K(x-y)e^{iuy}dy=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iux}k(u)\,d\sigma(u),$$

where the $k(u)$ is the Fourier transform of $K(x)$ and a bounded continuous function on the real line. Repeating the same arguments as the proof of Theorem 3 we get

(3.9) $$\overline{\lim_{l\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|g(t)|^2dt=\frac{1}{2\pi}\sum|k(\lambda)|^2\,|\sigma(\lambda+0)-\sigma(\lambda-0)|^2.$$

Hence if $g(x)=K*f\sim 0$, we obtain

$$\frac{1}{2\pi}\sum|k(\lambda)|^2\,|\sigma(\lambda+0)-\sigma(\lambda-0)|^2=0$$

and we can conclude that the condition $\sigma(\lambda+0)-\sigma(\lambda-0)>0$ implies $k(\lambda)=0$. Thus we get the formular (i).

(ii) Secondly we shall prove

(3.10) $$\Lambda_*(f)\supseteq\overline{\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0)\}}$$

This is immediate by the fact $\Lambda_*(f)$ to be closed and the formula (i).

(iii) Lastly we shall prove

(3.11) $$\Lambda_*(f)=\overline{\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0\}}$$

If there is an $\lambda_0\in\Lambda_*(f)-\overline{\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0\}}$, there exist an nighbourhood of $\lambda_0$, $(\lambda_0-\delta,\lambda_0+\delta)$ where the $\sigma(u)$ is continuous. Let $k_{\lambda_0}(u)$ be an ordinary triangular function on this interval. Let $K_{\lambda_0}(x)$ be its Fourier transform. Let us put

$$g_{\lambda_0}(x)=\int_{-\infty}^{\infty}K_{\lambda_0}(x-y)f(y)\,dy.$$

Then we get

$$\overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} \frac{1}{l}\int_x^{x+l} |g_{\lambda_0}(t)|^2 dt = \sum |k_{\lambda_0}(\lambda)|^2 |\sigma(\lambda+0)-\sigma(\lambda-0)|^2 = 0.$$

On the other hand we have $k_{\lambda_0}(u) \neq 0$ at $u=\lambda_0$ by the definition of $k_{\lambda_0}(u)$. Therefore we get $\lambda_0 \in \Lambda_*(f)$. This reads a contradiction. Thus we get the formula (iii).

**Theorem 5.** *Let $f(x)$ be represented by the formula* (3. 1). *Let us assume that $K(x)$ belong to the class $L_1$ and its Fourier transform vanish on the set of $\Lambda_*(f)$. Then we get*

$$K*f\sim 0. \tag{3.12}$$

*Proof of Theorem* 5. This is a corollary to the formula (3. 9.)

**4. The almost periodic function in the sense of Weyl.** For almost perioic function there always exist

$$a(\lambda) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}dt. \tag{4.1}$$

Then we have

**Theorem 6.** *Let $f(x)$ be an almost periodic function in the sense of Weyl. Let us put*

$$\Lambda_0(f) = \{\lambda \mid a(\lambda) \neq 0\}. \tag{4.2}$$

*Then we have*

$$\overline{\Lambda_0(f)} = \Lambda_{Wy}(f). \tag{4.3}$$

*Proof of Theorem* 6. (i) Proof of $\overline{\Lambda_0(f)} \subseteqq \Lambda_{Wy}(f)$. We have by the Wiener formula

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon)-s(u-\varepsilon)\}\,du = \sqrt{2\pi}\,a(\lambda).$$

Therefore for any positive number $\delta$ we get

$$|a(\lambda)| \leqq \overline{\lim_{\varepsilon\to 0}} \sup_{-\infty<x<\infty}\left(\frac{1}{2\varepsilon}\int_{\lambda-\delta}^{\lambda+\delta}|s(u+\varepsilon,x)-s(u-\varepsilon,x)|^2 du\right)^{\frac{1}{2}}.$$

Thus we attain $\Lambda_0(f) \subseteqq \Lambda_{Wy}(f)$. Since $\Lambda_{Wy}(f)$ is a closed set we conclude $\overline{\Lambda_0(f)} \subseteqq \Lambda_{Wy}(f)$.

(ii) Proof of $\Lambda_{Wy}(f) \subseteqq \overline{\Lambda_0(f)}$. We shall prove that if $\lambda \in \overline{\Lambda_0(f)}$ then $\lambda \notin \Lambda_{Wy}(f)$ too. From $\lambda \in \overline{\Lambda_0(f)}$ there exists an neighbourhood of $\lambda$, $(\lambda-\delta, \lambda+\delta)$ which doen not contain any element of $\Lambda_0(f)$. On the other hand since $f$ is an almost periodic, there exist an trigonometric polynomial

(4. 4) $$p_N(x)=\sum_1^N a_n e^{i\lambda_n x} \qquad \lambda_n\in\Lambda_0(f)\,(n=1,2,\cdots,N)$$

such that

(4. 5) $$\overline{\lim_{T\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{2T}\int_{-T}^{T}|f(t+x)-p_N(t+x)|^2dt\to 0 \qquad (N\to\infty.)$$

Therefore by the one-sided Wiener formula we get

$$\overline{\lim_{\varepsilon\to 0}}\sup_{-\infty<x<\infty}\frac{1}{2\varepsilon}\int_{-\infty}^{\infty}|\{s(u+\varepsilon,x)-s(u-\varepsilon,x)\}-\{s_N(u+\varepsilon,x)-s_N(u-\varepsilon,x)\}|^2du\to 0\,(N\to\infty)$$

where we mean by $s_N(u)$ the Fourier-Wiener transform of $p_N(x)$. Since $s_N(u$ doen contain only elements of $\Lambda_0(f)$ as an spectre we get

$$s_N(u+\varepsilon,x)-s_N(u-\varepsilon,x)=\sum_1^N a_n e^{i\lambda_n x}\sqrt{2\pi}\,\chi_n(u),$$

where $\chi_n(u)$ is an characteristic function of an interval $[\lambda_n-\varepsilon,\lambda_n+\varepsilon]$. Thus we can conclude

$$s_N(u+\varepsilon,x)-s_N(u-\varepsilon,x)=0 \qquad \text{on } (\lambda-\delta,\lambda+\delta).$$

Therefore we get

$$\overline{\lim_{\varepsilon\to 0}}\sup_{-\infty<x<\infty}\frac{1}{2\pi}\int_{\lambda-\delta}^{\lambda+\delta}|s(u+\varepsilon,x)-s(u-\varepsilon,x|^2du=0.$$

Hence we have $\lambda\notin\Lambda_{Wy}(f)$. Thus we have completed the Proof of Theorem 6.

**Theorem 7.** *Let $f(x)$ be a bounded almost periodic function in the sense of Weyl. Let $K(x)$ be an integrable function and its Fourier transform vanish on the set of $\Lambda_*(f)$. Then we have*

(4. 6) $$K*f\sim 0.$$

*Proof of Theorem* 7. By the hypotheses we get

$$\int_{-\infty}^{\infty}K(y)p_N(x-y)dy=\sum_1^N a_n\int_{-\infty}^{\infty}K(y)e^{i\lambda_n(x-y)}dy=0.$$

Thus we get

$$g(x)=\int_{-\infty}^{\infty}K(y)f(x-y)dy=\int_{-\infty}^{\infty}K(y)\{f(x-y)-p_N(x-y)\}\,dy$$

and

$$\frac{1}{l}\int_u^{u+l}|g(x)|^2dx\leqq\left|\int_{-\infty}^{\infty}K(y)dy\right|^2\sup_{-\infty<u<\infty}\frac{1}{l}\int_u^{u+l}|f(y)-p_N(y)|^2dy.$$

Therefore we get by (4. 5)

$$\overline{\lim_{l\to\infty}} \sup_{-\infty<u<\infty} \frac{1}{l}\int_u^{u+l} |g(x)|^2 dx = 0.$$

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