A CHARACTERIZATION OF STRONGLY SEPARABLE ALGEBRAS*)

By

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§ 1. In his paper [2] T. Kanzaki introduced the notion of strongly separable algebras over a commutative ring and obtained an interesting characterization of such algebras [2, Theorem 1].

Throughout the present note, $A \ni 1$ will represent always an algebra over a commutative ring $R \ni 1$, and C the center of A.

Let P be the set of elements $\sum x_i \otimes y_i$ in $A \underset{R}{\otimes} A$ such that $\sum x_i x \otimes y_i = \sum x_i \otimes x y_i$ for all x in A, and let φ be the A–A-(module-) homomorphism of $A \underset{R}{\otimes} A$ into A defined by $\varphi(\sum x_i \otimes y_i) = \sum x_i y_i$. If $\varphi(P)$ contains 1, then A is said to be a strongly separable (R-) algebra. (The definition is somewhat different from the original one in [2]. But, as is easily seen, the two definitions are equivalent).

The purpose of the present note is to give an another characterization of a strongly separable algebra A when it is R-finitely generated and projective (Theorem 2).

§ 2. A is said to be a Frobenius (resp. symmetric) algebra if A is a finitely generated, projective R-module and there exists an A-isomorphism: $A_A \cong \mathrm{Hom}_R(A, R)_A$ (resp. ${}_AA_A \cong {}_A\mathrm{Hom}_R(A, R)_A$), where $\mathrm{Hom}_R(A, R)$ is regarded as an A–A-module by the following operations:

$$bfa(x) = f(axb) \qquad a, b, x \in A, \ f \in \mathrm{Hom}_R(A, R)$$

At first we shall quote here the following theorem which is due to Kanzaki [2, Theorem 1].

Theorem 1. *A is a strongly separable (R-) algebra if and only if A is a separable (R-) algebra and $A = C \oplus [A, A]$, as C-module, where $[A, A]$ is the C-submodule of A generated by all $xy - yx$ (x, y in A).*

Lemma 1. *Let $B \ni 1$ be a ring, and Z the center of B. If $B = Z \oplus [B, B]$ as Z-module, then $\mathrm{Hom}_Z(B, Z)^B$ (the set of elements f in $\mathrm{Hom}_Z(B, Z)$ such that $af = fa$ for all a in B) is a free Z-module of rank 1.*

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Proof. Let f be an element in $\mathrm{Hom}_Z(B, Z)^B$. Then $f(xy-yx)=0$ for all x, y in B. Since $B=Z\oplus[B, B]$ f is uniquely determined by the effect on Z, namely, by $f(1)$. Thus we see that $\mathrm{Hom}_Z(B, Z)^B$ is Z-isomorphic to Z.

Lemma 2. *Let A be R-finitely generated and projective. If A is a strongly separable (R-) algebra, then A is a symmetric (R-) algebra.*

Proof. By Theorem 1 and Theorem 2.3 [1], A is a central separable C-algebra. Then, by Theorem 3.1 (c) [1], we have an A–A-isomorphism:

$$A \underset{C}{\otimes} \mathrm{Hom}_C(A, C)^A \cong \mathrm{Hom}_C(A, C),$$

where the left side term is regarded as an A–A-module by the operations on the first factor. Since $\mathrm{Hom}_C(A, C)^A$ is a free C-module of rank 1 by Theorem 1 and Lemma 1, the above isomorphism induces naturally the A–A-isomorphism

$${}_AA_A \cong \mathrm{Hom}_C(A, C).$$

Thus A is a symmetric C-algebra. By Proposition A. 4 [1], C is a commutative Frobenius whence symmetric R-algebra. Then by Theorem 2 [3] we can conclude that A is a symmetric R-algebra.

Theorem 2. *Let A be R-finitely generated and projective. Then, A is a strongly separable R-algebra if and only if there exist elements $r_1, r_2, \cdots, r_n$; $l_1, l_2, \cdots, l_n$ in A and a homomorphism h in $\mathrm{Hom}_R(A, R)$ such that*

$$(*) \qquad a = \sum r_i h(l_i a) = \sum h(a r_i) l_i, \quad h(xy)=h(yx)$$

for all a, x, y in A and that $\sum r_i l_i$ is a unit in C.

Proof. Let A be a strongly separable R-algebra. Then by Lemma 2 A is a symmetric algebra, whence by Theorem 1 [3] there exist elements $r_1, r_2, \cdots, r_n$; $l_1, l_2, \cdots, l_n$ in A and h in $\mathrm{Hom}_R(A, R)$ such that $(*)$ hold for all a, x, y in A. When this is the case, the mapping

$$A \ni a \longrightarrow ha \in \mathrm{Hom}_R(A, R)$$

gives an A-A-isomorphism of ${}_AA_A$ onto ${}_A\mathrm{Hom}_R(A, R)_A$.

Now, consider the A–A-isomorphism:

$${}_AA \underset{R}{\otimes} A_A \longrightarrow {}_A\mathrm{Hom}_R\big(\mathrm{Hom}_R(A, R), A\big)_A \longrightarrow {}_A\mathrm{Hom}_R(A, A)_A$$

defined by

$$x\otimes y \longrightarrow \big(f=ha \longrightarrow f(x)y=h(ax)y\big) \longrightarrow \big(a \longrightarrow f(x)y=h(ax)y\big)$$

$$x, y, a\in A, \quad f\in \mathrm{Hom}_R(A, R).$$

Then, as is easily seen, the image of P in $\mathrm{Hom}_R(A, A)$ is just $\mathrm{Hom}_A(A, A)=A_r$, the right multiplication ring of A. Since the image of $\sum r_i \otimes l_i a$ is

$$\left(x \longrightarrow \sum h(xr_i) l_i a = xa\right) = a_r\,,$$

we see that P is the set $\{\sum r_i \otimes l_i a | a \in A\}$. Since A is a strongly separable R-algebra there exists an element a_0 in A such that $\sum r_i l_i a_0 = 1$. Noting here that $\sum r_i l_i$ is in the center of A, we see that $\sum r_i l_i$ is a unit in C. The converse part of the theorem is almost obvious.

References

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