RELATIONS BETWEEN TWO MARTIN TOPOLOGIES ON A RIEMANN SURFACE

By

Zenjiro KURAMOCHI

Let R be a Riemann surface. Let G be a domain in R with relative boundary ∂G of positive capacity. Let $U(z)$ be a positive superharmonic function in G such that the Dirichlet integral $D(\min(M, U(z)))<\infty$ for every M. Let D be a compact domain in G. Let ${}_DU^M(z)$ be the lower envelope of superharmonic functions $\{U_n(z)\}$ such that $U_n(z)\geqq\min(M, U(z))$ on $D+\partial G$ except a set of capacity zero, $U_n(z)$ is harmonic in $G-D$ and $U_n(z)$ has M.D.I. (minimal Dirichlet integral) $\leqq D(\min(M, U(z))<\infty$ over $G-D$ with the same value as $U_n(z)$ on $\partial G+\partial D$. Then ${}_DU^M(z)$ is uniquely determined. Put ${}_DU(z)=\lim_{M=\infty} {}_DU^M(z)$. If for any compact domain D ${}_DU(z)=U(z)$ or ${}_DU(z)\leqq U(z)$, we call $U(z)$ a full harmonic (F.H.) or a full superharmonic (F.S.H.) in G respectively. If $U(z)$ is an F.S.H. in G and $U(z)=0$ on ∂G except a set of capacity zero, $U(z)$ is called an F_0.S.H. in G. Let $U(z)$ be an F.S.H. in G. Then ${}_DU(z)\uparrow$ as $D\uparrow$. For a non compact domain D, put $U(z)=\lim_{n=\infty} {}_{G_n\cap D}U(z)$, where $\{G_n\}$ is an exhaustion of G with compact relative boundary ∂G_n $(n=0,1,2\cdots)$.

$\mathfrak{M}^f(U(z))$ *of an* F_0.S.H. $U(z)$ *in* G. Let D be a domain in G. Suppose there exists at least one C_1-function $V(z)$ in $G-D$ such that $V(z)=1$ on D, $=0$ on ∂G except a set of capacity zero and $D(V(z))<\infty$. Let $\omega(D, z, G)$ be a harmonic function in $G-D$ such that $\omega(D, z, G)=1$ on D, $=0$ on ∂G except a set of capacity zero and $\omega(D, z, G)$ has M.D.I. over $G-D$. We call $\omega(D,z,G)$ a C.P. (capacitary potential) of D. Let $U(z)$ be an F_0.S.H. in G. Then $D_{Cg_M}({}_{g_M}U(z)) =MD(\omega(g_M, z, G))\uparrow$ as $M\to 0$,[1] where $g_M=E[z\colon U(z)>M]$.

Put $\mathfrak{M}^f(U(z))=\lim_{M\to} \frac{1}{2\pi} D_{Cg_M}({}_{g_M}U(z))$.

$\mathfrak{M}^f(U(z))$ *of an* F.S.H. $U(z)$ *in* G. For any compact domain D in G, if we can define functions $U_n(z)$ such that $U_n(z)$ is superharmonic in G, $U_n(z)$ is harmonic in $G-D$, $U_n(z)\geqq\min(M, U(z))$ on D, $U_n(z)=0$ on ∂G except a set of capacity zero and $U_n(z)$ has M.D.I. over $G-D$. Let ${}^0U^M(z)$ be the lower envelope of $\{U_n(z)\}$. Put ${}^0_DU(z)=\lim_{M=\infty} {}^0_DU^M(z)$ (clearly ${}^0_DU(z)\leqq {}_DU(z)$).

---

1) Z. Kuramochi: Superharmonic functions in a domain of a Riemann surface. Nagoya Math. J., to appear.

Since D is compact, ${}^{0}_{D}U(z)=0$ on ∂G except a set of capacity zero. For a non compact domain D, ${}^{0}_{D}U(z)$ is defined as ${}_{D}U(z)$. For $U(z)$, put $\mathfrak{M}^{f}(U(z))=\lim_{n=\infty}\mathfrak{M}^{f}$ $({}^{0}_{G_n}U(z))$, where $\{G_n\}$ is an exhaustion of G with compact relative boundary ∂G_n.

Let $\{R_n\}$ with compact relative boundary ∂R_n $(n=0,1,2,\cdots)$ Let $U(z)$ be an F_0.S.H. in $R-R_0$ such that $U(z)=0$ on ∂R_0. Consider $R-R_0$ as G. Then ${}_{D}U(z)$ is defined. In this case we say that ${}_{D}U(z)$ is defined relative to $R-R_0$. It is clear that the mapping $U(z)\rightarrow{}_{D}U(z)$ depends on the domain (G or $R-R_0$) in which ${}_{D}U(z)$ is defined. In the following we use ${}_{D}U(z)$ relative to $R-R_0$ which will be denoted by ${}^{R}_{D}U(z)$ to distinguish from ${}_{D}U(z)$ (relative to G). We understand ${}_{D}U(z)$ (without R on D) means ${}_{D}U(z)$ of $U(z)$ relative to G.

Martin topologies on $R-R_0$ and on a subdomain $G\subset(R-R_0)$. Let $N(z,p)$ bs an N-Green's function of G such that $N(z,p)$ is positively harmonic in $G-p$, $N(z,p)=0$ on ∂G except a set of capacity zero, $N(z,p)$ has a logarithmic singularity at p and $N(z,p)$ has M. D. I. (where Dirichlet integral is taken with respect to $N(z,p)+\log|z-p|$ in a neighbourhood of p). We suppose N-Martin topology is defined on $G+B$ using $N(z,p)$,s and the distance between p_1 and p_2 is given as

$$\delta(p_1,p_2)=\sup_{z\in D}\left|\frac{N(z,p_1)}{1+N(z,p_1)}-\frac{N(z,p_2)}{1+N(z,p_2)}\right|,$$

where D is a fixed compact domain and B is the set of the ideal boundary. Let $L(z,p)$ be an N-Green's function of $R-R_0$ with pole at p. Then also N-Martin topology is introduced on $R-R_0+B^L$ with metric:

$$\delta(p_1,p_2)=\sup_{z\in R_1}\left|\frac{L(z,p_1)}{1+L(z,p_1)}-\frac{L(z,p_2)}{1+L(z,p_2)}\right|,$$

where B^L is the set of the ideal boundary points.
In the following for simplicity we call above two topogies L and N-topologies.

Let $p\in R-R_0+B^L_1$ (B^L_1 is the set of minimal boundary points of $R-R_0$). If ${}^{R}_{CG}L(z,p)<L(z,p)$ (CG is thin at p), we denote by $p\overset{L}{\in}G$. Then

Theorem 1. *Suppose* $p\in R-R_0+B^L_1$ *and* $p\overset{L}{\in}G$. *Then* $U(z,p)=L(z,p)-N(z,p)$ *is an* F_0.S.H. *in* G *with* $D(\min(M,U(z)))\leqq 2\pi M$, *whence* $\mathfrak{M}^{f}(U(z,p))\leqq 1$.

Proof. $Nz,p):p\in R-R_0+B^L$ is contintinuous on ∂G except p. Hence ${}^{R}_{CG}L(z,p)=L(z,p)$ on ∂G and $U(z,p)=0$ on ∂G except a set of capacity zero.
Case 1. $p\in G$. In this case, clearly $U(z,p)=N(z,p)$ and $D(\min(M,U(z,p))\leqq 2\pi M$.
Case 2. $p\in\partial G$. Put $G_n=G+v_n(p)$, Then $CG_n\uparrow CG$ and ${}^{R}_{CG_n}L(z,p)\uparrow{}^{R}_{CG}L(z,p)$

as $n\rightarrow\infty$, where $v_n(p)=E\left[z: \operatorname{dist}(z,p)<\frac{1}{n}\right]$. By $p\in G_n$, we have

$$D(\min(M, U(z,p))\leqq\varliminf_n D(\min(M, L(z,p)-{}_{CG_n}^{R}L(z,p)))\leqq 2\pi M$$

Case 3. $p\in B_1^L-B_s^L$. In this case it was proved[2] $D(\min(M, U(z,p))\leqq 2\pi M$, where B_s^L is the set of singular points, i.e. set of point p such that $\omega(p,z,R-R_0)>0$ and B_1^L is the set of minimal boundary points of $R-R_0$.

Case 4. $p\in B_s^L$. It was proved only $D(U(z,p))<\infty$ but as case 3 it can be proved $D(\min(M, U(z,p)))\leqq 2\pi M$.

Hence ${}_D^RU(z,p)$ can be defined. Now ${}_{CG+D}^{R}({}_{CG}^{R}L(z,p))={}_{CG}^{R}L(z,p)$ by $CG+D)\subset CG$ and ${}_{CG+D}^{R}L(z,p)\leqq L(z,p)$. Hence ${}_DU(z,p)={}_{CG+D}^{R}(L(z,p)-{}_{CG}^{R}L(z,p))={}_{CG+D}^{R}L(z,p)-{}_{CG}^{R}L(z,p)$[3] $\leqq L(z,p)-{}_{CG}^{R}L(z,p)=U(z,p)$. By $D(\min(M, U(z,p)))\leqq 2\pi M$ we have at once $\mathfrak{M}^f(U(z,p))\leqq 1$. Thus $U(z,p)$ is an F_0.S.H. in G with $\mathfrak{M}^f(U(z,p))\leqq 1$.

Lemma 1. *1). Let $p_i\in R-R_0$ and $p_i\overset{L}{\rightarrow}p\in R-R_0+B^L$ (p_i tends to p relative to L-topology). Then $L(z,p)-\varliminf_i {}_{CG}^{R}L(z,p_i)\leqq L(z,p)-{}_{CG}^{R}L(z,p)$.*

2). Let $p_i\overset{L}{\rightarrow}p^\alpha\in R-R_0+B_1^L$ and $p_o\overset{M}{\rightarrow}p^\beta\in G+B: p_i\in G$. Then

$$N(z,p^\beta)=(1-a)\,(L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha)):\quad 1\geqq a\geqq 0.$$

Proof of 1). For any $\varepsilon>0$ we can find a number n_0 such that ${}_{CG}^{R}L(z,p^\alpha)\leqq {}_{CG\cap R_n}^{R}L(z,p^\alpha)+\varepsilon$ for $n\geqq n_0$. Since $L(z,p_i)\rightarrow L(z,p^\alpha)$ on $CG\cap R_n$, $\varliminf_i {}_{CG}^{R}L(z,p_i)\geqq\varliminf_i {}_{CG\cap R_n}^{R}L(z,p_i)\geqq {}_{CG}^{R}L(z,p^\alpha)-\varepsilon$. Let $\varepsilon\rightarrow 0$. Then we have (1).

Proof of 2). $L(z,p_i)-{}_{CG}^{R}L(z,p_i)=N(z,p_i)$ in G for $p_i\in G$. By the assumption $\lim_i L(z,p_i)$ and $\lim_i N(z,p_i)$ exist, whence $\lim_i {}_{CG}^{R}L(z,p_i)$ exists. We denote this limit by $U(z)$. Let μ be a canonical mass distribution[4] of $U(z)$ on $R-R_0+B_1^L$. Assume μ has a positive mass in int $(G\cap Cv_n(p^\alpha))$ (int G means the interior of G relative to L-topology and $v_n(p^\alpha)$ is a neighbouhood of p^α relative to L-topology). Then we can find a number n_0 such that G_{n_0} has a positive mass on $\bar{G}_{n_0}\cap Cv_n(p^\alpha)$, where $G_n=E\left[z\in R-R_0+B^L: \operatorname{dist}(z, CG)>\frac{1}{n}\right]$. Since $\operatorname{dist}(CG+v_{n+i}(p^\alpha), G_{n_0}-v_n(p^\alpha)))>0$,

2) Z. Kuramochi: Correspondence of boundaries of Riemann surfaces. Journ. Fac. Sci. Hokkaido Uni., XVII (1963). See page 101.

3) If $p\in G$, $U(z,p)=N(z,p)$, we suppose $p\in B^G$. Then $L(z,p)$ is harmonic in $R-R_0$, whence $\sup L(z,p)<\infty$ on a compact domin D and it is clear ${}_DU(z)={}_{CG+D}^{R}(L(z,p)-{}_{CG}^{R}L(z,p))$. If D is non compact, consider $D\cap G_n$ and let $n\rightarrow\infty$.

4) Z. Kuramochi: Potentials on Riemann surfaces. Journ. Fac. Sci. Hokkaido Univ., XVI (1962).

$${}_{CG+v_{n+i}(p^\alpha)}^{R}U(z)<U(z).^{5)}$$

Hence by ${}_{CG+v_{n+i}(p^\alpha)}L(z,p^\alpha)=L(z,p^\alpha)$ (for $p^\alpha\in R-R+B_1^L$) we have

$$\begin{aligned}N(z,p^\beta)&=L(z,p^\alpha)-U(z)>{}_{CG+v_{n+i}(p^\alpha)}^{R}L(z,p^\alpha)-{}_{CG+v_{n+i}(p^\alpha)}^{R}U(z)\\&={}_{CG+v_{n+i}(p^\alpha)}^{R}(L(z,p^\alpha)-U(z))={}_{v_{n+i}(p^\alpha)}(L(z,p^\alpha)-U(z)).\qquad(1)\end{aligned}$$

On the other hand, $L(z,p^\alpha)-U(z)=N(z,p^\beta)$ is an F_0.S.H. in G, whence

$${}_{v_{n+i}(p^\alpha)}(L(z,p^\alpha)-U(z))\leqq L(z,p^\alpha)-U(z).\qquad(2)$$

(1) contradicts (2). Hence $\mu=0$ on $Cv_n(p)\cap\operatorname{int}G$. Let $n\to\infty$. Then $\mu=0$ except on $p+CG$. *put* $V(z)=\int L(z,p)d\mu'(p)$, where μ' is the restriction of μ on CG. Let a be the mass of μ at p. Then $1\geqq a\geqq 0$, ${}_{CG}^{R}V(z)=V(z)$ and $U(z)=V(z)+aL(z,p^\alpha)$. Now $V(z)=(1-a)L(z,p^\alpha)$ on ∂G excpt a set of capacity zero. Hence $V(z)={}_{CG}^{R}V(z)=(1-a){}_{CG}^{R}L(z,p^\alpha)$. Thus $U(z)=(1-a){}_{CG}^{R}L(z,p^\alpha)+aL(z,p^\alpha)$ and

$$N(z,p^\beta)=L(z,p^\alpha)-\lim_i {}_{CG}^{R}L(z,p_i)=(1-a)((L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha)).$$

We denote by $\overset{L}{B}(G)$ the set of points p such that $p\in R-R_0+B_1^L$, $p\in B$ and $p\overset{L}{\in}G$. Clearly $\overset{L}{B}(G)$ is an F_σ set relative to L-topology by the upper semicontinuity of $L(z,\mathrm{p})-{}_{CG}^{R}L(z,p)$ and *if* $p\in\partial G$, $p\in\overset{L}{B}(G)$ *if and only if* p *is an irregular point for the Dirichlet problem in* G *by Lemma 1. (2).*

Lemma 2. *Let* $p_i\in\overset{L}{B}(G)+\overset{L}{G}$ *and* $p_1\neq p_2$. *Then* $L(z,p_1)-{}_{CG}^{R}L(z,p_1)\neq L(z,p_2)-{}_{CG}^{R}L(z,p_2)$.

Assume $L(z,p_1)-{}_{CG}^{R}L(z,p_1)=L(z,p_2)-{}_{CG}^{R}L(z,p_2)=U(z)$. Let n be a number such that $\operatorname{dist}(v_n(p_1),v_n(p_2))>0$, where $v_n(p_i)$ is a neigebouhood of p_i relative to L-topology. Now $p_2\overset{L}{\in}v_n(p_2)$ imply

$$(G\cap v_n(p_2))\overset{L}{\ni}p_2.^{6)}$$

Let $V_n=G-v_n(p_1)$. Then $V_n\supset(\mathrm{G}\cap v_n(p_2))\overset{L}{\ni}p_2$. Whence

$${}_{CV_n}^{R}L(z,p)<L(z,p).$$

By $CV_n\supset CG$ we have ${}_{CV_n}^{R}({}_{CG}^{R}L(z,p_i))={}_{CG}^{R}L(z,p_i):i=1,2$. Now ${}_{CV_n}^{R}L(z,p)\downarrow$ as $n\to\infty$ by $CV_n\downarrow$. Hence there exist a point z_0 *in* V_{n_0}, a number n_0 and a const. $\delta>0$ such that ${}_{CV_n}^{R}L(z_0,p_2)<L(z_0,p_2)-\delta$ for $n\geqq n_0$. Hence

$$\begin{aligned}{}_{CV_n}^{R}(U(z_0))&={}_{CN_n}^{R}L(z_0,p_2)-{}_{CG}^{R}L(z_0,p_2)<{}_{CG}^{R}L(z_0,p_2)-{}_{CG}^{R}L(z_0,p_2)-\delta\\&=U(z_0)-\delta:n\geqq n_0.\qquad(3)\end{aligned}$$

5) See page 60 of 4).
6) See page 99 of 2).

By $CV_n+(v_n(p_1)\cap CG)\supset v_n(p_1)$, we have

$$ {}_{CV_n}^{R}L(z,p_1)+{}_{v_n(p_1)\cap CG}^{R}L(z,p_1)\geqq {}_{v_n(p_1)}^{R}L(z,p_1)=L(z,p_1). $$

We proved if a domain $\Omega \overset{L}{\in} p$, $\lim_{n} {}_{v_n(p)\cap CG}^{R}L(z,p)=0$.[7] Hence for any $\varepsilon>0$ there exists a number n' such that ${}_{CV_n}^{R}L(z_0,p_1)\geqq L(z_0,p)-\varepsilon$ for $n\geqq n'$. Hence

$$ \begin{aligned} {}_{CV_n}^{R}U(z_0) &= {}_{CV_n}^{R}L(z_0,p_1)-{}_{CV_n}^{R}({}_{CG}^{R}L(z_0,p_1))={}_{CV_n}^{R}L(z_0,p_1)-{}_{CG}^{R}L(z_0,p_1) \\ &\geqq L(z_0,p)-{}_{CG}^{R}L(z_0,p_1)-\varepsilon=U(z_0)-\varepsilon, \qquad \text{for } n\geqq n'. \qquad (4) \end{aligned} $$

By (3) and (4) $U(z_0)-\delta\geqq U(z_0)-\varepsilon$. This is a contradiction. Hence $L(z,p_1)-{}_{CG}^{R}L(z,p_1)\neq L(z,p_2)-{}_{CG}^{R}L(z,p_2)$.

Let p^α be a point in $G+\overset{L}{B}(G)$. If there exists a sequence $\{p_i\}$ such that $p_i\overset{L}{\to}p^\alpha$ and $p_i\overset{M}{\to}p^\beta\in G+B$, we say that p^β lies on p^α. We denote the set of points p lying on p^α by $\mathfrak{p}(p^\alpha)$. Then

Lemma 3. *Let $p^\alpha\in G+\overset{L}{B}(G)$. Then $\mathfrak{p}(p^\alpha)$ contains only one point p^β of $G+B_1$ and $L(z,p^\alpha)-{}_{CG}^{R}L(z,p)=N(z,p^\beta)$, where B_1 is the set of minimal boundary points of G relative to N-topology. We denote such p^β by $f(p_\alpha)$.*

Let $p_i\overset{L}{\to}p^\alpha$ and $p_i\overset{M}{\to}p^\beta$. Then by Lemma 1.2) $N(z,p^\beta)=(1-a_\beta)(L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha))$. Hence any function $N(z,p^\beta)$ corresponding to p^α is a submultiple of a fixed function and there exists at most one minimal or inner point $p^{\beta'}$ of $G+B_1$ in $\mathfrak{p}(p^\alpha)$ such that $\mathfrak{M}(p^{\beta'})=1$ ($\mathfrak{M}(p^{\beta'})=\mathfrak{M}^f(N(z,p^{\beta'}))=1$ is a necessary condition for $p^{\beta'}$ to be minimal).[8] Let $p^\alpha\in G+\overset{L}{B}(G)$ and $v_n(p^\alpha)$ be a neighbourhood of p^α relative to L-topology and $\bar{v}_n(p^\alpha)$ be the closure of $v_n(p_\alpha)$ relative by M-topology. Then by $p\in G+\overset{L}{B}(G)$ $L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha)=\delta_\beta N(z,p^\beta)$: $\delta_\beta=\dfrac{1}{1-a_\beta}$ and by ${}_{v_n(p^\alpha)}L(z,p^\alpha=L(z,p)$ and $CG+v_n(p^\alpha)\supset CG$ we have

$$ \begin{aligned} \delta_\beta N(z,p^\beta) &= L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha)={}_{CG+v_n(p^\alpha)}(L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha)) \\ &= \delta_{\beta_{\bar{v}_n(p^\alpha)}}N(z,p^\beta). \end{aligned} $$

Let $n\to\infty$. Then $N(z,p^\beta)={}_FN(z,p^\beta)>0$, where $F=\bigcap_{n>0}\bar{v}_n(p_\alpha)$ is a M-closed set, whence $N(z,p^\beta)$ is representable by a canonical mass distribution on F.[9] This implies $\mathfrak{p}(p^\alpha)$ contains at least one point in $G+B_1$. Thus $\mathfrak{p}(p^\alpha)$ contains only one point p^* in $G+B_1$ and $(1-a^*)(L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha))=N(z,p^*)$. On the other hand, $\mathfrak{M}^f(L(z,p)-{}_{CG}^{R}L(z,p))\leqq 1$ by Theorem 1 and $\mathfrak{M}^f(N(z,p^*))=1$.

7) See 6).
8) See Lemma 4 of 1).
9) See 5).

Hence $a^*=0$ and $L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha)=N(z,p^*)$.

Theorem 2. *Let p^β be a point in $G+B_1$. Let $f^{-1}(p^\beta)$ be the set of points p in $R-R_0+B^L$ (not only in $G+\overset{L}{B}(G)$) such that $L(z,p)-{}_{CG}^{R}L(z,p)=N(z,p^\beta)$. Then $f^{-1}(p^\beta)$ consists of only one point $p\in G+\overset{L}{B}(G)$. Hence the mapping $f(p^\alpha)$: $p^\alpha\in G+\overset{L}{B}(G)$ is one-to-one manner between $G+\overset{L}{B}(G)$ and $G+B_1$ and further $f^{-1}(p^\beta)$ is a continuous function of p^β in $G+B_1$, but $f(p^\alpha)$ is not necessarily continuous in $G+\overset{L}{B}(G)$.*

Let $p\in f^{-1}(p^\beta)$. Then $L(z,p)-{}_{CG}^{R}L(z,p)$ is minimal in G and is equal to $N(z,p^\beta):p\in G+B_1$. There exists a canonical distribution $\mu(p^\alpha)$ on $R-R_0+B_1^L$ such that $L(z,p)=\int L(z,p^\alpha)d\mu(p^\alpha)$. Hence

$$N(z,p^\beta)=L(z,p)-{}_{CG}^{R}L(z,p)=\int(L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha))d\mu(p^\alpha).^{10)}$$

Now by Lemma 3 $L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha)=N(z,q)$ is minimal in G, where $p^\alpha\in G+\overset{L}{B}(G)$ and $q=f(p^\alpha)$. Clearly $L(z,p^\alpha)-{}_{CG}^{R}L(z,p^\alpha)=0$ for $p^\alpha\notin G+\overset{L}{B}(G)$. Since $N(z,p^\beta)$ is minimal $\mu(p^\alpha)$ must be a point mass a at $p'\in R-R_0+B_1^L$ and clearly $p'\in G+\overset{L}{B}(G)$. Hence $N(z,p^\beta)=a(L(z,p')-{}_{CG}^{R}L(z,p)):a>0$. But $\mathfrak{M}^f(N(z,p^\beta))=1$ and $\mathfrak{M}(L(z,p^\alpha)-{}_{CG}^{R}L(z,p))\leqq 1$ by Theorem 1, hence $a=1$ and $N(z,p^\beta)=L(z,p')-{}_{CG}^{R}L(z,p'):p'\in G+\overset{L}{B}(G)$.

Suppose there exist two points p_1 and p_2 in $G+\overset{L}{B}(G)$ such that $L(z,p_i)-{}_{CG}^{R}L(z,p_i)=N(z,p^\beta)$: $i=1,2$. Then by Lemma 2 $p_1=p_2$. Thus $f^{-1}(p^\beta)$ is uniquely determined and $f^{-1}(p^\beta)\in G+\overset{L}{B}(G)$.

We show $f^{-1}(p^\beta)$ is continuous in $G+B_1$. Let $p_i^\beta\in G+B_1$ and $p_i^\beta\overset{M}{\to}p^\beta\in G+B_1$ as $i\to\infty$ and let $p_i^\alpha=f^{-1}(p_i^\beta)$. Then $\{p_i^\alpha\}$ has at least one limiting point p in $\overline{R-R_0}+B^L$, since $R-R_1+B^L$ is compact. Let $\{p_j^\alpha\}$ be a subsequence of $\{p_i^\alpha\}$ such that $p_j^\alpha\to p$ and $p_j^\beta\to p^\beta:p_j^\beta=f(p_j^\alpha)$. Then $\lim_j L(z,p_j^\alpha)=L(z,p)$, $\lim_j N(z,p_j^\beta)=N(z,p^\beta)$ and $\lim_j {}_{CG}^{R}L(z,p_j^\alpha)$ exists, i.e. $L(z,p)-\lim_j {}_{CG}^{R}L(z,p_j^\alpha)=N(z,p^\beta)$. Let $p'=f^{-1}(p^\beta)$. Then $L(z,p')-{}_{CG}^{R}L(z,p')=N(z,p^\beta)$ and $p'\in G+\overset{L}{B}(G)$. By $\lim_j {}_{CG}^{R}L(z,p_j^\alpha)\geqq {}_{CG}^{R}(\lim_j L(z,p_j^\alpha))={}_{CG}^{R}L(z,p)$, we have

$$L(z,p)-{}_{CG}^{R}L(z,p)\geqq L(z,p)-\lim_j {}_{CG}^{R}L(z,p_j^\alpha)=N(z,p^\beta).$$

Let $\mu(q)$ be a canonical mass distribution of $L(z,p)$ on $R-R_0+B_1^L$. Then $L(z,p)=\int L(z,q)d\mu(q)$ and $\int d\mu(q)=1$ by $\mathfrak{M}^f(L(z,p))=\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}L(z,p)ds$

10) Becaus $\int {}_{CG}L(z,p)d\mu(p)={}_{CG}(\int L(z,p)d\mu(p))$. See Theorem 1 of 1).

$=\int d\mu(p)=1$. Now

$$L(z,p)-{}_{CG}^{R}L(z,p)=\int(L(z,q)-{}_{CG}^{R}L(z,q))d\mu(q)=\int N(z,q^\beta)\delta(q)d\mu(q),$$

where $\delta(q)=1$ or 0 according as $q\in G+\overset{L}{B}(G)$ or not and $q^\beta=f(q)\subset G+B_1$. Hence $\mathfrak{M}^f(L(z,p)-{}_{CG}^{R}L(z,p))=\int\delta(q)d\mu(q)$ by Theorem 6.[11] On the other hand, by $N(z,p^\beta)\leqq L(z,p)-{}_{CG}^{R}L(z,p)$, $\mathfrak{M}^f(N(z,p^\beta))=1\leqq\mathfrak{M}^f(L(z,p)-{}_{CG}^{R}L(z,p))\leqq 1$ by Theorem 1. Hence $\delta(q)=1$ if $\mu(q)>0$ and $\int d\mu(q)=1=\mathfrak{M}^f(L(z,p)-{}_{CG}^{R}L(z,p))$. Both $L(z,p)-{}_{CG}^{R}L(z,p)$ and $N(z,p^\beta)$ are F_0.S.H.s in G. Let $V_M=E[z:L(z,p)-{}_{CG}^{R}L(z,p)>M]$ and $V'_M=E[z:N(z,p)>M$. Then $V_M\supset V'_M$. $\mathfrak{M}^f(N(z,p^\beta))=\dfrac{MD(\omega(V'_M,z,G))}{2\tau}=1$ for any M, since $N(z,p^\beta)$ is minimal and ${}_{V'_M}N(z,p^\beta)=N(z,p^\beta)$. Also $1=\mathfrak{M}^f(L(z,p)-{}_{CG}^{R}L(z,p))\geqq\dfrac{MD(\omega(V_M,z,G))}{2\tau}\geqq\dfrac{MD(\omega(V'_N,z,G))}{2\pi}=1$, because $MD(\omega(V_M,z,G))\uparrow$ as $M\to 0$. Hence $V_M\supset V'_M$ and $\omega(V_M,z,G)=\omega(V'_M,z,G)$ for any M. This implies $L(z,p)-{}_{CM}^{R}L(z,p)=N(z,p^\beta)$ and $p=f^{-1}(p^\beta)=p'\in G+\overset{L}{B}(G)$. Since any subsequence $p_j^\alpha\to p'$, $\{p_i^\alpha\}$ converges to $f^{-1}(p_\beta)$ as $p_i^\beta\to p^\beta$.

We show $f(p^\alpha)$ is not necessarily continuous. Let $R-R_0$ be $E[0<|z|<1]=\Omega$, and F be a closed set on the real axis such that $z_0=0$ is an irregular point for the Dirichlet problem of $G=\Omega-F$, where $F=\sum_{K=0}^{\infty}F_K$ and F_K is a segment. Then $L(z,p)$ of Ω and $N(z,p)$ of G are Green's functions $G(z,p)$ and $G'(z,p)$ of Ω and G respectively. Then by Lemma 3 there exists a sequence $\{p_i\}$ such that $G(z,p_i)$ converges to a function $G'(z,p^\beta)$ with $\mathfrak{M}^f(G'(z,p^\beta))=1$ and $p_i\to z_0$. Hence $p^\beta=f(z_0)$. Let p_0 be a fixed point in G. Let q_i be a point such that q_i is so near F_i that $G'(p_0,q_i)\leqq\dfrac{1}{i}$. Then $\lim_i G(z,q_i)=0$. For any i we can find $G'(z,p'_i)$ such that p'_i lies on a curvc connecting p_i and q_i and that $G(p_0,p'_i)\to a(G'(p_0,p^\beta)$ as $i\to\infty$, where $0<a<1$. Also we choose a subsequence $\{p'_j\}$ from $\{p'_i\}$ so that $p'_i\to z_0$ (relative to L-topology) and $G'(z,p'_j)\to p^{\beta'}$ (relative to M-topology): $p^{\beta'}\neq p^\beta$. Then since $p'_j\in G'_j$, $p'_j=f(p'_j)$ and $p'_j\overset{L}{\to}z_0$ but $f(p'_j)\overset{M}{\to}p^{\beta'}\neq p^\beta=f(z_0)$. Hence $f(p)$ is not continuous at z_0.

We call the *harmonic dimension of* $p\in(\partial G+B)$ *relative* to G and $R-R_0$ the number of linearly independent F_0.S.H.s with finite $\mathfrak{M}^f$ in G and $R-R_0$ which are harmonic in G and $R-R_0$ respectively. Then by Lemma 1 we have the following

11) If μ is canonical, $\mathfrak{M}^f(U(z))=\int d\mu(p)$. See Theorem 6 of 1).

Corollarly. *Harmonic dimension of p relative to $R-R_0$ is equal to that of p relative to G.*

Applications to extremisations. Let $U(z)$ be an F_0.S.H. in $R-R_0$ with $\mathfrak{M}^f(U(z))<\infty$. Then there exists a canonical distribution μ such that $U(z)=\int L(z,p)d\mu(p)$[12] and $\int d\mu(p)=\mathfrak{M}^f(U(z))$. Put $V(z)=U(z)-{}_{CG}^{R}U(z)$. Then $V(z)=\int(L(z,p)-{}_{CG}^{R}L(z,p))d\mu(p)=\int N(z,q)\delta(p)d\mu(p)$, where $q=f(p)$ and $\delta(p)=1$ or 0 according as $p\in G+\overset{L}{B}(G)$ or not. Hence $V(z)$ is an F.S.H. in G with $\mathfrak{M}^f(V(z))\leqq\int\delta(p)d\mu(p)<\infty$ and $U(z)-V(z)={}_{CG}U(z)$ is full harmonic in G. We denote $V(z)$ by ${}_{in\,ex}U(z)$. Let $V'(z)$ be an F.S.H. in G with $\mathfrak{M}^f(V(z))<\infty$. Then $V(z)$ is a potential such that $V(z)=\int_{G+B_1}N(z,q)d\mu(q)$[13] and $\int d\mu(q)=\mathfrak{M}^f(V'(z))$. Put $U'(z)=\int L(z,p)d\mu(q)$, where $p=f^{-1}(q)$. Then $U'(z)$ is an F_0.S.H. in $R-R_0$ with $\mathfrak{M}^f(U'(z))\leqq\int d\mu(q)$ and $U'(z)-V(z)$ is full harmonic in G. We denote $U'(z)$ by ${}_{ex}V'(z)$. Then $U'(z)-{}_{CG}U'(z)=V'(z)$.

Let $\{G_n\}$ be an exhaustionof G with compact relative boundary ∂G_n. Since ${}_{G_n}V'(z)$ is full harmonic in $G-\overline{G}_n$, the solution of Neumann's problem (to obtain an F_0.S.H. $W(z)$ in $R-R_0$ such that $W(z)-{}_{G_n}V'(z)$ is full harmonic in G_{n+i+j} and $W(z)$ is full harmonic in $G-G_{n+i}$) can be obtained by smooting process by dist$(\partial G_{n+i}, \partial G_{n+i+j})>0$ for a given singularity of ${}_{G_n}V(z)$ in $\overline{G}_n$ and its solution is unique. It is evident that this solution coincides with ${}_{ex}({}_{G_n}V'(z))$. Clearly ${}_{ex}({}_{G_n}V'(z))\uparrow$ as $n\to\infty$. On the other hand, $f^{-1}(p): p\in G+B_1$ is continuous, we have ${}_{ex}(V'(z))=\lim_{n=\infty}({}_{ex\,G_n}V'(z))$. Hence ${}_{ex}V'(z)$ *is the least* F_0.S.H. *in $R-R_0$ such that ${}_{ex}V'(z)-V(z)$ is full harmonic in G.* We have easily the following

Theorem 3. 1). *Let $U(z)$ de an* F_0.S.H. *in $R-R_0$ with $\mathfrak{M}^f(U(z))<\infty$. Then ${}_{ex}({}_{in\,ex}U(z))\leqq U(z)$ and ${}_{ex}({}_{in\,ex}U(z))=U(z)$ if and only if the canonical distribution* of *$U(z)$ has no mass on CG.*

2). *Let $V(z)$ be an* F.S.H. *in G with $\mathfrak{M}^f(V(z))<\infty$. Then*

$$ {}_{in\,ex}({}_{ex}V(z))=V(z)\,. $$

Department of Mathematics
Hokkaido University

(Received June 30, 1966)

12) See 4).
13) See 1).