ON MIXED PROBLEMS FOR REGULARLY HYPERBOLIC SYSTEMS

By

Taira SHIROTA and Kazuo ASANO

1. Introduction. Let Ω be a domain with a bounded boundary Γ of $\boldsymbol{R}^n$. We consider the mixed problems for the hyperbolic system of equations

$$(1)\qquad \sum_{j=1}^{k} a_{ij}\left(x, \frac{\partial}{\partial t}, D_x\right) u_j = f_i, \qquad i=1, \cdots, k, \quad (t, x) \in (0, T)\times\Omega,$$

$$\left(u_j(0, x), \frac{\partial u_j}{\partial t}(0, x), \cdots, \frac{\partial^{m_j-1} u_j}{\partial t^{m_j-1}}(0, x)\right) \in D\left(a^{\frac{m_j}{2}}\right) \times D\left(a^{\frac{m_j-1}{2}}\right)$$

$$\times \cdots \times D(a^{\frac{1}{2}}), \qquad j=1, \cdots, k,$$

with boundary conditions

$$(D)\qquad u_j|_\Gamma = au_j|_\Gamma = \cdots = a^{\frac{m_j}{2}-1} u_j|_\Gamma = 0\,, \qquad j=1, \cdots, k$$

or

$$(N)\qquad \left(\frac{\partial}{\partial n} + \rho(x)\right) u_j\Big|_\Gamma = \left(\frac{\partial}{\partial n} + \rho(x)\right) au_j\Big|_\Gamma = \cdots$$

$$= \left(\frac{\partial}{\partial n} + \rho(x)\right) a^{\frac{m_j}{2}-1} u_j\Big|_\Gamma = 0\,, \qquad j=1, \cdots, k,$$

respectively, where $a_{ij}\left(x, \frac{\partial}{\partial t}, D_x\right) = \delta_{ij} \frac{\partial^{m_j}}{\partial t^{m_j}} + a^1_{ij}(x, D) \frac{\partial^{m_j-1}}{\partial t^{m_j-1}} + \cdots + a^{m_j}_{ij}(x, D)$ $+$(*lower order terms*), $a^h_{ij}(x, D) = \sum_{|\alpha|=h} a^{ij}_\alpha(x) D^\alpha$, $D=(D_1, \cdots, D_n)$, $D_l = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_l}$, δ_{ij} is a Kronecker delta, m_j is even for all j, and let all of the roots $\tau_s(x, \xi)$ $(s=1, \cdots, m_1+\cdots+m_k)$ with respect to τ of the characteristic equation $\det\left(a^0_{ij}(x, \tau, \xi)\right) = 0$ be pure imaginary and distinct mutually, not zero uniformly for $x \in \bar{\Omega}$ and $|\xi|=1$, where $a^0_{ij}(x, \tau, \xi)$ is the principal part of $a_{ij}(x, \tau, \xi)$. Furthermore let $a(x, D)$ be a uniformly elliptic operator such that $a(x, D) = -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} +$(*lower order terms*), $a_{ij}(x) = a_{ji}(x)$ are real-valued functions and $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqslant \delta|\xi|^2$ $(\delta>0)$ for $x\in\bar{\Omega}$ and $\xi\in\boldsymbol{R}^n$. Moreover $D(a) = H^2(\Omega)\cap H^1_0(\Omega)$ or $D(a) = \left\{u \in H^2(\Omega)\,; \left(\frac{\partial}{\partial n} + \rho(x)\right) u\Big|_\Gamma = 0\right\}$ according as we consider Dirichlet type boundary conditions (D) or Neumann ones (N) respectively, $\frac{\partial}{\partial n}$ is the conormal derivative of $a(x, D)$ at the boundary Γ and $\rho(x) \in C^{\infty}(\Gamma)$.

Furthermore we assume that after applying (any) coordinate transformation $(U \cap \Omega, U \cap \Gamma) \to (\boldsymbol{R}^n_+ = \{y \in \boldsymbol{R}^n;\ y_n > 0\},\ \boldsymbol{R}^{n-1} = \{y \in \boldsymbol{R}^n;\ y_n = 0\})$ such that on the boundary the conormal direction of $a(x, D)$ is changed into the normal direction, the coefficients of the principal part of (1) containing odd power of $\frac{\partial}{\partial y_n}$ are zero on the boundary $\boldsymbol{R}^{n-1}$.

In the present paper we will extend the theory of Calderón-Zygmund's singular integral operator to that with boundary conditions and as one of its application we will prove that the above mixed problems are solved by means of the method of semi group analogous to the cases of second order. In §2.1 we construct the theory of the singular integral operators with boundary conditions on $\boldsymbol{R}^n_+$ and next we extend it to one on the domain Ω with bounded boundary in §2.2. Here we avoid to use the principle of reflection as we can as possible, because it will be expected that the theory of our singular integral operators is extended to that with respect to elliptic operators of higher order. Finally we show the Energy inequality and the existence of resolvent in §3.

Concerning the mixed problems it seems to us that J. Leray asked what ones for hyperbolic systems of equations of higher order of Petrowski-Leray-Gårding were well-posed ([11], the third part, Introduction). Later extending Cauchy-Kowalewski theorem to mixed problem G. F. D. Duff [7] treated the mixed problems for the case of single equation and $a = \Delta$ in the quarter space (see also S. Mizohata [13] and S. Miyatake [12]). With the progress of researches of boundary value problems for elliptic equations, applying them to the mixed problems for hyperbolic equations has been considered. In particular S. Agmon [1] showed that the certain mixed problems where the coefficients of the differential operators are constant and the domain is quadrant were well-posed. On the other hand the mixed problems for systems of first order have been developed by Friedrichs-Lax-Phillips, Agranovič and so on. However, it seems to us to be confronted with a large difficulty in the process to reduce the operators of higher order to the system of first order theoretically. Furthermore concerning the difficulty of the mixed problems A. A. Dezin [6] pointed out from the general viewpoint of partial differential equations (see also [9]).

For the simplicity of the description we assume that the coefficients of the operators are sufficiently smooth and bounded with their derivatives in

$\boldsymbol{R}^n$ and the boundary Γ is also sufficiently smooth. Furthermore we will be concerned with only single equations (see Remarks in §3). In this case we write (1) as

$$(1') \qquad Lu = a\left(x, \frac{\partial}{\partial t}, D_x\right)u = \left(\frac{\partial^{2m}}{\partial t^{2m}} + a_1(x, D)\frac{\partial^{2m-1}}{\partial t^{2m-1}} + \cdots + a_{2m}(x, D)\right)u + (\textit{lower order terms})u = f, \qquad (t, x) \in (0, T) \times \Omega .$$

This article is the extension and details of our previous papers [3] and [18]. Finally we note that the necessity for the condition about coefficients of equations mentioned above is obtained by [19].

§ 2. Singular integral operators with boundary conditions

2. 1. Singular integral operators with boundary conditions defined on $\boldsymbol{R}^n_+$.

Definition 1. *Let Ξ_4^∞ be the set of $\sigma(x, \xi)$ such that $\sigma(x, \xi) \in C^{4,\infty}_{x,\xi}(\overline{\boldsymbol{R}^n_+} \times (\boldsymbol{R}^n - \{0\}))$ (this means that $\sigma(x, \xi)$ is defined on $\overline{\boldsymbol{R}^n_+} \times (\boldsymbol{R}^n - \{0\})$ and all derivatives of σ with respect to x and ξ such that the order with respect to x is not higher than four are continuous), $\sigma(x, \lambda\xi) = \sigma(x, \xi)$ for $\lambda > 0$ and for every integer $s (\geqslant 0)$ we have the following estimate.*

$$(2) \qquad \sum_{\substack{|\mu| \leqslant 4 \\ |\nu| \leqslant s}} \sup_{\substack{x \in \overline{\boldsymbol{R}^n_+} \\ |\xi| = 1}} \left| \left(\frac{\partial}{\partial x}\right)^\mu \left(\frac{\partial}{\partial \xi}\right)^\nu \sigma(x, \xi) \right| \equiv M_s(\sigma) < \infty .$$

Let $\Sigma = \{\xi \in \boldsymbol{R}^n;\ |\xi| = 1\}$ and let $\{Y_{l,m}(\xi)\}$ be a complete orthonormal system in $L^2(\Sigma)$ such that every $Y_{l,m}(\xi)$ be a real polynomial of spherical harmonic of homogeneous order l and $\int_\Sigma Y_{l,m}(\xi) d\Sigma = 0$.

Then for $\sigma(x, \xi) \in \Xi_4^\infty$, we can expand it such that

$$(3) \qquad \sigma(x, \xi) = \sum_{l,m} \sigma_{l,m}(x) Y_{l,m}(\xi) \qquad \text{in} \quad L^2(\Sigma),$$

here

$$\sigma_{l,m}(x) = \int_\Sigma \sigma(x, \xi) Y_{l,m}(\xi) d\Sigma .$$

Moreover, as well known we obtain the following estimates.

$$(4) \qquad \begin{cases} \sup\limits_{|\xi|=1} \left| \left(\dfrac{\partial}{\partial \xi}\right)^\nu Y_{l,m}(\xi) \right| \leqslant c(n, \nu) l^{\frac{1}{2}(n-2)+|\nu|} , \\ \text{the number of } m \text{ is of order } l^{n-2} \text{ as } l \to \infty , \\ \sum\limits_{|\mu| \leqslant 4} \sup\limits_{x \in \overline{\boldsymbol{R}^n_+}} \left| \left(\dfrac{\partial}{\partial x}\right)^\mu \sigma_{l,m}(x) \right| \leqslant c(n, k) M_{2k}(\sigma) l^{-2k+\frac{n}{2}}, \text{ where } k \text{ is an} \\ \text{arbitrary non negative integer.} \end{cases}$$

Definition 2. *Let $\mathfrak{A}$ be the algebra generated by $\sigma(x, \xi)$ and $f(x) \dfrac{\xi_n}{|\xi|}$*

with the following properties: $\sigma(x,\xi)\in\Xi_4^\infty$, $\sigma(x,\xi',\xi_n)=\sigma(x,\xi',-\xi_n)$, $f(x)\in\mathfrak{B}^4(\overline{\boldsymbol{R}_+^n})$ *and* $f(x',0)=0$, *here* x,ξ *will denote* $x=(x_1,\cdots,x_{n-1},x_n)=(x',x_n)$, $\xi=(\xi_1,\cdots,\xi_{n-1},\xi_n)=(\xi',\xi_n)$. *For* $\alpha(x,\xi)\in\mathfrak{A}$ *consider the singular integral operator* $\alpha(x,D)$ *as follows:*

$$\alpha(x,D)u=F'\alpha(x,\xi)F\tilde{u}\,|_{x_n>0}\qquad \text{for}\quad u\in L^2(\boldsymbol{R}_+^n),$$

where
$$(Fv)(\xi)=\int_{\boldsymbol{R}^n}e^{-ix\xi}v(x)dx\,,$$
$$(F'v)(\xi)=(2\pi)^{-n}\int_{\boldsymbol{R}^n}e^{ix\xi}v(x)dx\qquad \text{for}\quad v\in L^2(\boldsymbol{R}^n)\,,\ x\xi=\sum_{j=1}^n x_j\xi_j$$

$i=\sqrt{-1}$, *and* $\tilde{u}(x)$ *is the extended function of* $u(x)$ *to* $\boldsymbol{R}^n$ *by defining it to be the odd function or the even one with respect to* x_n.

Furthermore for $\alpha(x,\xi)\in\mathfrak{A}$, *let* $M_s(\alpha)=min\left(M_s(\sigma)+\sum_{j=1}^p M_s\left(\sigma_j(x,\xi)f_j(x)\times\frac{\xi_n}{|\xi|}\right)\right)$, *where* $\alpha(x,\xi)=\sigma(x,\xi)+\sum_{j=1}^p\sigma_j(x,\xi)f_j(x)\frac{\xi_n}{|\xi|}\in\mathfrak{A}$. *Then* $M_s(\alpha+\alpha_2)\leqslant M_s(\alpha_1)+M_s(\alpha_2)$ *and* $M_s(\alpha_1\alpha_2)\leqslant M_s(\alpha_1)M_s(\alpha_2)$.

Definition 3. *Let* Λ_+^2, Λ_-^2 *and* $\Lambda^2=-\Delta$, *whose definition domains are* $\left\{u\in H^2(\boldsymbol{R}_+^n);\ \frac{\partial u}{\partial x_n}(x',0)=0\right\}$, $H^2(\boldsymbol{R}_+^n)\cap H_0^1(\boldsymbol{R}_+^n)$ *and* $H^2(\boldsymbol{R}^n)$, *respectively. Then they are selfadjoint operators on* $L^2(\boldsymbol{R}_+^n)$, $L^2(\boldsymbol{R}_+^n)$ *and* $L^2(\boldsymbol{R}^n)$, *respectively. Let* $\Lambda_+=(\Lambda_+^2)^{\frac{1}{2}}$, $\Lambda_-=(\Lambda_-^2)^{\frac{1}{2}}$ *and* $\Lambda=(\Lambda^2)^{\frac{1}{2}}$ *then we see that* $D(\Lambda_+)=H^1(\boldsymbol{R}_+^n)$, $D(\Lambda_-)=H_0^1(\boldsymbol{R}_+^n)$ *and* $D(\Lambda)=H^1(\boldsymbol{R}^n)$.

Lemma 1. *Let* $u(x)\in D(\Lambda_\pm)$ *and* $\alpha(x,\xi)\in\mathfrak{A}$. *Then* $\alpha(x,D)u$, $\alpha(x,D)^*u\in D(\Lambda_\pm)$ *where in Definition 2 we assume that* $\tilde{u}(x)$ *is the odd extension or even one of* $u(x)$ *respectively according as the case where we consider* Λ_- *or* Λ_+.

In what follows $\tilde{u}(x)$ will be defined as above.

Proof. We will show only the assertion for $D(\Lambda_-)$, because the proof for $D(\Lambda_+)$ is more obvious than that for $D(\Lambda_-)$.

Let

$$F^-u(\xi)=\int_{\boldsymbol{R}_+^n}e^{-ix'\xi'}\sin(x_n\xi_n)u(x',x_n)\,dx\,,\ (\text{for }\ \xi_n\geqslant 0)$$
$$F^{-\prime}u(\xi)=4(2\pi)^{-n}\int_{\boldsymbol{R}_+^n}e^{ix'\xi'}\sin(x_n\xi_n)u(x',x_n)dx\qquad \text{for}\ \ u\in L^2(\boldsymbol{R}_+^n).$$

Then we find that $u\in D(\Lambda_-)$ if and only if $F^-u(\xi)$ and $|\xi|F^-u(\xi)\in L^2(\boldsymbol{R}_+^n)$. For if $u(x)\in D(\Lambda_-)$ then from $F^-u(\xi)=\frac{i}{2}F\tilde{u}(\xi)_{|\xi_n>0}$ it follows that $F^-u(\xi)$ and $|\xi|F^-u(\xi)\in L^2(\boldsymbol{R}_+^n)$. Conversely let $F^-u(\xi)$ and $|\xi|F^-u(\xi)\in L^2(\boldsymbol{R}_+^n)$. Then $F\tilde{u}(\xi)$ and $|\xi|F\tilde{u}(\xi)\in L^2(\boldsymbol{R}^n)$, from which $\tilde{u}(x)\in H^1(\boldsymbol{R}^n)$ and $u(x)\in H^1(\boldsymbol{R}_+^n)$. Next, to show $u(x)\in H_0^1(\boldsymbol{R}_+^n)$ let us set partial Fourier transformation F_1^- such that for $u(x', x_n)\in L^2(\boldsymbol{R}_+^n)$

$$(F_1^- u)(\xi', x_n)=\int_{-\infty}^{\infty} e^{-ix'\xi'} u(x', x_n)dx' .$$

Then we see that

$$(F_1^- u)(\xi', x_n)=\frac{\pi}{2}\int_0^{\infty}\sin (x_n\xi_n)(F^- u)(\xi', \xi_n)d\xi_n .$$

Furthermore from the hypothesis it follows

$$\int_0^{\infty}|(F^- u)(\xi', \xi_n)|d\xi_n$$
$$\leqslant\left(\int_0^{\infty}(1+\xi_n^2)|(F^- u)(\xi', \xi_n)|^2 d\xi_n\right)^{\frac{1}{2}}\left(\int_0^{\infty}(1+\xi_n^2)^{-1}d\xi_n\right)^{\frac{1}{2}}<\infty ,$$

and

$$|(F_1^- u)(\xi', x_n)|^2$$
$$\leqslant c\int_0^{\infty}(1+\xi_n^2)|(F^- u)(\xi', \xi_n)|^2 d\xi_n\int_0^{\infty}(1+\xi_n^2)^{-1}d\xi_n \quad \text{for } x_n>0 \text{ and } a.e.\ \xi',$$

from which it implies by virtue of Lebesgue's theorem

$$(F_1^- u)(\xi', 0)=\frac{\pi}{2}\lim_{x_n\to 0}\int_0^{\infty}\sin (x_n\xi_n)(F^- u)(\xi', \xi_n)d\xi_n$$
$$=\frac{\pi}{2}\int_0^{\infty}\lim_{x_n\to 0}\sin (x_n\xi_n)(F^- u)(\xi', \xi_n)d\xi_n=0 \qquad \text{for } a.e.\,\xi',$$

hence

$$\lim_{x_n\to 0}\int_{-\infty}^{\infty}|(F_1^- u)(\xi', x_n)|^2 d\xi'=\int_{-\infty}^{\infty}\lim_{x_n\to 0}|(F_1^- u)(\xi', x_n)|^2 d\xi' \tag{5}$$
$$=\int_{-\infty}^{\infty}|(F_1^- u)(\xi', 0)|^2 d\xi'=0.$$

On the other hand we see that

$$\int_{-\infty}^{\infty}|u(x', x_n)|^2 dx'=(2\pi)^{-n+1}\int_{-\infty}^{\infty}|(F_1^- u)(\xi', x_n)|^2 d\xi' \quad \text{for } \ x_n>0. \tag{6}$$

From (5) and (6) it follows

$$\lim_{x_n\to 0}\int_{-\infty}^{\infty}|u(x', x_n)|^2 dx'=0 ,$$

that is, $u(x)\in H_0^1(\boldsymbol{R}_+^n)$.

Now we will prove $\alpha(x, D)u\in D(\Lambda_-)$ for $u\in D(\Lambda_-)$ using the equivalent relation stated above. Let $\alpha(x, \xi)=f(x)\frac{\xi_n}{|\xi|}$ and $f(x', 0)=0$. Then

$$\alpha(x, D)u = F'\alpha(x, \xi)F\tilde{u}\Big|_{x_n>0} = f(x)F'\frac{\xi_n}{|\xi|}F\tilde{u}\Big|_{x_n>0}$$

From $f(x', 0)=0$ it implies $\alpha(x, D)u \in H_0^1(\boldsymbol{R}_+^n)$.

Furthermore let $\alpha(x, \xi) \in \Xi_4^\infty$ and $\alpha(x, \xi', \xi_n)=\alpha(x, \xi', -\xi_n)$. Then by virtue of (3), (4) and regarding that $\alpha(x, \xi)$ is an even function with respect to ξ_n we see that

$$(7) \qquad \alpha(x, \xi) = \sum_{l,m} \alpha_{l,m}(x) Y_{l,m}^+(\xi)$$

where $Y_{l,m}^+(\xi) = \frac{1}{2}\Big(Y_{l,m}(\xi', \xi_n) + Y_{l,m}(\xi', -\xi_n)\Big)$ and that it converges uniformly on $(x, \xi) \in \overline{\boldsymbol{R}_+^n} \times \Big(\boldsymbol{R}^n - \{0\}\Big)$.

Then we see by the characterization mentioned above that every $\alpha_{l,m}(x) Y_{l,m}(D) u \in D(\Lambda_-)$, because $F'\alpha_{l,m}(x) Y_{l,m}^+(\xi) F\tilde{u}|_{x_n>0} = \alpha_{l,m}(x) F^{-\prime} Y_{l,m}^+(\xi) F^- u$, $Y_{l,m}^+(\xi) F^- u(\xi)$ and $|\xi| Y_{l,m}^+(\xi) F^- u(\xi) \in L^2(\boldsymbol{R}_+^n)$. Hence we have only to show that $\sum_{l,m} \alpha_{l,m}(x) Y_{l,m}^+(D) u$ converges in $H^1(\boldsymbol{R}_+^n)$. But it is easily seen from the fact that this expression converges in $L^2(\boldsymbol{R}_+^n)$ by virtue of (4). Furthermore from (4) and by the fact that $Y_{l,m}^+(D)\Lambda_- u = \Lambda_- Y_{l,m}^+(D) u$ for $u \in D(\Lambda_-)$ we see that above expression converges in $H^1(\boldsymbol{R}_+^n)$.

We also find that $\alpha(x, D)^* u \in D(\Lambda_-)$ for $u \in D(\Lambda_-)$. To prove this let $\alpha(x, \xi)=f(x)\frac{\xi_n}{|\xi|}$, $f(x', 0)=0$. Then for every $v \in C_0^\infty(\boldsymbol{R}_+^n)$

$$\Big(\alpha(x, D)^* u, \Lambda_- v\Big)_{x_n>0} = \Big(u, \alpha(x, D)\Lambda_- v\Big)_{x_n>0} = \left(u, f(x)\frac{1}{\sqrt{-1}}\frac{\partial v}{\partial x_n}\right)_{x_n>0}$$
$$= \frac{1}{\sqrt{-1}}\left(\frac{\partial}{\partial x_n}(\overline{f(x)}u), v\right)_{x_n>0},$$

from which we find that

$$\Big|(\alpha(x, D)^* u, \Lambda_- v)_{x_n>0}\Big| \leqslant C\|u\|_{1, L^2(\boldsymbol{R}_+^n)}\|v\|_{x_n>0},$$

where $\|u\|_{x_n>0}^2 = \int_{\boldsymbol{R}_+^n} |u|^2 dx$ and $(u, v)_{x_n>0} = \int_{\boldsymbol{R}_+^n} u\bar{v}\,dx$ for $u, v \in L^2(\boldsymbol{R}_+^n)$.

This implies $\alpha(x, D)^* u \in D(\Lambda_-^*) = D(\Lambda_-)$.

Furthermore let $\alpha(x, \xi) \in \Xi_4^\infty$ and $\alpha(x, \xi', \xi_n)=\alpha(x, \xi', -\xi_n)$. Then we find that $\alpha(x, D)^* u = \sum_{l,m} Y_{l,m}^+(D)\alpha_{l,m}(x) u$. Moreover every term on the right is in $D(\Lambda_-)$ and similarly to the case stated above this expression converges in

$H^1(\boldsymbol{R}^n_+)$. This implies $\alpha(x, \mathrm{D})^* u \in D(\Lambda_-)$. q.e.d.

Theorem 1. *Let $\alpha(x, \xi)$ and $\beta(x, \xi) \in \mathfrak{A}$. Then for $u(x) \in D(\Lambda_\pm)$ we obtain the following estimates:*

$$\left\| \left(\alpha(x, D)\Lambda_\pm - \Lambda_\pm \alpha(x, D)\right) u \right\|_{x_n>0} \leqslant C M_{2n}(\alpha) \|u\|_{x_n>0}, \tag{8}$$

$$\left\| \left(\alpha(x, D)^* \Lambda_\pm - \Lambda_\pm \alpha(x, D)^*\right) u \right\|_{x_n>0} \leqslant C M_{2n}(\alpha) \|u\|_{x_n>0}, \tag{9}$$

$$\left\| \left(\alpha(x, D)^* - \alpha^\#(x, D)\right) \Lambda_\pm u \right\|_{x_n>0} \leqslant C M_{2([\frac{3}{2}n]+1)}(\alpha) \|u\|_{x_n>0}, \tag{10}$$

$$\left\| \left(\alpha(x, D)\beta(x, D) - (\alpha\circ\beta)(x, D)\right) \Lambda_\pm u \right\|_{x_n>0} \leqslant C M_{2([\frac{3}{2}n]+1)}(\alpha) M_{2n}(\beta) \|u\|_{x_n>0}, \tag{11}$$

where $\alpha^\#(x, D)$, $(\alpha\circ\beta)(x, D)$ are the operators with symbol $\overline{\alpha(x, \xi)}$, $\alpha(x, \xi)\beta(x, \xi)$, respectively, [ ] denotes Gauss symbol and the constant C depends only on dimension n.

In the following C will be used to denote various constants depending only on n.

To prove Theorem 1 the following Lemmas 2–11 will be needed. Hereafter the proofs of Theorem 1 and Lemmas 2–10 except Lemma 11 will be carried out only the case of Λ_+. In the case of Λ_- the proofs are obtained by regarding $\tilde{u}(x)$ in the proofs mentioned below as the odd function.

Notation. *For $a(x) \in \mathfrak{B}^k(\boldsymbol{R}^n_+)$, k: non negative integer, let $|a|_k$ be the norm of $a(x)$ such that*

$$|a|_k = \sum_{|\alpha| \leq k} \sup_{x \in \boldsymbol{R}^n_+} \left| \left(\frac{\partial}{\partial x}\right)^\alpha a(x) \right|,$$

and for $a(x) \in \mathfrak{B}^{k+\delta}(\boldsymbol{R}^n_+)$, k: non negative integer and δ is a constant such that $0 < \delta < 1$, let $|a|_{k+\delta}$ be the norm of $a(x)$ such that

$|a|_{k+\delta} = |a|_k +$ *Hölder constant of order δ.*

Lemma 2. *Let $u(x) \in D(\Lambda_+)$. Then $\Lambda_+ u = \Lambda \tilde{u}|_{x_n>0}$.*

Proof. Similarly to the case of $D(\Lambda_-)$ in Lemma 1, let

$$F^+ u(\xi) = \int_{\boldsymbol{R}^n_+} e^{-ix'\xi'} \cos(x_n \xi_n) u(x', x_n) dx,$$

$$F^{+\prime} u(\xi) = 4(2\pi)^{-n} \int_{\boldsymbol{R}^n_+} e^{ix'\xi'} \cos(x_n \xi_n) u(x', x_n) dx \qquad \text{for } u(x) \in L^2(\boldsymbol{R}^n_+).$$

Then

$$(F^+u)(\xi) = \frac{1}{2}(F\tilde{u})(\xi)\Big|_{\xi_n>0}$$

and
$$(F^{+\prime}u)(\xi) = 2(F'\tilde{u})(\xi)\,|_{\xi_n>0}\,.$$

From our definitions it follows that

$$F^+(\Lambda_+u)(\xi) = |\xi|\,(F^+u)(\xi) \qquad (\xi_n>0)\,,$$

and that for $v(x) \in D(\Lambda)$

$$F(\Lambda v)(\xi) = |\xi|\,(Fv)\,(\xi)\,.$$

Thus it implies

$$\Lambda_+u = F^{+\prime}F^+\Lambda_+u = F^{+\prime}\Big(|\xi|\,(F^+u)(\xi)\Big) = 2F'\Big(|\xi|\,(\widetilde{F^+u})(\xi)\Big)\Big|_{x_n>0}$$
$$= F'\Big(|\xi|F\tilde{u}(\xi)\Big)\Big|_{x_n>0} = \Lambda\tilde{u}\Big|_{x_n>0}\,.$$

Lemma 3. *Let* $a(x) \in \mathfrak{B}^4(\overline{\boldsymbol{R}^n_+})$. *For* $u(x) \in D(\Lambda_+)$

$$\Big\|\Big(a(x)\Lambda_+ - \Lambda_+ a(x)\Big)u\Big\|_{x_n>0} \leqslant C\,|a|_4\,\|u\|_{x_n>0}\,.$$

Proof. Let us decompose $a(x)$ as follows:

$$a(x) = a_1(x') + a_2(x')x_n + a_3(x', x_n)x_n^2 = b_1(x) + b_2(x)\,,$$

here
$$b_1(x) = a_1(x') + a_3(x', x_n)x_n^2\,, \qquad b_2(x) = a_2(x')x_n\,.$$

Furthermore choose $\varphi(x) \in \mathfrak{B}(\overline{\boldsymbol{R}^n_+})$ such that $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(x) = 1$ for $0 \leqslant x_n \leqslant 1$ and $\varphi(x) = 0$ for $x_n \geqslant 2$.

Then

$$a(x) = \varphi(x)b_1(x) + \varphi(x)b_2(x) + \Big(1-\varphi(x)\Big)a(x)$$
$$\equiv a_4(x) + a_5(x) + a_6(x)\,.$$

From our decomposition we see that

$$\tilde{a}_4(x) \in \mathfrak{B}^2(\boldsymbol{R}^n)\,,$$

$$\tilde{a}_5(x) = \begin{cases} a_2(x')\,|x_n| & \text{for } \ |x_n| \leqslant 1\,, \\ 0 & \text{for } \ |x_n| \geqslant 2\,, \end{cases} \qquad a_5(x) \in \mathfrak{B}^3(\overline{\boldsymbol{R}^n_+})\,,$$

and
$$\tilde{a}_6(x) \in \mathfrak{B}^4(\boldsymbol{R}^n)\,.$$

Now by virtue of Lemma 2 it implies

$$\Big(a(x)\Lambda_+ - \Lambda_+ a(x)\Big)u$$
$$= \Big(\tilde{a}(x)\Lambda_+ - \Lambda_+ \tilde{a}(x)\Big)\tilde{u}\Big|_{x_n>0}$$
$$= (\tilde{a}_4\Lambda - \Lambda\tilde{a}_4)\tilde{u}|_{x_n>0} + (\tilde{a}_5\Lambda - \Lambda\tilde{a}_5)\tilde{u}|_{x_n>0} + (\tilde{a}_6\Lambda - \Lambda\tilde{a}_6)\tilde{u}|_{x_n>0}\,.$$

Because of $\tilde{a}_4$ and $\tilde{a}_6 \in \mathfrak{B}^{1+\delta}(\boldsymbol{R}^n)$, the first and the last terms are bounded by $C\|\tilde{u}\|_{x_n>0}(=C\|u\|_{x_n>0})$ from the theory of the ordinary singular integrals. Hence to complete the proof of Lemma 3 it suffices to prove the following

Lemma 4. *Let* $a(x) \in \mathfrak{B}^{1+\delta}(\overline{\boldsymbol{R}^n_+})$ *such that*

$$a(x) = \begin{cases} a_1(x')x_n & \text{for } 0 \leqslant x_n \leqslant 1\,, \\ 0 & \text{for } x_n \geqslant 2\,. \end{cases}$$

Then for $u(x) \in D(\Lambda_+)$

$$\Big\| \Big(a(x)\Lambda_+ - \Lambda_+ a(x)\Big)u \Big\|_{x_n>0} \leqslant C|a|_{1+\delta}\|u\|_{x_n>0}\,.$$

Proof. Since $C_0^\infty(\overline{\boldsymbol{R}^n_+})$ (this means the set of all functions $u(x)$ such that u is C^∞ function with compact support contained in $\overline{\boldsymbol{R}^n_+}$) is dense in $D(\Lambda_+)$ it suffices to prove this lemma for $u \in C_0^\infty(\overline{\boldsymbol{R}^n_+})$.

Here we remark that hereafter we may prove following Lemmas 5–11 for $u(x) \in C_0^\infty(\overline{\boldsymbol{R}^n_+})$.

Let R_j be the operator from $L^2(\boldsymbol{R}^n)$ into itself such that

$$(FR_jv)(\xi) = \frac{1}{\sqrt{-1}} \cdot \frac{\xi_j}{|\xi|}(Fv)(\xi) \qquad \text{for} \quad v(x) \in L^2(\boldsymbol{R}^n).$$

From this it follows that

$$\Lambda\tilde{u} = \sum_{j=1}^n \frac{\partial}{\partial x_j} R_j\tilde{u} = \sum_{j=1}^n R_j \frac{\partial}{\partial x_j}\tilde{u}\,,$$

and $$\Big(\tilde{a}(x)\Lambda - \Lambda\tilde{a}(x)\Big)\tilde{u} = \sum_{j=1}^n \Big(\tilde{a}(x)R_j - R_j\tilde{a}(x)\Big)\frac{\partial}{\partial x_j}\tilde{u} - \sum_{j=1}^n R_j \frac{\partial\tilde{a}}{\partial x_j}\tilde{u}\,.$$

Where $\dfrac{\partial\tilde{a}}{\partial x_j}$ is not continuous but bounded, so each of the second term on the right is bounded in L^2-sense. Therefore we may investigate each of the first term on the right.

To this end we set

$$v_\varepsilon(x) = \int_{|x-y|\geqslant\varepsilon} \Big(\tilde{a}(x) - \tilde{a}(y)\Big)R_j(x-y)\frac{\partial}{\partial y_j}\tilde{u}(y)dy$$

$$= \int_{|x-y|=\varepsilon} \big(\tilde{a}(x) - \tilde{a}(y)\big) R_j(x-y)\tilde{u}(y) \cos \gamma_j dS_\varepsilon$$
$$+ \int_{|x-y|\geqslant\varepsilon} \frac{\partial \tilde{a}}{\partial y_j}(y) R_j(x-y)\tilde{u}(y)dy$$
$$+ \int_{|x-y|\geqslant\varepsilon} \big(\tilde{a}(x) - \tilde{a}(y)\big)\left(\frac{\partial}{\partial x_j} R_j\right)(x-y)\tilde{u}(y)dy$$
$$\equiv v_\varepsilon^1(x) + v_\varepsilon^2(x) + v_\varepsilon^3(x),$$

where $S_\varepsilon = \left\{x \in \boldsymbol{R}^n\ ;\ |x| = \varepsilon\right\}$.

Then we may show the following estimates:

$$\lim_{\varepsilon \downarrow 0} \|v_\varepsilon^i(x)\|_{x_n>0} \leqslant C|a|_{1+\delta}\|u\|_{x_n>0} \qquad (i=1, 2, 3).$$

We at first estimate $v_\varepsilon^1(x)$.

$$v_\varepsilon^1(x) = \int_{|x-y|=\varepsilon} \frac{\tilde{a}(x) - \tilde{a}(y)}{|x-y|} \cdot |x-y| R_j(x-y)\tilde{u}(y) \cos \gamma_j dS_\varepsilon\,.$$

Hence

$$|v_\varepsilon^1(x)| \leqslant |\tilde{a}(x)|_1 C \int_{S_1} \varepsilon \cdot \frac{1}{\varepsilon^n} |\tilde{u}(x+\varepsilon\omega)| \varepsilon^{n-1} dS_1.$$
$$= C|a(x)|_1 \int_{S_1} |\tilde{u}(x+\varepsilon\omega)|\, dS_1\,.$$

The right hand tends to $C|a(x)|_1\omega_n|\tilde{u}(x)|$ as $\varepsilon \to 0$, where $\omega_n = \int_{S_1} dS_1$

Therefore

$$\lim_{\varepsilon\to 0} \|v_\varepsilon^1(x)\|_{x_n>0} \leqslant C|a(x)|_1 \|u(x)\|_{x_n>0}\,.$$

Next $v_\varepsilon^2(x)$ tends to $v.p. R_j(x) * \left(\frac{\partial \tilde{a}}{\partial x_j}\tilde{u}\right)(x)$ as $\varepsilon\to 0$, hence

$$\lim_{\varepsilon\to 0} \|v_\varepsilon^2(x)\|_{x_n>0} \leqslant |\tilde{a}(x)|_1 \|\tilde{u}(x)\|_{x_n>0} = |a(x)|_1 \|u(x)\|_{x_n>0}.$$

Finally we estimate $v_\varepsilon^3(x)$. We decompose it as follows:

$$v_\varepsilon^3(x) = \int_{\varepsilon\leqslant|x-y|<1} dy + \int_{|x-y|\geqslant 1} dy$$
$$= v_\varepsilon^4(x) + v^5(x).$$

From $|v^5(x)| \leqslant 2|\tilde{a}(x)|_0 \left|\left(\frac{\partial}{\partial x_j} R_j(x)\right)_{|x|\geqslant 1}\right| * |\tilde{u}(x)|$ and $\frac{\partial}{\partial x_j} R_j(x) = O(|x|^{-n-1})$,

it follows

$$\|v^5(x)\|_{x_n>0} \leqslant 2|\tilde{a}(x)|_0 \left(\int_{|x|\geqslant 1} \left| \frac{\partial}{\partial x_j} R_j \right| dx \right) \cdot \|\tilde{u}(x)\|_{x_n>0} \leqslant C\,|a(x)|_0 \|u(x)\|_{x_n>0}\,.$$

Furthermore we estimate $v^4_\varepsilon(x)$.

We see that $\tilde{a}(x)=a(x)$ and $\tilde{a}(y)=a(y)$ on $\varepsilon \leqslant |x-y|<1$ where $x_n \geqslant 1$. Hence by virtue of the theory of ordinary singular integrals it follows

$$\lim_{\varepsilon\to 0} \|v^4_\varepsilon(x)\|_{x_n>1} = \lim_{\varepsilon\to 0} \left(\int_{\{x\in \boldsymbol{R}^n;\, x_n>1\}} |v^4_\varepsilon(x)|^2 dx \right)^{\frac{1}{2}} \leqslant C|a|_{1+\delta} \|u\|_{x_n>0}\,.$$

Moreover we estimate $v^4_\varepsilon(x)$ where $0<x_n<1$. By $\tilde{\tilde{a}}(x)$ set the odd extension of $a(x)$ to $\boldsymbol{R}^n$ with respect to x_n. Then

$$\begin{aligned} v^4_\varepsilon(x) &= \int_{\varepsilon\leqslant|x-y|<1} \left(\tilde{a}(x)-\tilde{\tilde{a}}(y)\right) \left(\frac{\partial}{\partial x_j} R_j\right)(x-y)\tilde{u}(y)dy \\ &\quad + \int_{\varepsilon\leqslant|x-y|<1} \left(\tilde{\tilde{a}}(y)-\tilde{a}(y)\right) \left(\frac{\partial}{\partial x_j} R_j\right)(x-y)\tilde{u}(y)dy \\ &\equiv v^6_\varepsilon(x) + v^7_\varepsilon(x)\,. \end{aligned}$$

As $\tilde{a}(x)=\tilde{\tilde{a}}(x)$ on $x_n>0$ and $\tilde{\tilde{a}}(x)\in \mathfrak{B}^{1+\delta}(\boldsymbol{R}^n)$, $v^6_\varepsilon(x)$ is estimated similarly to the case where $x_n \geqslant 1$ of $v^4_\varepsilon(x)$. Hence

$$\lim_{\varepsilon\to 0} \|v^6_\varepsilon(x)\|_{0<x_n<1} = \lim_{\varepsilon\to 0} \left(\int_{\{x\in \boldsymbol{R}^n;\, 0<x_n<1\}} |v^6_\varepsilon(x)|^2 dx \right)^{\frac{1}{2}} \leqslant C|a|_{1+\delta} \|u\|_{x_n>0}\,.$$

To finish the proof we may estimate only $v^7_\varepsilon(x)$. As $\tilde{a}(y)=\tilde{\tilde{a}}(y)$ on $y^n \geqslant 0$,

$$\begin{aligned} v^7_\varepsilon(x) &= 2\int_{\substack{\varepsilon_1\leqslant|x-y|<1 \\ y_n<0}} a_1(y')y_n \left(\frac{\partial}{\partial x_j} R_j\right)(x-y)\tilde{u}(y)dy \\ &= 2\int_{\substack{\varepsilon_1\leqslant|x'-y',x_n+y_n|<1 \\ y_n>0}} a^1(y')y_n \left(\frac{\partial}{\partial x_j} R_j\right)(x'-y',\, x_n+y_n)\tilde{u}(y',\, -y_n)\,dy\,, \end{aligned}$$

where $\varepsilon_1 = \text{distance}\left(x,\, \{x_n=0\}\right)$.

Hence

$$\begin{aligned} |v^7_\varepsilon(x)| &\leqslant |a|_1 C \int_{\boldsymbol{R}^n_+} y_n \left(|x'-y'|+x_n+y_n\right)^{-n-1} |\tilde{u}(y',\, -y_n)|\,dy \\ &\leqslant C|a|_1 \int_{\boldsymbol{R}^n_+} \left(|x'-y'|+x_n+y_n\right)^{-n} |\tilde{u}(y',\, -y_n)|\,dy\,. \end{aligned}$$

Using Hilbert's inequality, we see that

$$\|v^7_\varepsilon(x)\|_{x_n>0} \leqslant C|a|_1 \|u(x)\|_{x_n>0}\,.$$

From the above statements, by passing to the limit, we find that

$$\left\| \left(a(x)\Lambda_+ - \Lambda_+ a(x) \right) u \right\|_{x_n>0} \leqslant C|a|_{1+\delta}\|u\|_{x_n>0} \qquad \text{for} \quad u \in D(\Lambda_+).$$

Lemma 5. *Let* $a(x) \in \mathfrak{B}^4(\overline{\boldsymbol{R}}^n_+)$, $A(\xi) \in \Xi^\infty_4$ *and* $A(\xi', \xi_n) = A(\xi', -\xi_n)$. *Then for* $u(x) \in D(\Lambda_+)$

$$\left\| \left(a(x)A(D)\Lambda_+ - A(D)\Lambda_+ a(x) \right) u \right\|_{x_n>0} \leqslant C\,|a|_4 M_{n+2}(A)\,\|u\|_{x_n>0}\,.$$

Proof. We define $\tilde{A}(D)$ as follows:

$$\tilde{A}(D)v = F'A(\xi)Fv \qquad \text{for} \quad v \in L^2(\boldsymbol{R}^n)\,.$$

Then $A(D)u = \tilde{A}(D)\tilde{u}|_{x_n>0}$ for $u \in L^2(\boldsymbol{R}^n_+)$. From this and Lemma 2 it follows

$$\begin{aligned} &\left\| \left(a(x)A(D)\Lambda_+ - A(D)\Lambda_+ a(x) \right) u \right\|_{x_n>0} \\ &= \left\| \sum_{j=1}^n \left(\tilde{a}(x)\tilde{A}(D)R_j - \tilde{A}(D)R_j\tilde{a}(x) \right) \frac{\partial \tilde{u}}{\partial x_j} - \sum_{j=1}^n \tilde{A}(D)R_j \frac{\partial \tilde{a}}{\partial x_j}\tilde{u} \right\|_{x_n>0} \\ &\leqslant \sum_{j=1}^n \left\| \left(\tilde{a}(x)\tilde{A}(D)R_j - \tilde{A}(D)R_j\tilde{a}(x) \right) \frac{\partial \tilde{u}}{\partial x_j} \right\|_{x_n>0} + \sum_{j=1}^n \left\| \tilde{A}(D)R_j \frac{\partial \tilde{a}}{\partial x_j}\tilde{u} \right\|_{x_n>0}. \end{aligned}$$

Second term on the right is estimated as follows:

$$\sum_{j=1}^n \left\| \tilde{A}(D)R_j \frac{\partial \tilde{a}}{\partial x_j}\tilde{u} \right\|_{x_n>0} \leqslant C|a|_1 M_0(A)\,\|u\|_{x_n>0}\,.$$

Hence we shall estimate the first term on the right. For $v \in L^2(\boldsymbol{R}^n)$, $F\left(\tilde{A}(D)R_j v\right) = \frac{1}{\sqrt{-1}} A(\xi)\frac{\xi_j}{|\xi|}Fv$ and $A(\xi)\frac{\xi_j}{|\xi|}$ is of positive homogeneous degree zero. From this, provided $\int_\Sigma A(\xi)\frac{\xi_j}{|\xi|}d\Sigma = c_1 \neq 0$, dividing $\frac{1}{\sqrt{-1}}A(\xi)\frac{\xi_j}{|\xi|}$ into $A_1(\xi) + \frac{c_1}{\omega_n}$, it follows that there exists a function $f(x) \in \mathcal{E}^k(\boldsymbol{R}^n - \{0\})$ of homogeneous degree $(-n)$ such that

$$F'\left(A_1(\xi)\right)(x) = v.p.f(x)\,, \qquad \int_{S_1} f(x)dS_1 = 0 \qquad \text{and}$$

$$\begin{aligned} \sum_{|\alpha| \leqslant k} \sup_{|x| \geqslant 1} \left| D^\alpha f(x) \right| &\leqslant C \sum_{|\alpha| \leqslant n+1+k} \sup_{|\xi| \geqslant 1} \left| D^\alpha A_1(\xi) \right| \\ &\leqslant C \sum_{|\alpha| \leqslant n+1+k} \sup_{|\xi| \geqslant 1} \left| D^\alpha A(\xi) \right|. \end{aligned}$$

Replacing $R_j(x)$ in Lemma 4 by $f(x)$ obtained above and making use of $k=1$ in the above expression, we can prove the following estimate similarly to that used in the proof of Lemma 4

$$\left\| \left((\tilde{a}(x)\tilde{A}(D)R_j - \tilde{A}(D)R_j\tilde{a}(x) \right) \frac{\partial \tilde{u}}{\partial x_j} \right\|_{x_n>0} \leqslant C|a|_4 M_{n+2}(A)\|u\|_{x_n>0}\,.$$

Therefore

$$\left\|\Big(a(x)A(D)\Lambda_{+}-A(D)\Lambda_{+}a(x)\Big)u\right\|_{x_n>0}\leqslant C|a|_4M_{n+2}(A)\|u\|_{x_n>0}. \quad \text{q.e.d.}$$

The following Corollary is obtained by the same method used above.

Corollary. *Let* $a(x)\in\mathfrak{B}^{1+\delta}(\overline{\boldsymbol{R}_{+}^{n}})$ *and* $A(\xi)\in\Xi_{4}^{\infty}$. *Then for* $u\in D(\Lambda)$

$$\left\|\Big(\tilde{a}(x)\tilde{A}(D)\Lambda-\Lambda\tilde{A}(D)\tilde{a}(x)\Big)u\right\|\leqslant C|a|_{1+\delta}M_{n+2}(A)\|u\|.$$

Now, for $\eta(\xi)=\dfrac{\xi_n}{|\xi|}$ define the operators $\eta_1(D)$ and $\eta(D)$ as follows:

$$\eta_1(D)\tilde{u}=F'\eta(\xi)F\tilde{u}, \quad \eta(D)u=\eta_1(D)\tilde{u}\Big|_{x_n>0} \qquad \text{for} \quad u\in L^2(\boldsymbol{R}_{+}^{n}).$$

Furthermore for convenience of calculations set $\eta^2(D)$ as follows:

$$\eta^2(D)u=F'\Big(\eta(\xi)\Big)^2F\tilde{u}\Big|_{x_n>0} \qquad \text{for} \quad u\in L^2(\boldsymbol{R}^{n}).$$

Lemma 6. *Let* $a(x)\in\mathfrak{B}^{4}(\overline{\boldsymbol{R}_{+}^{n}})$. *Then for* $u\in D(\Lambda_{+})$

$$\left\|\Big(\eta(D)a(x)-a(x)\eta(D)\Big)\Lambda_{+}u\right\|_{x_n>0}\leqslant C|a|_4\|u\|_{x_n>0}.$$

Proof. From $\eta(D)\Lambda_{+}u=\dfrac{1}{\sqrt{-1}}\dfrac{\partial\tilde{u}}{\partial x^n}\Big|_{x_n>0}$ and Lemma 3 it follows

$$\begin{aligned}
&\left\|\Big(\eta(D)a(x)-a(x)\eta(D)\Big)\Lambda_{+}u\right\|_{x_n>0}\\
&\leqslant\left\|\eta(D)\Big(a(x)\Lambda_{+}-\Lambda_{+}a(x)\Big)u\right\|_{x_n>0}+\|\eta(D)\Lambda_{+}a(x)u-a(x)\eta(D)\Lambda_{+}u\|_{x_n>0}\\
&\leqslant C|a|_4\|u\|_{x_n>0}+\left\|\frac{\partial}{\partial x_n}\widetilde{a(x)u}-a(x)\frac{\partial\tilde{u}}{\partial x_n}\right\|_{x_n>0}\\
&\leqslant C|a|_4\|u\|_{x_n>0}+C|a|_1\|u\|_{x_n>0}\leqslant C|a|_4\|u\|_{x_n>0}.
\end{aligned}$$

Lemma 7. *Let* $f(x)\in\mathfrak{B}^{2}(\overline{\boldsymbol{R}_{+}^{n}})$, $f(x',0)=0$, $A(\xi)\in\Xi_{4}^{\infty}$ *and* $A(\xi',\xi_n)=A(\xi',-\xi_n)$. *Then for* $u(x)\in D(\Lambda_{+})$

$$\left\|\Big(f(x)\eta(D)A(D)-A(D)f(x)\eta(D)\Big)\Lambda_{+}u\right\|_{x_n>0}\leqslant C|f|_{1+\delta}M_{n+2}(A)\|u\|_{x_n>0}.$$

Proof. We see that

$$\begin{aligned}
\eta(D)A(D)\Lambda_{+}u&=F'\Big(\eta(\xi)A(\xi)|\xi|F\tilde{u}\Big)\Big|_{x_n>0}=F'\Big(A(\xi)\xi_nF\tilde{u}\Big)\Big|_{x_n>0}\\
&=F'\left(A(\xi)F\left(\frac{1}{\sqrt{-1}}\frac{\partial\tilde{u}}{\partial x_n}\right)\right)\Big|_{x_n>0}=\frac{1}{\sqrt{-1}}\tilde{A}(D)\left(\frac{\partial\tilde{u}}{\partial x_n}\right)\Big|_{x_n>0}
\end{aligned}$$

and $A(D)f(x)\eta(D)\Lambda_{+}u=\dfrac{1}{\sqrt{-1}}\tilde{A}(D)\left(\tilde{\tilde{f}}(x)\dfrac{\partial\tilde{u}}{\partial x_n}\right)\Big|_{x_n>0}$ for $\dfrac{\partial\tilde{u}}{\partial x_n}$ is the odd function, from which and the theory of ordinary singular integrals it implies

$$\begin{aligned}&\left\|\big(f(x)\eta(D)A(D)-A(D)f(x)\eta(D)\big)\Lambda_+u\right\|_{x_n>0}\\&=\left\|\frac{1}{\sqrt{-1}}\left(\tilde{\tilde{f}}(x)\tilde{A}(D)\frac{\partial\tilde{u}}{\partial x_n}-\tilde{A}(D)\tilde{\tilde{f}}(x)\frac{\partial\tilde{u}}{\partial x_n}\right)\right\|_{x_n>0}\\&\leqslant C\left|\tilde{\tilde{f}}(x)\right|_{1+\delta}M_{n+2}(A)\,\|\tilde{u}\|\leqslant C\left|f(x)\right|_{1+\delta}M_{n+2}(A)\,\|u\|_{x_n>0}\,.\end{aligned}$$

Lemma 8. *Let $f(x)$ be as in Lemma 7. Then for $u(x)\in D(\Lambda_+)$*

$$\left\|\big(f(x)\eta(D)\Lambda_+-\Lambda_+f(x)\eta(D)\big)u\right\|_{x_n>0}\leqslant C|f|_{1+\delta}\|u\|_{x_n>0}\,.$$

Proof. From that $\eta_1(D)\tilde{u}$ is the odd function it follows

$$\begin{aligned}&\left\|\big(f(x)\eta(D)\Lambda_+-\Lambda_+f(x)\eta(D)\big)u\right\|_{x_n>0}\\&=\|\tilde{\tilde{f}}(x)\eta_1(D)\Lambda\tilde{u}-\Lambda\tilde{\tilde{f}}(x)\eta_1(D)\tilde{u}\|_{x_n>0}\\&\leqslant\left\|\big(\tilde{\tilde{f}}(x)\Lambda-\Lambda\tilde{\tilde{f}}(x)\big)\eta_1(D)\tilde{u}\right\|_{x_n>0}\leqslant C|f|_{1+\delta}\|u\|_{x_n>0}\,.\end{aligned}$$

Lemma 9. *Let $f(x)$ be as in Lemma 7. Then for $u(x)\in D(\Lambda_+)$*

$$\left\|\big(f(x)\eta(D)\big)^*-\bar{f}(x)\eta(D)\big)\Lambda_+u\right\|_{x_n>0}\leqslant C|f|_{1+\delta}\|u\|_{x_n>0}.$$

Proof. Let $v(x)\in C_0^\infty(\boldsymbol{R}_+^n)$. Then

$$\begin{aligned}&\Big(\big(\big(f(x)\eta(D)\big)^*-\bar{f}(x)\eta(D)\big)\Lambda_+u,\,v\Big)_{x_n>0}\\&=\big(u,\,\Lambda_+f(x)\eta(D)v\big)_{x_n>0}-\big(\bar{f}(x)\eta(D)\Lambda_+u,\,v)\big)_{x_n>0}\\&=\Big(u,\,\big(\Lambda_+f(x)\eta(D)-f(x)\eta(D)\Lambda_+\big)v\Big)_{x_n>0}\\&\quad+\big(u,f(x)\eta(D)\Lambda_+v\big)_{x_n>0}-\big(\bar{f}(x)\eta(D)\Lambda_+u,\,v\big)_{x_n>0}\,.\end{aligned}$$

From Lemma 8 the absolute value of the first term on the right is bounded by $C|f|_{1+\delta}\|u\|_{x_n>0}\|v\|_{x_n>0}$. The rest terms on the right are calculated as follows:

$$\begin{aligned}&\big(u,f(x)\eta(D)\Lambda_+v\big)_{x_n>0}-\big(\bar{f}(x)\eta(D)\Lambda_+u,\,v\big)_{x_n>0}\\&=\left(u,f(x)\frac{1}{\sqrt{-1}}\frac{\partial\tilde{v}}{\partial x_n}\right)_{x_n>0}-\left(\bar{f}(x)\frac{1}{\sqrt{-1}}\frac{\partial\tilde{u}}{\partial x_n},\,v\right)_{x_n>0}.\end{aligned}$$

From this it follows

$$\left|\left(u, f(x)\eta(D)\Lambda_+ v\right)_{x_n>0} - \left(\bar{f}(x)\eta(D)\Lambda_+ u, v\right)_{x_n>0}\right| \leqslant C|f|_1 \|u\|_{x_n>0}\|v\|_{x_n>0}.$$

Hence

$$\left|\left(\left(\left(f(x)\eta(D)\right)^* - \bar{f}(x)\eta(D)\right)\Lambda_+ u, v\right)_{x_n>0}\right| \leqslant C|f|_{1+\delta}\|u\|_{x_n>0}\|v\|_{x_n>0}.$$

Because that $C_0^\infty(\boldsymbol{R}_+^n)$ is dense in $L^2(\boldsymbol{R}_+^n)$, it follows

$$\left\|\left(\left(f(x)\eta(D)\right)^* - \bar{f}(x)\eta(D)\right)\Lambda_+ u\right\|_{x_n>0} \leqslant C|f|_{1+\delta}\|u\|_{x_n>0}.$$

Lemma 10. *Let $f(x)$ be as in Lemma 7. Then for $u(x)\in D(\Lambda_+)$*

$$\left\|\left(\eta(D)f(x)\eta(D) - f(x)\eta^2(D)\right)\Lambda_+ u\right\|_{x_n>0} \leqslant C|f|_{1+\delta}\|u\|_{x_n>0}.$$

Proof. We see that

$$\eta(D)f(x)\eta(D)\Lambda_+ u = F'\eta(\xi)F\left(\tilde{\tilde{f}}(x)\frac{1}{\sqrt{-1}}\frac{\partial\tilde{u}}{\partial x_n}\right)\Bigg|_{x_n>0}$$

and $\eta^2(D)\Lambda_+ u = F'\eta(\xi)^2|\xi|F\tilde{u}|_{x_n>0} = F'\eta(\xi)F\dfrac{1}{\sqrt{-1}}\dfrac{\partial\tilde{u}}{\partial x_n}\Big|_{x_n>0}$, from which it follows

$$\begin{aligned}
&\left\|\left(\eta(D)f(x)\eta(D) - f(x)\eta^2(D)\right)\Lambda_+ u\right\|_{x_n>0} \\
&\leqslant \left\|\left(F'\eta(\xi)F\tilde{\tilde{f}}(x) - \tilde{\tilde{f}}(x)F'\eta(\xi)F\right)\frac{\partial\tilde{u}}{\partial x_n}\right\|_{x_n>0} \\
&\leqslant \left\|\left(R_n\tilde{\tilde{f}}(x) - \tilde{\tilde{f}}(x)R_n\right)\frac{\partial\tilde{u}}{\partial x_n}\right\|_{x_n>0} \leqslant C|\tilde{\tilde{f}}|_{1+\delta}\|\tilde{u}\| \leqslant C|f|_{1+\delta}\|u\|_{x_n>0}.
\end{aligned}$$

q. e. d.

Now decompose $Y_{l,m}(\xi)$ as follows:

$$\begin{aligned}
Y_{l,m}(\xi) &= Y_{l,m}(\xi', \xi_n) \\
&= \frac{1}{2}\left(Y_{l,m}(\xi', \xi_n) + Y_{l,m}(\xi', -\xi_n)\right) + \frac{1}{2}\left(Y_{l,m}(\xi', \xi_n) - Y_{l,m}(\xi', -\xi_n)\right) \\
&\equiv Y_{l,m}^+(\xi', \xi_n) + Y_{l,m}^-(\xi', \xi_n).
\end{aligned}$$

Then we find that every $Y_{,m}^+(\xi', \xi_n)\left((Y_{l,m}^-(\xi', \xi_n)\right)$ becomes an even (odd) function with respect to ξ_n and $\int_\Sigma Y_{l,m}^+(\xi)d\Sigma = 0 = \int_\Sigma Y_{l,m}^-(\xi)d\Sigma$.

Lemma 11. *We have the following estimate for every l, m and α.*

$$\sup_{|\xi|=1}\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha Y_{l,m}^\pm(\xi', \xi_n)\right| \leqslant \sup_{|\xi|=1}\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha Y_{l,m}(\xi', \xi_n)\right|.$$

Proof. We see that

$$\sup_{|\xi|=1}\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha} Y_{l,m}(\xi', \xi_n)\right| = \sup_{|\xi|=1}\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha} Y_{l,m}(\xi', -\xi_n)\right|$$

for every l, m and α. From which it implies the desired inequality. q. e. d.

We shall prove Theorem 1 using the preceding Lemmas.

Proof of Theorem 1. Similarly to the proofs of Lemmas 3–10, we may calculate this theorem for $u(x)\in C_0^\infty(\overline{\boldsymbol{R}^n_+})$ and $v(x)\in C_0^\infty(\boldsymbol{R}^\infty_+)$ if it needs. Without loss of generality, it suffices to prove for $\alpha(x, \xi)=a(x, \xi)f(x)\frac{\xi_n}{|\xi|}$ and $\beta(x, \xi)=b(x, \xi)g(x)\frac{\xi_n}{|\xi|}$ $(\in\mathfrak{A})$. By definition 2 $a(x, \xi)$ is the even function with respect to ξ_n. Therefore, as stated in the proof of Lemma 1 we see that $a(x, \xi)=\sum_{l,m} a_{l,m}(x)Y_{l,m}(\xi)=\sum_{l,m} a_{l,m}(x)Y^+_{l,m}(\xi)$. We furthermore remark that $\Lambda_+ Y^+_{l,m}(D)u=Y^+_{l,m}(D)\Lambda_+u$ for $u\in D(\Lambda_+)$.

Proof of (8). We see that

$$\begin{aligned}\alpha(x, D)u &= F'f(x)\eta(\xi)a(x, \xi)F\tilde{u}|_{x_n>0}\\ &= F'f(x)\eta(\xi)\sum_{l,m} a_{l,m}(x)Y^+_{l,m}(\xi)F\tilde{u}|_{x_n>0}\\ &= \sum_{l,m} a_{l,m}(x)f(x)\eta(D)Y^+_{l,m}(D)u\,.\end{aligned}$$

From this

$$\begin{aligned}&\alpha(x, D)\Lambda_+ - \Lambda_+\alpha(x, D)\\ &\quad= \sum_{l,m} a_{l,m}(x)f(x)\eta(D)Y^+_{l,m}(D)\Lambda_+ - \Lambda_+\sum_{l,m} a_{l,m}(x)f(x)\eta(D)Y^+_{l,m}(D)\\ &\quad= \sum_{l,m} a_{l,m}(x)\Big(f(x)\eta(D)\Lambda_+ - \Lambda_+f(x)\eta(D)\Big)Y^+_{l,m}(D)\\ &\qquad+\sum_{l,m}\Big(a_{l,m}(x)\Lambda_+ - \Lambda_+a_{l,m}(x)\Big)f(x)\eta(D)Y^+_{l,m}(D)\,.\end{aligned}$$

Hence applying (2), (4) and Lemma 11 to all the terms, Lemma 8 to the first term and Lemma 3 to the second term on the right we find that

$$\begin{aligned}&\Big\|\Big(\alpha(x, D)\Lambda_+ - \Lambda_+\alpha(x, D)\Big)u\Big\|_{x_n>0}\\ &\quad\leqslant CM_{2n}(a)\sum_l l^{-2n+\frac{n}{2}}l^{n-2}|f|_{1+\delta}l^{\frac{1}{2}(n-2)}\|u\|_{x_n>0}\\ &\quad+CM_{2n}(a)\sum_l l^{-2n+\frac{n}{2}}l^{n-2}|f|_0 l^{\frac{1}{2}(n-2)}\|u\|_{x_n>0}\leqslant CM_{2n}(\alpha)\|u\|_{x_n>0}\,.\end{aligned}$$

Proof of (9). Using (8), we see that

$$\left|\left(\left(\alpha(x, D)^* \Lambda_+ - \Lambda_+ \alpha(x, D)^*\right) u, v\right)_{x_n>0}\right|$$

$$= \left|\left(u, \left(\Lambda_+ \alpha(x, D) - \alpha(x, D) \Lambda_+\right) v\right)_{x_n>0}\right| \leqslant CM_{2n}(\alpha) \|u\|_{x_n>0} \|v\|_{x_n>0} .$$

Therefore

$$\left\|\left(\alpha(x, D)^* \Lambda_+ - \Lambda_+ \alpha(x, D)^*\right) u\right\|_{x_n>0} \leqslant CM_{2n}(\alpha) \|u\|_{x_n>0} .$$

Proof of (10). We see that

$$\left(\alpha(x, D)^* - \alpha^\#(x, D)\right) \Lambda_+$$

$$= \sum_{l,m} \left(Y_{l,m}^+(D) \left(f(x)\eta(D)\right)^* \bar{a}_{l,m}(x) - \bar{a}_{l,m}(x) f(x)\eta(D) Y_{l,m}^+(D)\right) \Lambda_+$$

$$= \sum_{l,m} \left(Y_{l,m}^+(D)\left(f(x)\eta(D)\right)^* \left(\bar{a}_{l,m}(x)\Lambda_+ - \Lambda_+ \bar{a}_{l,m}(x)\right)\right.$$

$$+ \sum_{l,m} Y_{l,m}^+(D) \left(\left(f(x)\eta(D)\right)^* - f(x)\eta(D)\right) \Lambda_+ \bar{a}_{l,m}(x)$$

$$+ \sum_{l,m} \left(Y_{l,m}^+(D) f(x)\eta(D) - f(x)\eta(D) Y_{l,m}^+(D)\right) \Lambda_+ \bar{a}_{l,m}(x)$$

$$+ \sum_{l,m} f(x)\eta(D) \left(Y_{l,m}^+(D)\Lambda_+ \bar{a}_{l,m}(x) - \bar{a}_{l,m}(x) Y_{l,m}^+(D)\Lambda_+\right)$$

$$+ \sum_{l,m} f(x) \left(\eta(D)\bar{a}_{l,m}(x) - \bar{a}_{l,m}(x)\eta(D)\right) \Lambda_+ Y_{l,m}^+(D) .$$

Hence applying (2), (4) and Lemma 11 to all the terms, Lemma 3 to the first term, Lemma 9 to the second term, Lemma 7 to the third term, Lemma 5 to the fourth term and Lemma 6 to the last term, we find that

$$\left\|\left(\alpha(x, D)^* - \alpha^\#(x, D)\right) \Lambda_+ u\right\|_{x_n>0}$$

$$\leqslant C\sum_l l^{\frac{1}{2}(n-2)} |f|_0 \, l^{-2n+\frac{n}{2}} l^{n-2} M_{2n}(a) \|u\|_{x_n>0}$$

$$+ C\sum_l l^{\frac{1}{2}(n-2)} |f|_{1+\delta} \, l^{-2n+\frac{n}{2}} l^{n-2} M_{2n}(a) \|u\|_{x_n>0}$$

$$+ C\sum_l |f|_{1+\delta} \, l^{\frac{1}{2}(n-2)+n+2} l^{-2([\frac{3}{2}n]+1)+\frac{n}{2}} l^{n-2} M_{2([\frac{3}{2}n]+1)}(a) \|u\|_{x_n>0}$$

$$+ C|f|_0 \sum_l l^{-2([\frac{3}{2}n]+1)+\frac{n}{2}} l^{n-2} l^{\frac{1}{2}(n-2)+n+2} M_{2([\frac{3}{2}n]+1)}(a) \|u\|_{x_n>0}$$

$$+ C|f|_0 \sum_l l^{-2n+\frac{n}{2}} l^{n-2} l^{\frac{1}{2}(n-2)} M_{2n}(a) \|u\|_{x_n>0}$$

$$\leqslant CM_{2([\frac{3}{2}n]+1)}(\alpha) \|u\|_{x_n>0} .$$

Proof of (11). We see that

$$(\alpha\circ\beta)(x, D)u = F'f(x)\eta(\xi)a(x, \xi)g(x)\eta(\xi)b(x, \xi)F\tilde{u}|_{x_n>0}$$
$$=f(x)g(x)F'\big(\eta(\xi)\big)^2 a(x, \xi)b(x, \xi)F\tilde{u}|_{x_n>0}$$
$$=f(x)g(x)F'\big(\eta(\xi)\big)^2 \sum_{l,m} a_{l,m}(x)Y_{l,m}(\xi)\sum_{p,q} b_{p,q}(x)Y_{p,q}(\xi)F\tilde{u}|_{x_n>0}$$
$$=\sum_{l,m}\sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)g(x)\eta^2(D)Y^+_{l,m}(D)Y^+_{p,q}(D)u.$$

From this

$$\big(\alpha(x, D)\beta(x, D)-(\alpha\circ\beta)(x, D)\big)$$
$$=\sum_{l,m} a_{l,m}(x)f(x)\eta(D)Y^+_{l,m}(D)\sum_{p,q} b_{p,q}(x)g(x)\eta(D)Y^+_{p,q}(D)\Lambda_+$$
$$-\sum_{l,m}\sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)g(x)\eta^2(D)Y^+_{l,m}(D)Y^+_{p,q}(D)\Lambda_+$$
$$=\sum_{l,m}\sum_{p,q} a_{l,m}(x)f(x)\eta(D)Y^+_{l,m}(D)b_{p,q}(x)\big(g(x)\eta(D)\Lambda_+-\Lambda_+g(x)\eta(D)\big)Y^+_{p,q}(D)$$
$$+\sum_{l,m}\sum_{p,q} a_{l,m}(x)f(x)\eta(D)Y^+_{l,m}(D)\big(b_{p,q}(x)\Lambda_+-\Lambda_+b_{p,q}(x)\big)g(x)\eta(D)Y^+_{p,q}(D)$$
$$+\sum_{l,m}\sum_{p,q} a_{l,m}(x)f(x)\eta(D)\big(Y^+_{l,m}(D)\Lambda_+b_{p,q}(x)-b_{p,q}(x)Y^+_{l,m}(D)\Lambda_+\big)$$
$$\times g(x)\eta(D)Y^+_{p,q}(D)$$
$$+\sum_{l,m}\sum_{p,q} a_{l,m}(x)f(x)\big(\eta(D)b_{p,q}(x)-b_{p,q}(x)\eta(D)\big)\Lambda_+Y^+_{l,m}(D)g(x)\eta(D)Y^+_{p,q}(D)$$
$$+\sum_{l,m}\sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)\eta(D)Y^+_{l,m}(D)\big(\Lambda_+g(x)\eta(D)-g(x)\eta(D)\Lambda_+\big)Y^+_{p,q}(D)$$
$$+\sum_{l,m}\sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)\eta(D)\big(Y^+_{l,m}(D)g(x)\eta(D)-g(x)\eta(D)Y^+_{l,m}(D)\big)$$
$$\times\Lambda_+Y^+_{p,q}(D)$$
$$+\sum_{l,m}\sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)\big(\eta(D)g(x)\eta(D)-g(x)\eta^2(D)\big)\Lambda_+Y^+_{l,m}(D)Y^+_{p,q}(D).$$

Hence applying (2,) (4) and Lemma 11 to all the terms, Lemma 8 to the first term, Lemma 3 to the second term, Lemma 5 to the third term, Lemma 6 to the fourth term, Lemma 8 to the fifth term, Lemma 7 to the sixth term and Lemma 10 to the last term, similarly to the proof of (10) we find that

$$\left\|\big(\alpha(x, D)\beta(x, D)-(\alpha\circ\beta)(x, D)\big)\Lambda_+u\right\|_{x_n>0}\leqslant CM_{2([\frac{3}{2}n]+1)}(\alpha)M_{2n}(\beta)\|u\|_{x_n>0}.$$

q. e. d.

Furthermore we consider the singular integral operators with boundary conditions on $\boldsymbol{R}^n_+$ for more extended symbol.

Definition 4. *Let* $\mathfrak{A}$ *be the set* $\alpha(x, \xi)$ *such that*

$\alpha(x, \xi)=\sum_{i=1}^{\infty}\alpha_i(x, \xi)$, $\alpha_i(x, \xi)\in\mathfrak{A}$ *and for every integer* $s(\geqslant 0)$, *there exists* $M_s(\alpha)(<\infty)$ *such that*

$$\sum_{i=1}^{\infty}M_s(\alpha_i)\leqslant M_s(\alpha)\,. \tag{12}$$

For $\alpha(x, \xi)\in\overline{\mathfrak{A}}$ *consider the singular integral operator* $\alpha(x, D)$ *as follows*:

$$\alpha(x, D)u=\sum_{i=1}^{\infty}\alpha_i(x, D)u \quad in \quad L^2(\boldsymbol{R}_+^n) \qquad for \quad u\in L^2(\boldsymbol{R}_+^n).$$

Here we remark that $\alpha(x, \xi)\in\Xi_4^{\infty}$ is in $\overline{\mathfrak{A}}$ if and only if $\alpha(x', 0, \xi)$ is an even function with respect to ξ_n.

Theorem 2. *For the symbols* $\alpha(x, \xi)$ *and* $\beta(x, \xi)\in\overline{\mathfrak{A}}$, $\alpha(x, D)u$, $\alpha(x, D)^* u \in D(\Lambda_\pm)$ *for* $u\in D(\Lambda_\pm)$ *and the statements of Theorem* 1 *are also valid.*

Proof. The first statement is shown by virtue of Lemma 1, (12) and Theorem 1. The last statement is shown by virtue of Theorem 1 and (12) regarding $\alpha_i(x, \xi)$ and $\beta_i(x, \xi)\in\mathfrak{A}$. Hence we prove only (11).

$$\begin{aligned}
&\left\|\Big(\alpha(x, D)\beta(x, D)-(\alpha\circ\beta)(x, D)\Big)\Lambda_\pm u\right\|_{x_n>0}\\
&\leqslant\left\|\Big(\sum_{i,j}\alpha_i(x, D)\beta_j(x, D)-\sum_{i,j}(\alpha_i\circ\beta_j)(x, D)\Big)\Lambda_\pm u\right\|_{x_n>0}\\
&\leqslant\sum_{i,j}\left\|\Big(\alpha_i(x, D)\beta_j(x, D)-(\alpha_i\circ\beta_j)(x, D)\Big)\Lambda_\pm u\right\|_{x_n>0}\\
&\leqslant C\sum_{i,j}M_{2([\frac{3}{2}n]+1)}(\alpha_i)\,M_{2n}(\beta_j)\,\|u\|_{x_n>0}\\
&\leqslant C\sum_{i}M_{2([\frac{3}{2}n]+1)}(\alpha_i)\sum_{j}M_{2n}(\beta_j)\|u\|_{x_n>0}\\
&\leqslant CM_{2([\frac{3}{2}n]+1)}(\alpha)M_{2n}(\beta)\|u\|_{x_n>0}\,. \qquad\qquad \text{q. e. d.}
\end{aligned}$$

Lemma 12. (Kohn and Nirenberg see [8])

Let $\sigma(x, \xi)\in\overline{\mathfrak{A}}$ *and* $\sigma(x, \xi)\geqslant d>0$ *for* $x\in\overline{\boldsymbol{R}_+^n}$ *and* $|\xi|=1$. *Then for sufficiently large* $k>0$

$$\left\|\Big(\sigma(x, D)\Lambda_\pm+k\Big)u\right\|_{x_n>0}^2\geqslant(d-\varepsilon)^2\|\Lambda_\pm u\|_{x_n>0}^2+(k-\rho)^2\|u\|_{x_n>0}^2$$

for $u\in D(\Lambda_\pm)$, *where* ε *is an arbitrary small positive number*, $k>\rho$ *and* ρ *depends on* n, σ *and* ε.

Proof. We see that

$$\begin{aligned}
&\left\|\Big(\sigma(x, D)\Lambda_\pm+k\Big)u\right\|_{x_n>0}^2\\
&\quad=\|\sigma\Lambda_\pm u\|_{x_n>0}^2+k\Big((\sigma\Lambda_\pm u, u)_{x_n>0}+(u, \sigma\Lambda_\pm u)_{x_n>0}\Big)+k^2\|u\|_{x_n>0}^2\,.
\end{aligned}$$

We at first estimate the first term on the right. Regarding that $\sigma(x, D)=\sigma^{\#}(x, D)$ since $\sigma(x, \xi)$ is real,

$$(\sigma\Lambda_{\pm}u, \sigma\Lambda_{\pm}u)_{x_n>0}=\Big(\sigma\Lambda_{\pm}u, (\sigma^{\#}-\sigma^{*})\Lambda_{\pm}u\Big)_{x_n>0}$$

$$+\Big((\sigma\cdot\sigma-\sigma\circ\sigma)\Lambda_{\pm}u, \Lambda_{\pm}u)\Big)_{x_n>0}+(\sigma\circ\sigma\Lambda_{\pm}u, \Lambda_{\pm}u)_{x_n>0}.$$

The first two terms are bounded by $\varepsilon_1\|\Lambda_{\pm}u\|^2_{x_n>0}+c(\varepsilon_1)\|u\|^2_{x_n>0}$ by virtue of Theorem 2. Let us estimate the last term on the right. From the assumption we can choose $\sigma_1(x,\xi)\in\bar{\mathfrak{A}}$ such that $\sigma(x, \xi)^2=(d-\varepsilon')^2+(\sigma_1(x, \xi))^2$, $d>\varepsilon'>0$, $\sigma_1(x, \xi)\geqslant\varepsilon'>0$ for $x\in\overline{\boldsymbol{R}^n_+}$, $|\xi|=1$.

Then

$$(\sigma\circ\sigma\Lambda_{\pm}u, \Lambda_{\pm}u)_{x_n>0}$$

$$=(d-\varepsilon')^2\|\Lambda_{\pm}u\|^2_{x_n>0}+(\sigma_1\circ\sigma_1\Lambda_{\pm}u, \Lambda_{\pm}u)_{x_n>0}$$

$$=(d-\varepsilon')^2\|\Lambda_{\pm}u\|^2_{x_n>0}+\Big((\sigma_1\circ\sigma_1-\sigma_1\cdot\sigma_1)\Lambda_{\pm}u, \Lambda_{\pm}u\Big)_{x_n>0}$$

$$+\Big(\sigma_1\Lambda_{\pm}u, (\sigma_1^{*}-\sigma_1^{\#})\Lambda_{\pm}u\Big)_{x_n>0}+(\sigma_1\Lambda_{\pm}u, \sigma_1\Lambda_{\pm}u)_{x_n>0}.$$

The second and the third terms on the right are also bounded by $\varepsilon_1\|\Lambda_{\pm}u\|^2_{x_n>0}+c(\varepsilon_1)\|u\|^2_{x_n>0}$ by virtue of Theorem 2. Therefore

$$\|\sigma\Lambda_{\pm}u\|^2_{x_n>0}\geqslant(d-\varepsilon')^2\|\Lambda_{\pm}u\|^2_{x_n>0}-2\varepsilon_1\|\Lambda_{\pm}u\|^2_{x_n>0}-c(\varepsilon_1)\|u\|^2_{x_n>0}.$$

Next we estimate $(\sigma\Lambda_{\pm}u, u)_{x_n>0}+(u, \sigma\Lambda_{\pm}u)_{x_n>0}$. From the assumption $\sigma(x, \xi)\geqslant d>0$, we see that $\Big(\sigma(x,\xi)\Big)^{\frac{1}{2}}\in\bar{\mathfrak{A}}$.

From this

$$(\sigma\Lambda_{\pm}u, u)_{x_n>0}=(\sigma^{\frac{1}{2}}\circ\sigma^{\frac{1}{2}}\Lambda_{\pm}u, u)_{x_n>0}$$

$$=(\Lambda_{\pm}\sigma^{\frac{1}{2}}u, \sigma^{\frac{1}{2}}u)_{x_n>0}+\Big((\sigma^{\frac{1}{2}\#}-\sigma^{\frac{1}{2}*})\Lambda_{\pm}\sigma^{\frac{1}{2}}u, u\Big)_{x_n>0}$$

$$+\Big(\sigma^{\frac{1}{2}}(\sigma^{\frac{1}{2}}\Lambda_{\pm}-\Lambda_{\pm}\sigma^{\frac{1}{2}})u, u\Big)_{x_n>0}+\Big((\sigma^{\frac{1}{2}}\circ\sigma^{\frac{1}{2}}-\sigma^{\frac{1}{2}}\cdot\sigma^{\frac{1}{2}})\Lambda_{\pm}u, u\Big)_{x_n>0}.$$

By virtue of Theorem 2 the last three terms are bounded by $C\|u\|^2_{x_n>0}$. From $(\Lambda_{\pm}\sigma^{\frac{1}{2}}u, \sigma^{\frac{1}{2}}u)_{x_n>0}>0$ it follows that

$$(\sigma\Lambda_{\pm}u, u)_{x_n>0}+(u, \sigma\Lambda_{\pm}u)_{x_n>0}\geqslant-C\|u\|^2_{x_n>0}. \tag{13}$$

Therefore

$$\|(\sigma\Lambda_{\pm}+k)u\|^2_{x_n>0}\geqslant(d-\varepsilon')^2\|\Lambda_{\pm}u\|^2_{x_n>0}-2\varepsilon_1\|\Lambda_{\pm}u\|^2_{x_n>0}-c(\varepsilon_1)\|u\|^2_{x_n>0}$$

$$-Ck\|u\|^2_{x_n>0}+k^2\|u\|^2_{x_n>0}.$$

By setting k sufficiently large and ε_1 sufficiently small (>0), we obtain the desired estimate. q.e.d.

2.2. Singular integral operators with boundary conditions defined on Ω.

We at first describe the coordinate transformations, from which we reduce above ones to singular integral operators with boundary conditions on $\boldsymbol{R}^n_+$.

Lemma 13. *There exist a finite covering $\{U_\alpha\}$ of $\bar{\Omega}$ and diffeomorphisms T_α from U_α into $\boldsymbol{R}^n$ such that the following* 1), 2) *and* 3) *are satisfied*:

1) *If $U_\alpha \cap \Gamma \neq \phi$, then $T_\alpha(U_\alpha \cap \Omega) \subset \boldsymbol{R}^n_+$, $T_\alpha(U_\alpha \cap \Gamma) \subset \boldsymbol{R}^{n-1}$ and $T_\alpha(U_\alpha \cap \Gamma)$ contains origin.*

2) *Let $a_0(x, D)$ be the principal part of $a(x, D)$ and let $b_\alpha(y, \xi) \equiv \sum_{i,j} b^{(\alpha)}_{ij}(y)\xi_i\xi_j |\xi|^{-2} = a_0(T^{-1}_\alpha(y), (dT^*_\alpha)(\xi))|\xi|^{-2}$ for $y \in T_\alpha(U_\alpha)$ and $\xi \in \boldsymbol{R}^n - \{0\}$, then $b^{(\alpha)}_{nj}(y) = 0$ for $y_n = 0$ and for $n \neq j$, that is, the conormal direction of $a(x, D)$ at Γ transfers to the normal direction of the surface $\boldsymbol{R}^{n-1}$, here dT_α is the differential from the tangent space T_x to the tangent space $T_{T_\alpha(x)}$ and dT^*_α is the dual differential of dT_α from $T^*_{T_\alpha(x)}$ to T^*_x.*

3) *If $U_\alpha \cap U_\beta \cap \Gamma \neq \phi$ and $y \in T_\alpha(U_\alpha \cap U_\beta)$, then the n-th component of $T_\beta \cdot (T^{-1}_\alpha(y))$ is equal to y_n.*

Proof. We can choose a function $\varphi(x)$ such that $\varphi(x) \in C^\infty(\boldsymbol{R}^n)$, grad $\varphi(x) \neq 0$ on $U(\Gamma)$ and $\varphi(x) = 0$ if and only if $x \in \Gamma$. Furthermore for $x_0 \in \Gamma$ there exists a diffeomorphism S' from some neighborhood of origin $U'(0)$ $(\subset \boldsymbol{R}^{n-1})$ into Γ such that $S'(0) = x_0$.

Now, for any $y' \in U'(0)$ we construct a curve $S(y', y_n)$ transversal to Γ through $S'(y') \in \Gamma$. To this end it suffices to solve the following ordinary differential equations:

$$\frac{dS_i(y', y_n)}{dy_n} = \sum_{j=1}^n a^{(0)}_{ij}\big(S(y', y_n)\big)\frac{\partial\varphi}{\partial x_j}\big(S(y', y_n)\big) \qquad i = 1, \cdots, n, \tag{14}$$

$$S(y', 0) = S'(y') \qquad \text{for} \quad y' \in U'(0).$$

Suppose that $S(y', y_n)$ is not transversal at $S'(y')$, that is, there exists $\lambda = (\lambda, \cdots, \lambda_{n-1}) \in \boldsymbol{R}^{n-1}$ such that

$$\sum_{j=1}^n a^{(0)}_{ij}(x)\frac{\partial\varphi}{\partial x_j}(x) = \sum_{k=1}^{n-1} \lambda_k \frac{\partial S'_i}{\partial y_k}(x) \qquad i = 1, \cdots, n,$$

where $\quad x = S'(y') = \big(S'_1(y'), \cdots, S'_n(y')\big).$

From $\varphi(S'(y')) = 0$ it follows that

$$\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} \frac{\partial S'_i}{\partial y_k} = 0 \qquad \text{for} \quad k=1, \cdots, n-1 .$$

Hence

$$\sum_i \frac{\partial \varphi}{\partial x_i} \sum_j a_{ij}^{(0)}(x) \frac{\partial \varphi}{\partial x_j} = \sum_i \frac{\partial \varphi}{\partial x_i} \sum_k \lambda_k \frac{\partial S'_i}{\partial y_k} = \sum_k \lambda_k \sum_i \frac{\partial \varphi}{\partial x_i} \frac{\partial S'_i}{\partial y_k} = 0,$$

which is a contradiction for grad $\varphi \neq 0$ and $a(x, \xi)$ is elliptic. Therefore $S(y', y_n)$ is transversal at $S'(y')$.

Consequently, if we set $U(0)=U'(0) \times \{y_n ; |y_n| < \varepsilon\}$ for sufficiently small $\varepsilon(>0)$, then we see that S indicate a diffeomorphism from $U(0)$ into $\boldsymbol{R}^n$.

Denote $U=S(U(0))$ and the inverse transformation of S by T. Then we find that a finite covering $\{U_\alpha\}$ and diffeomorphisms T_α are obtained as Γ is compact. Consequently, from the construction 1) is shown and 3) is found from the facts that $a_{ij}(x)$ and $\varphi(x)$ are given in global and from the uniqueness of solution of the equation (14). Now we show 2).

From $\quad b_\alpha(y, \xi) = a_0\Big(T_\alpha^{-1}(y), (dT_\alpha^*)(\xi)\Big)$

$$= \sum_{i,j} \Big(\sum_{k,l} a_{k,l}\Big(S_\alpha(y)\Big) \frac{\partial T_{\alpha_i}}{\partial x_k}\Big(S_\alpha(y)\Big) \frac{\partial T_{\alpha_j}}{\partial x_l}\Big(S_\alpha(y)\Big) \xi_i \xi_j\Big) \qquad \text{for} \quad |\xi|=1$$

it follows that

$$b_{nj}^{(\alpha)}(y) = \sum_{k,l} a_{k,l}\Big(S_\alpha(y)\Big) \frac{\partial T_{\alpha_n}}{\partial x_k}\Big(S_\alpha(y)\Big) \frac{\partial T_{\alpha_j}}{\partial x_l}\Big(S_\alpha(y)\Big), \tag{15}$$

where $\qquad T_\alpha(x) = \Big(T_{\alpha_1}(x), \cdots, T_{\alpha_n}(x)\Big).$

Furthermore from $\varphi(S_\alpha(y', 0))=0$ we find that there exists a function $\psi(y) \in C^\infty(\boldsymbol{R}^n)$ such that $\psi(y', 0) \neq 0$, and $\varphi(S_\alpha(y', y_n)) = y_n \psi(y)$, that is, $\varphi(x) = T_{\alpha_n}(x) \psi(T_\alpha(x))$.

From this and (14) it follows that

$$\begin{aligned} \frac{\partial S_{\alpha_i}}{\partial y_n}(y', 0) &= \sum_j a_{ij}^{(0)}\Big(S_\alpha(y', 0)\Big) \frac{\partial (T_{\alpha_n}(x) \psi(T_\alpha(x)))}{\partial x_j} \Big(S_\alpha(y', 0)\Big) \\ &= \sum_j a_{ij}^{(0)}\Big(S_\alpha(y', 0)\Big) \frac{\partial T_{\alpha_n}}{\partial x_j} \Big(S_\alpha(y', 0)\Big) \psi(y', 0) . \end{aligned} \tag{16}$$

By $(\tilde{a}_{ij})$ set the inverse matrix of $(a_{ij}^{(0)})$. Then from (16)

$$\frac{\partial T_{\alpha_n}}{\partial x_k}\Big(S_\alpha(y', 0)\Big) = \sum_i \tilde{a}_{ki}\Big(S_\alpha(y', 0)\Big) \frac{\partial S_{\alpha_i}}{\partial y_n}(y', 0) \cdot \Big(\psi(y', 0)\Big)^{-1} .$$

Substituting this into (15) setting $y_n=0$, it follows that

$$
\begin{aligned}
b_{nj}^{(\alpha)}(y',0) &= \sum_{k,l} a_{k,l}^{(0)}\big(S_\alpha(y',0)\big) \sum_i \tilde{a}_{ki}\big(S_\alpha(y',0)\big) \frac{\partial S_{\alpha i}}{\partial y_n}(y',0) \\
&\quad \times \frac{\partial T_{\alpha j}}{\partial x_l}\big(S_\alpha(y',0)\big)\cdot\big(\psi(y',0)\big)^{-1} \\
&= \sum_{i,l} \delta_{il} \frac{\partial S_{\alpha i}}{\partial y_n}(y',0) \frac{\partial T_{\alpha j}}{\partial x_l}\big(S_\alpha(y',0)\big)\cdot\big(\psi(y',0)\big)^{-1} \\
&= \sum_l \frac{\partial S_{\alpha l}}{\partial y_n}(y',0) \frac{\partial T_{\alpha j}}{\partial x_l}\big(S_\alpha(y',0)\big)\cdot\big(\psi(y',0)\big)^{-1} \\
&= \delta_{jn}\cdot\big(\psi(y',0)\big)^{-1}. \qquad \text{q. e. d.}
\end{aligned}
$$

We now take a partition of unity $\{\varphi_i\}$ with respect to $\bar{\Omega}$ such that $\sum_{i=1}^{N}\varphi_i(x)=1$ on $\bar{\Omega}$, $\varphi_i u\in D(a)$ for $u\in D(a)$ and $\{\mathrm{Supp}\,\varphi_i\}$ is a star-finite refinement of $\{U_\alpha\}$ which is defined in Lemma 13 and also if $\mathrm{Supp}\,\varphi_j\cap\Gamma\neq\phi$, then $(\mathrm{Supp}\,\varphi_j)^0\cap\Gamma\neq\phi$, here G^0 is an interior of G.

Definition 5. *Let $\mathfrak{A}_\Omega$ be the set of $\sigma(x,\xi)$ such that $\sigma(x,\xi)\in\Xi_4^\infty(\bar{\Omega})\times(\boldsymbol{R}^n-\{0\}))$ and for Supp $\varphi_j\cap\Gamma\neq\phi$ and Supp $\varphi_i\cap$Supp $\varphi_j\neq\phi$*

$$
(T_{\alpha^*}\circ\varphi_i)\cdot\tilde{\sigma}\big(T_\alpha^{-1}(y),\ (dT_\alpha^*)(\xi)\big)\cdot(T_{\alpha^*}\circ\varphi_j)\in\mathfrak{A}\,.
$$

In the above by $\Xi_4^\infty(\bar{\Omega}\times(\boldsymbol{R}^n-\{0\})$ we mean the set being replaced $\overline{\boldsymbol{R}_+^n}$ by $\bar{\Omega}$ in Definition 1, $U_\alpha\supset$Supp $\varphi_i\cup$Supp φ_j, $(T_{\alpha^*}\circ u)(x)=u(T_\alpha^{-1}(x))$ and $\tilde{\sigma}(T_\alpha^{-1}(y),(dT_\alpha^*)(\xi)\big)=\psi(y)\,\sigma\big(T_\alpha^{-1}(y),(dT_\alpha^*)(\xi)\big)+(1-\psi(y)\big)\sigma\big(T_\alpha^{-1}(0),(dT_\alpha^*)(\xi)\big)$ for $\psi(y)\in C_0^\infty\big(T_\alpha(U_\alpha)\big)$ such that $\psi(y)=1$ for $y\in T_\alpha$(Supp $\varphi_i\cup$Supp φ_j).

Definition 6. *For $\sigma(x,\xi)\in\mathfrak{A}_\Omega$ consider the singular integral operator $\sigma(x,D)$ as follows: for $u\in L^2(\Omega)$*

$$
\sigma(x,D)u=\sum_{ij}\sigma_{ij}(x,D)u\,.
$$

Where if Supp $\varphi_j\cap\Gamma=\phi$, we set

$$
\sigma_{ij}(x,D)u=\varphi_i(x)\tilde{\sigma}(x,D)\varphi_j(x)u\,,
$$

here $\quad\tilde{\sigma}(x,\xi)=\psi_1(x)\sigma(x,\xi)+\big(1-\psi_1(x)\big)\sigma(x_0,\xi)\quad$ for $\quad\psi_1(x)\in C_0^\infty(\boldsymbol{R}^n)$

such that $\psi_1(x)=1$ for $x\in$Supp φ_j, Supp $\varphi_j\subset\Omega$, $x_0\in$(Supp $\varphi_j)^0$ and $\tilde{\sigma}(x,D)$ is the ordinary singular integral operator. Furthermore if Supp $\varphi_j\cap\Gamma\neq\phi$ and Supp $\varphi_i\cap$Supp $\varphi_j\neq\phi$, we set

$$
\sigma_{ij}(x,D)u=T_\alpha^*\circ\Big((T_{\alpha^*}\circ\varphi_i)\cdot\tilde{\sigma}\big(T_\alpha^{-1}(y),(dT_\alpha^*)(D_y)\big)(T_{\alpha^*}\circ\varphi_j u)\Big),
$$

here $U_\alpha \supset Supp\,\varphi_i \cup Supp\,\varphi_j$, $\tilde{\sigma}$ *is defined by Definition* 5 (*see Definition* 4) *and* $(T_\alpha^* \circ w)(x) = w(T_\alpha(x))$. *Finally if* $Supp\,\varphi_j \cap \Gamma \neq \phi$ *and* $Supp\,\varphi_i \cap Supp\,\varphi_j = \phi$, *we set*

$$\sigma_{ij}(x, D)u = 0 .$$

Definition 7. *Let* $\tilde{\Lambda}_\pm$ *be as follows: for* $u \in D(a^{\frac{1}{2}})$

$$\tilde{\Lambda}_\pm u = \sum_{i,j} \tilde{\Lambda}_{\pm ij} u .$$

Where if $Supp\,\varphi_j \cap \Gamma = \phi$, *we define*

$$\tilde{\Lambda}_{\pm ij} u = \varphi_i(x) \widetilde{\sqrt{a_0}}(x, D) \Lambda \varphi_j(x) u ,$$

here $\widetilde{\sqrt{a_0}}(x, \xi) = \psi_1(x) \sqrt{a_0}\left(x, \frac{\xi}{|\xi|}\right) + \left(1 - \psi_1(x)\right) \sqrt{a_0}\left(x_0, \frac{\xi}{|\xi|}\right)$, $\psi_1(x)$ *is as in Definition* 6 *and* $\widetilde{\sqrt{a_0}}(x, D)$ *is the ordinary singular integral operator. Furthermore if* $Supp\,\varphi_j \cap \Gamma \neq \phi$ *and* $Supp\,\varphi_i \cap Supp\,\varphi_j \neq \phi$, *we define*

$$\tilde{\Lambda}_{\pm ij} u = T_\alpha^* \circ \left((T_{\alpha *} \circ \varphi_i) \widetilde{\sqrt{b_\alpha}}(y, D) \Lambda_\pm T_{\alpha *} \circ \varphi_j u\right),$$

here $U_\alpha \supset Supp\,\varphi_i \cup Supp\,\varphi_j$, $\widetilde{\sqrt{b_\alpha}}(y, D)$ *is defined by Lemma* 13 *and Definition* 4 *like as Definition* 6. *Finally if* $Supp\,\varphi_j \cap \Gamma \neq \phi$ *and* $Supp\,\varphi_i \cap Supp\,\varphi_j = \phi$, *we define*

$$\tilde{\Lambda}_{\pm i,j} u = 0 .$$

Then we shall show that the definitions of $\sigma(x, D)$ and $\tilde{\Lambda}_\pm$ are invariant with respect to the choice of U_α satisfying Lemma 13. To this end we may prove the following Lemmas 14 and 15, using theorem 7 in [15] (page 247).

Lemma 14. *For* $\sigma(x, \xi) \in \overline{\mathfrak{A}}$

$$\|\sigma(x, D) \Lambda_\pm u - \text{ordinary singular integral operator } \sigma(x, D) \Lambda u\|_{x_n > 0} \leqslant C \|u\|_{x_n > 0}$$

for $u \in D(a^{\frac{1}{2}})$ *with* $Supp\,u \subset \{x \in \boldsymbol{R}_+^n ; x_n \geqslant d > 0\}$.

Lemma 15. *Let* T *be a diffeomorphism from* $(\boldsymbol{R}_x^n, \boldsymbol{R}_{+x}^n)$ *to* $(\boldsymbol{R}_y^n, \boldsymbol{R}_{+y}^n)$ *and satisfy* 3) *of Lemma* 13. *Then for* $\sigma(x, \xi) \in \overline{\mathfrak{A}}$, $\varphi_1(x), \varphi_2(x) \in C_0^\infty(\boldsymbol{R}_x^n)$

$$\|T_* \circ \left(\varphi_2 \sigma(x, D) \Lambda_\pm \varphi_1 u\right) - (T_* \circ \varphi_2)\left(\tau(y, D) \Lambda_\pm T_* \circ \varphi_1 u\right)\|_{y_n > 0} \leqslant C \|u\|_{x_n > 0} \text{ for}$$

$u \in D(a^{\frac{1}{2}})$, *here* $\tau(y, \eta) = \dfrac{\tau_0(T^{-1}(y), (dT^*)(\eta))}{|\eta|} \in \overline{\mathfrak{A}}$ *and* $\tau_0(x, \xi) = \sigma(x, \xi)|\xi|$ (*see the remark next to Definition* 4).

Proof of Lemma 14. We see that

$\sigma(x, D) \Lambda_\pm u = F' \sigma_1(x, \xi) |\xi| F \tilde{u}|_{x_n > 0}$, where $\sigma_1(x, \xi)$ is the extension of $\sigma(x, \xi)$

such that $\sigma_1(x, \xi)\in \Xi_4^\infty(\boldsymbol{R}^n\times(\boldsymbol{R}^n-\{0\}))$.

Set
$$u_1(x)=\begin{cases}\tilde{u}(x) & \text{for} \quad x_n\geqslant d\,,\\ 0 & \text{for} \quad x_n<d\,,\end{cases}$$

and
$$u_2(x)=\tilde{u}(x)-u_1(x)\,,$$

then
$$F'\sigma(x, \xi)|\xi|F\tilde{u}=F'\sigma(x, \xi)|\xi|Fu_1+F'\sigma(x, \xi)|\xi|Fu_2.$$

The first term on the right is the ordinary singular integral operator $\sigma(x, D)\Lambda u$, and the symbol of the second term on the right may be considered as zero by virtue of Supp $u_2\subset\{x\in\boldsymbol{R}^n\,;\,x_n\leqslant -d\}$, from which and the theory of the ordinary singular integrals we obtain the desired estimate. q. e. d.

Proof of Lemma 15. When $\sigma(x,\xi)\in\mathfrak{A}$ and $\sigma(x, \xi', \xi_n)=\sigma(x, \xi', -\xi_n)$, we see that

$$\varphi_2(x)\sigma(x, D)\varphi_1 u=F'\Big(\widetilde{\varphi_2(x)}\tilde{\sigma}(x, D)F\widetilde{(\varphi_1 u)}\Big)\Big|_{x_n>0}\,,$$

where $\tilde{\sigma}(x, \xi)$ is an arbitrary extension of $\sigma(x, \xi)$ such that $\tilde{\sigma}(x, \xi)\in C^{4,\infty}_{x,\xi}(\boldsymbol{R}^n\times(\boldsymbol{R}^n-\{0\}))$ and $\tilde{\sigma}(x, \xi', \xi_n)=\tilde{\sigma}(x, \xi', -\xi_n)$. Furthermore $\tilde{\sigma}(x, D)$ is the ordinary singular integral operator.

From this and 3) of Lemma 13 it follows that

$$\Big\|T_*\circ\Big(\varphi_2\cdot\tilde{\sigma}(x, D)\Lambda\,\widetilde{\varphi_1 u}\Big)-(T_*\circ\varphi_2)\Big(\textit{ordinary singular integral operator}\ \tau(y, D_y)\Big)\Lambda(T_*\circ\widetilde{\varphi_1 u})\Big\|_{L^2(\boldsymbol{R}^n_y)}\leqslant C\|\tilde{u}\|_{L^2(\boldsymbol{R}^n_x)}$$ by virtue of the theory of Seeley.

Hence

$$\Big\|T_*\circ\Big(\varphi_2\cdot\sigma(x, D)\Lambda_\pm\varphi_1 u\Big)-(T_*\circ\varphi_2)\Big(\tau(y, D_y)\Lambda_\pm\,T_*\circ\varphi_1 u\Big)\Big\|_{y_n>0}\leqslant C\|\tilde{u}\|_{L^2(\boldsymbol{R}^n_x)}$$
$$\leqslant C\|u\|_{x_n>0} \qquad \text{for} \quad u\in D(a^{\frac{1}{2}})\,.$$

For $f(x)\dfrac{\xi_n}{|\xi|}\in\mathfrak{A}$, we see that the proposition holds by virtue of 3) of Lemma 13.

For $\sigma(x, \xi)=\sigma_1(x, \xi)f(x)\dfrac{\xi_n}{|\xi|}\in\mathfrak{A}$, we set $f(x)\eta(D)\Lambda_\pm u=v$. Then, as above, we see that

$$\Big\|T_*\circ\Big(\varphi_2\cdot\sigma_1(x, D)\varphi_1 v\Big)-(T_*\circ\varphi_2)\cdot\sigma_1\Big(T^{-1}(y), (dT^*)(D_y)\Big)(T_*\circ\varphi_1)v\Big\|_{y_n>0}$$
$$\leqslant C\|\tilde{v}\|_{-1, L^2(\boldsymbol{R}^n)} \qquad \text{for} \quad v\in L^2(\boldsymbol{R}^n_+)\,.$$

By virtue of $v=f(x)\frac{1}{\sqrt{-1}}\frac{\partial\tilde{u}}{\partial x_n}\Big|_{x_n>0}\in L^2(\boldsymbol{R}^n_+)$ and from Theorem 2, we obtain

$$\left\|T_*\circ\left(\varphi_2\cdot\sigma(x,D)\Lambda_\pm\varphi_1u\right)-(T_*\circ\varphi_2)\cdot\left(\tau(y,D)\Lambda_\pm(T_*\circ\varphi_1u)\right)\right\|_{y_n>0}\leqslant C(\|\tilde{v}\|_{-1,L^2(\boldsymbol{R}^n)}$$
$$+\|\tilde{u}\|_{L^2(\boldsymbol{R}^n)})=C\left(\left\|f(x)\frac{\partial\tilde{u}}{\partial x_n}\right\|_{-1,L^2(\boldsymbol{R}^n)}+\|\tilde{u}\|_{L^2(\boldsymbol{R}^n)}\right)\leqslant C\|\tilde{u}\|_{L^2(\boldsymbol{R}^n)}\leqslant C\|u\|_{x_n>0}.$$

q. e. d.

Furthermore we have the following Theorem and Lemmas similarly to Theorems and Lemmas obtained in §2.1.

Lemma 16. *For sufficiently large* $\lambda>0$

$$\|(\tilde{\Lambda}_\pm+\lambda)u\|^2_{L^2(\Omega)}\geqslant(\delta-\varepsilon)\|u\|^2_{1,L^2(\Omega)}+(\lambda-\rho)^2\|u\|^2_{L^2(\Omega)}$$

for $u\in D(a^{\frac{1}{2}})$, ε *is an arbitrary small positive number,* ρ *depends on* ε *and* n, *and* δ *is a constant determined by means of the ellipticity of* $a(x,D)$.

In the following λ is assumed to be a sufficiently large constant satisfing the above inequality and be fixed.

Proof. Let $u\in D(a)$. From the ellipticity of $a(x,D)$ we see that

$$(17)\qquad \left(a_0(x,D)u,u\right)_{L^2(\Omega)}+k\|u\|^2_{L^2(\Omega)}\geqslant\delta\|u\|^2_{1,L^2(\Omega)}$$

for some constant $k(>0)$.

To obtain the desired inequality we at first show that

$$(18)\qquad \left|\left(a_0(x,D)u,u\right)_{L^2(\Omega)}-\|\tilde{\Lambda}_\pm u\|^2_{L^2(\Omega)}\right|\leqslant\varepsilon\|u\|^2_{1,L^2(\Omega)}+c(\varepsilon)\|u\|_{L^2(\Omega)}.$$

Using the partition of unity described next to Lemma 13 $(a_0(x,D)u,u)_{L^2(\Omega)}$ goes over into

$$(19)\qquad \sum_{j,l}\left(a_0(x,D)\varphi_ju,\varphi_lu\right)_{L^2(\Omega)}+\sum_j\left([\varphi_j,a_0]u,u\right)_{L^2(\Omega)}.$$

The second term is bounded by $\varepsilon\|u\|^2_{1,L^2(\Omega)}+c(\varepsilon)\|u\|^2_{L^2(\Omega)}$ by means of the interpolation. We will estimate the first term.

We may suppose $\operatorname{Supp}\varphi_j\cap\operatorname{Supp}\varphi_l\neq\phi$. If $\operatorname{Supp}\varphi_j\cap\Gamma=\phi$ and $\operatorname{Supp}\varphi_l\cap\Gamma=\phi$, then, since $\varphi_ju\in D(a)$,

$$(20)\qquad \left(a_0(x,D)\varphi_ju,\varphi_lu\right)_{L^2(\Omega)}=(\widetilde{\sqrt{a_0}}\circ\widetilde{\sqrt{a_0}}\Lambda^2\varphi_ju,\varphi_lu)_{L^2(\boldsymbol{R}^n)}$$
$$=(\widetilde{\sqrt{a_0}}\Lambda\varphi_ju,\widetilde{\sqrt{a_0}}\Lambda\varphi_lu)_{L^2(\boldsymbol{R}^n)}+(\widetilde{\sqrt{a_0}}\Lambda\varphi_ju,(\widetilde{\sqrt{a_0}}^*-\widetilde{\sqrt{a_0}}^\#)\Lambda\varphi_lu)_{L^2(\boldsymbol{R}^n)}$$
$$+(\widetilde{\sqrt{a_0}}\circ\widetilde{\sqrt{a_0}}-\widetilde{\sqrt{a_0}}\cdot\widetilde{\sqrt{a_0}})\Lambda\varphi_ju,\Lambda\varphi_lu)_{L^2(\boldsymbol{R}^n)}$$
$$+(\widetilde{\sqrt{a_0}}\circ\widetilde{\sqrt{a_0}}\Lambda-\Lambda\widetilde{\sqrt{a_0}}\circ\widetilde{\sqrt{a_0}})\Lambda\varphi_ju,\varphi_lu)_{L^2(\boldsymbol{R}^n)}.$$

We see that the last three terms are bounded by $\varepsilon\|u\|^2_{1,L^2(\Omega)}+c(\varepsilon)\|u\|^2_{L^2(\Omega)}$ by means of the theory of the ordinary singular integrals and the interpolation.
Furthermore if $\operatorname{Supp}\varphi_j\cap\Gamma\neq\phi$ or $\operatorname{Supp}\varphi_l\cap\Gamma\neq\phi$, then

$$\left(a_0(x,D)\varphi_j u,\varphi_l u\right)_{L^2(\Omega)}=\left(Jb_\alpha(y,D)\Lambda^2_\pm T_{\alpha^*}\circ\varphi_j u,\ T_{\alpha^*}\circ\varphi_l u\right)_{y_n>0}$$

by using the transformation T_α such that $\operatorname{Supp}\varphi_j\cup\operatorname{Supp}\varphi_l\subset U_\alpha$, here J is a Jacobian (>0).
By virtue of the uniform ellipticity of b_α and Theorem 2 it is equal to

$$\begin{aligned}(21)\quad &\left(J\widetilde{\sqrt{b_\alpha}}(y,D)\Lambda_\pm T_{\alpha^*}\circ\varphi_j u,\widetilde{\sqrt{b_\alpha}}(y,D)\Lambda_\pm T_{\alpha^*}\circ\varphi_l u\right)_{y_n>0}\\ &+\left(J\widetilde{\sqrt{b_\alpha}}(y,D)\Lambda_\pm T_{\alpha^*}\circ\varphi_j u,\left(\widetilde{\sqrt{b_\alpha}}(y,D)^*-\widetilde{\sqrt{b_\alpha}}^{\#}(y,D)\right)\Lambda_\pm T_{\alpha^*}\circ\varphi_l u\right)_{y_n>0}\\ &+\left(\left(J(\widetilde{\sqrt{b_\alpha}}\circ\widetilde{\sqrt{b_\alpha}})(y,D)-\widetilde{\sqrt{b_\alpha}}(y,D)J\widetilde{\sqrt{b_\alpha}}(y,D)\right)\Lambda_\pm T_{\alpha^*}\circ\varphi_j u,\Lambda_\pm T_{\alpha^*}\circ\varphi_l u\right)_{y_n>0}\\ &+\left(\left(J(\widetilde{\sqrt{b_\alpha}}\circ\widetilde{\sqrt{b_\alpha}})(y,D)\Lambda_\pm-\Lambda_\pm J(\widetilde{\sqrt{b_\alpha}}\circ\widetilde{\sqrt{b_\alpha}})(y,D)\right)\Lambda_\pm T_{\alpha^*}\circ\varphi_j u,\ T_{\alpha^*}\circ\varphi_l u\right)_{y_n>0},\end{aligned}$$

and the last three terms are bounded by $\varepsilon\|\Lambda_\pm T_{\alpha^*}\circ\varphi_j u\|^2_{y_n>0}+\|T_{\alpha^*}\circ\varphi_l u\|^2_{y_n>0}$.

On the other hand for $u\in D(a^{\frac{1}{2}})$

$$\|\tilde\Lambda_\pm u\|^2_{L^2(\Omega)}=\sum_{i,j,k,l}(\tilde\Lambda_{\pm ij}u,\ \tilde\Lambda_{\pm kl}u)_{L^2(\Omega)}\ .$$

Let $\operatorname{Supp}\varphi_j\cap\Gamma=\phi$ and $\operatorname{Supp}\varphi_l\cap\Gamma=\phi$. Then

$$\begin{aligned}(22)\qquad &\sum_{i,k}(\tilde\Lambda_{\pm ij}u,\ \tilde\Lambda_{\pm kl}u)_{L^2(\Omega)}\\ &=\sum_{i,k}\left(\varphi_i(x)\widetilde{\sqrt{a_0}}(x,D)\Lambda\varphi_j u,\ \varphi_k(x)\widetilde{\sqrt{a_0}}(x,D)\Lambda\varphi_l u\right)_{L^2(\Omega)}\\ &=\left(\widetilde{\sqrt{a_0}}(x,D)\Lambda\varphi_j u,\ \widetilde{\sqrt{a_0}}(x,D)\Lambda\varphi_l u\right)_{L^2(\Omega)}.\end{aligned}$$

Furthermore let $\operatorname{Supp}\varphi_j\cap\Gamma\neq\phi$ and $\operatorname{Supp}\varphi_l\cap\Gamma\neq\phi$. Then

$$\begin{aligned}&(\tilde\Lambda_{\pm ij}u,\ \tilde\Lambda_{\pm kl}u)_{L^2(\Omega)}\\ &=\left(T^*_\alpha\circ\left((T_{\alpha^*}\circ\varphi_i)\widetilde{\sqrt{b_\alpha}}(y,D)\Lambda_\pm T_{\alpha^*}\circ\varphi_j u\right),\ T^*_\beta\circ\right.\\ &\qquad\times\left.\left((T_{\beta^*}\circ\varphi_k)\widetilde{\sqrt{b_\alpha}}(y,D)\Lambda_\pm T_{\beta^*}\circ\varphi_l u\right)\right)_{L^2(\Omega)},\end{aligned}$$

here $\operatorname{Supp}\varphi_i\cap\operatorname{Supp}\varphi_j\neq\phi$, $\operatorname{Supp}\varphi_k\cap\operatorname{Supp}\varphi_l\neq\phi$.
We may assume that $\operatorname{Supp}\varphi_i\cap\operatorname{Supp}\varphi_k\neq\phi$, otherwise this expression is equal to zero. Furthermore we may assume that $\{\operatorname{Supp}\varphi_i\}$ is a star-star finite refinement of $\{U_\alpha\}$ in addition to the properties described before. Then we can choose α, β as the same index. Hence using the coordinate transformation T_α and omitting lower order terms this expression goes over into

$$\left(J(T_{\alpha*}\circ\varphi_i)\sqrt{\widetilde{b_\alpha}}(y, D)\Lambda_\pm T_{\alpha*}\circ\varphi_j u,\ (T_{\alpha*}\circ\varphi_k)\sqrt{\widetilde{b_\alpha}}(y, D)\Lambda_\pm T_{\alpha*}\circ\varphi_l u\right)_{y_n>0}.$$

Adding up this expression with respect to i and k, it is equal to

$$\left(J\sqrt{\widetilde{b_\alpha}}(y, D)\Lambda_\pm T_{\alpha*}\circ\varphi_j u,\ \sqrt{\widetilde{b_\alpha}}(y, D)\Lambda_\pm T_{\alpha*}\circ\varphi_l u\right)_{y_n>0}. \tag{23}$$

By virtue of Lemma 14 and theorem 7 in [15] (page 247) we can also treat the calculation in the case where $\operatorname{Supp}\varphi_j\cap\Gamma=\phi$ or $\operatorname{Supp}\varphi_l\cap\Gamma=\phi$ as above. Hence from (19)—(23) we obtain (18).

We next show that for some $k_1>0$

$$(\tilde{\Lambda}_\pm u, u)_{L^2(\Omega)}+(u, \tilde{\Lambda}_\pm u)_{L^2(\Omega)}\geqslant -k_1\|u\|^2_{L^2(\Omega)}. \tag{24}$$

To prove this let $\psi_i(x)=\left(\varphi_i(x)\right)^{\frac{1}{2}}$. Then we see that $\{\psi_i(x)\}$ is also another partition of unity with respect to $\bar{\Omega}$ such that $\sum_i\left(\psi_i(x)\right)^2=1$, $\psi_i u\in D(a^{\frac{1}{2}})$ for $u\in D(a^{\frac{1}{2}})$, $\psi_i u$ and $\psi_i^2 u\in D(a)$ for $u\in D(a)$ and $\{\operatorname{Supp}\psi_i\}$ is a star-finite refinement of $\{U_\alpha\}$. Then

$$(\tilde{\Lambda}_\pm u, u)_{L^2(\Omega)}=\sum_h(\tilde{\Lambda}_\pm\psi_h u, \psi_h u)_{L^2(\Omega)}+\sum_h\left((\psi_h\tilde{\Lambda}_\pm-\tilde{\Lambda}_\pm\psi_h)u, \psi_h u\right)_{L^2(\Omega)}.$$

We estimate the first term on the right. If $\operatorname{Supp}\psi_h\cap\Gamma\neq\phi$, from Definition 7 we see that

$$\begin{aligned}(\tilde{\Lambda}_\pm\psi_h u, \psi_h u)_{L^2(\Omega)}&=\sum_{i,j}(\tilde{\Lambda}_{\pm ij}\psi_h u, \psi_h u)_{L^2(\Omega)}\\&=\sum_{i,j}\left(T_\alpha^*(T_{\alpha*}\circ\varphi_i)\sqrt{\widetilde{b_\alpha}}(y, D)\Lambda_\pm(T_{\alpha*}\circ\varphi_j\psi_h u), \psi_h u\right)_{L^2(\Omega)}\\&=\sum_{i,j}\left(J(T_{\alpha*}\circ\varphi_i)\sqrt{\widetilde{b_\alpha}}(y, D)\Lambda_\pm(T_{\alpha*}\circ\psi_h\varphi_j u), T_{\alpha*}\circ\psi_h u\right)_{y_n>0}\\&=\left(J\sqrt{\widetilde{b_\alpha}}(y, D)\Lambda_\pm(T_{\alpha*}\circ\psi_h u), T_{\alpha*}\psi_h u\right)_{y_n>0}.\end{aligned}$$

By virtue of (13) it follows from this

$$(\tilde{\Lambda}_{\pm ij}\psi_h u, \psi_h u)_{L^2(\Omega)}+(\psi_h u, \tilde{\Lambda}_{\pm ij}\psi_h u)_{L^2(\Omega)}\geqslant -k_1\|T_{\alpha*}\circ\psi_h u\|^2_{y_n>0}.$$

We can estimate above expression when $\operatorname{Supp}\psi_h\cap\Gamma=\phi$ similarly to the case when $\operatorname{Supp}\psi_h\cap\Gamma\neq\phi$.

Furthermore $\tilde{\Lambda}_\pm\psi_h-\psi_h\tilde{\Lambda}_\pm$ can be extended to be a bounded operator from $L^2(\Omega)$ into itself, which will be proved at the following Theorem 3. Hence we obtained (24).

Therefore from (17), (18) and (24) we have

$$\begin{aligned}
&\|(\tilde{\Lambda}_\pm+\lambda)u\|^2_{L^2(\Omega)} \\
&=\|\tilde{\Lambda}_\pm u\|^2_{L^2(\Omega)}+\lambda\Big((\tilde{\Lambda}_\pm u, u)_{L^2(\Omega)}+(u, \tilde{\Lambda}_\pm u)_{L^2(\Omega)}\Big)+\lambda^2\|u\|^2_{L^2(\Omega)} \\
&=\Big(a_0(x, D)u, u\Big)_{L^2(\Omega)}+\Big(\|\tilde{\Lambda}_\pm u\|^2_{L^2(\Omega)}-\Big(a_0(x, D)u, u\Big)_{L^2(\Omega)}\Big) \\
&\quad+\lambda\Big((\tilde{\Lambda}_\pm u, u)_{L^2(\Omega)}+(u, \tilde{\Lambda}_\pm u)_{L^2(\Omega)}\Big)+\lambda^2\|u\|^2_{L^2(\Omega)} \\
&\geqslant\delta\|u\|^2_{1,L^2(\Omega)}-k\|u\|^2_{L^2(\Omega)}-\varepsilon\|u\|^2_{1,L^2(\Omega)}-c(\varepsilon)\|u\|^2_{L^2(\Omega)}-\lambda k_1\|u\|^2_{L^2(\Omega)}+\lambda^2\|u\|^2_{L^2(\Omega)} \\
&=(\delta-\varepsilon)\|u\|^2_{1,L^2(\Omega)}+\Big(\lambda^2-k-c(\varepsilon)-\lambda k_1\Big)\|u\|^2_{L^2(\Omega)}\,.
\end{aligned}$$

We take ε sufficiently small such that $\delta>\varepsilon(>0)$, next take λ, ρ sufficiently large such that $\rho^2-k-c(\varepsilon)-\rho k_1>0$ and $\lambda>\rho$. Then

$$\|(\tilde{\Lambda}_\pm+\lambda)u\|^2_{L^2(\Omega)}\geqslant(\delta-\varepsilon)\|u\|_{1,L^2(\Omega)}+(\lambda-\rho)^2\|u\|^2_{L^2(\Omega)}\,.$$

For $u\in D(a^{\frac{1}{2}})$ we obtain the estimate by passing to the limit. q. e. d.

From Lemma 16 we may define $D(a^{\frac{1}{2}})$ as the definition demain of the closed operator $\tilde{\Lambda}_\pm$ and denote it by $D(\tilde{\Lambda}_\pm)$.

Lemma 17. *Let* $\alpha(x, \xi)\in\mathfrak{A}_\Omega$ *and* $u\in D(\tilde{\Lambda}_\pm)$. *Then* $\alpha(x, D)u$, $\alpha(x, D)^*u$ $\in D(\tilde{\Lambda}_\pm)$.

Proof. This is shown by virtue of Theorem 2, Definition 5 and Lemma 13. q. e. d.

From Lemmas 14, 15, 17 and Theorem 2 it implies the following theorem by the same consideration used by Seeley (see theorem 2 in [15] (page 262)).

Theorem 3. *For* $\alpha(x, \xi)$, $\beta(x, \xi)\in\mathfrak{A}_\Omega$ *the operators*

$$\alpha(x, D)\tilde{\Lambda}_\pm-\tilde{\Lambda}_\pm\alpha(x, D),\ \alpha(x, D)^*\tilde{\Lambda}_\pm-\tilde{\Lambda}_\pm\alpha(x, D)^*,$$
$$\Big(\alpha(x, D)^*-\alpha^\#(x, D)\Big)\tilde{\Lambda}_\pm,\ \Big((\alpha\circ\beta)(x, D)-\alpha(x, D)\beta(x, D)\Big)\tilde{\Lambda}_\pm$$

are extended to operators of $B(L^2(\Omega), L^2(\Omega))$, *where the symbols of* $\alpha^\#(x, D)$ *and* $(\alpha\circ\beta)(x, D)$ *are* $\overline{\alpha(x, \xi)}$ *and* $\alpha(x, \xi)\beta(x, \xi)$, *respectively.*

Furthermore we obtain the following lemmas needed for the next section.

Lemma 18. *Let* $u\in D(\tilde{\Lambda}_\pm)$. *Then* $u\in D(\tilde{\Lambda}^*_\pm)$ *and*

$$\|(\tilde{\Lambda}_\pm-\tilde{\Lambda}^*_\pm)u\|_{L^2(\Omega)}\leqslant C\|u\|_{L^2(\Omega)}\,.$$

Proof. It is seen from Definition 7, Lemmas 14, 15, 17 and Theorem 2 that the symbols of $\tilde{\Lambda}_{\pm ij}$ and $\tilde{\Lambda}^*_{\pm ij}$ are same. It implies this lemma.

Lemma 19. $\tilde{\Lambda}_\pm+\lambda$ *has a bounded inverse from* $L^2(\Omega)$ *into itself such that*

$$\|(\tilde{\Lambda}_\pm+\lambda)^{-1}u)\|_{L^2(\Omega)}\leqslant(\lambda-\rho)^{-1}\|u\|_{L^2(\Omega)}\,.$$

Proof. We take a partition of unity $\{\Phi_h\}$ with respect to $\bar{\Omega}$ such that $\sum_{\text{finite}} \Phi_h^2 = 1$ on $\bar{\Omega}$, $\Phi_h u, \Phi_h^2 u \in D(a)$ for $u \in D(a)$. Furthermore we assume that if $\operatorname{Supp} \Phi_h \cap \Gamma \neq \phi$ then $(\operatorname{Supp} \Phi_h)^0 \cap \Gamma \neq \phi$, $\{\operatorname{Supp} \Phi_h\}$ is a star-finite refinement of $\{U_\alpha\}$. Let us choose $x_h \in (\operatorname{Supp} \Phi_h)^0$ for every Φ_h and in particular $x_h \in (\operatorname{Supp} \Phi_h)^0 \cap \Gamma$ if $(\operatorname{Supp} \Phi_h) \cap \Gamma \neq \phi$. Then we may assume that if $U_\alpha \supset \operatorname{Supp} \Phi_h$, for any $y \in \overline{\boldsymbol{R}^n_+}$, $\sup_{|\xi|=1} |\sigma_\alpha(y, \xi) - \sigma_\alpha(y_h, \xi)| < \varepsilon_1$ for sufficiently small positive number ε_1, which is determined later. Where $y_h = T_\alpha(x_h)$, $\sigma_\alpha(y, D) = \sqrt{\widetilde{b_\alpha}}(y, D)$ if $\operatorname{Supp} \Phi_h \cap \Gamma \neq \phi$ and setting $x = y$, $\sigma_\alpha(y, D) = \sqrt{\widetilde{a_0}}(x, D)$ if $\operatorname{Supp} \Phi_h \cap \Gamma = \phi$.

Furthermore we take another partition of unity $\{\varphi_j\}$ with respect to $\bar{\Omega}$ described next to Lemma 13. Here we assume that for every Φ_h there exists (at least) one U_α such that $\bigcup_{i,j} \{\operatorname{Supp} \Phi_h \cup \operatorname{Supp} \varphi_i \cup \operatorname{Supp} \varphi_j ;\ \operatorname{Supp} \Phi_h \cap \operatorname{Supp} \varphi_i \neq \phi, \operatorname{Supp} \varphi_i \cap \operatorname{Supp} \varphi_j \neq \phi\} \subset U_\alpha$. Furthermore we may assume that if $\operatorname{Supp} \Phi_h \cap \Gamma = \phi$, then $U_\alpha \cap \Gamma = \phi$, hereafter we write one of such U_α by U_h.

Then we can write $\tilde{\Lambda}_\pm + \lambda$ as

$$(\tilde{\Lambda}_\pm + \lambda) u = \sum_{i,j} T_\alpha^* \circ \Big((T_{\alpha^*} \circ \varphi_i) \Big(\sigma_\alpha(y, D) \Lambda_{\pm\alpha} + \lambda \Big) (T_{\alpha^*} \circ \varphi_j u) \Big)$$

for $u \in D(a^{\frac{1}{2}})$, here $T_\alpha = I$ if $\operatorname{Supp} \varphi_j \cap \Gamma = \phi$, I is an identity operator and $\Lambda_{\pm\alpha} = \Lambda_\pm$.

As y_h is in $\boldsymbol{R}^{n-1}$ when $(\operatorname{Supp} \Phi_h)^0 \cap \Gamma \neq \phi$, $\sigma_h(y_h, \xi)$ is the even function with respect to ξ_n from the construction of $\sigma_h(y, \xi)$. Hence $\sigma_h(y_h, D) \Lambda_{\pm h} + \lambda$ has a bounded inverse from $L^2(\boldsymbol{R}^n_+)$ onto $D(\Lambda_\pm)$.

Then set

$$R \cdot = \sum_h T_h^* \circ \Big((T_{h^*} \circ \Phi_h) \Big(\sigma_h(y_h, D) \Lambda_{\pm h} + \lambda \Big)^{-1} T_{h^*} \circ \Phi_h \cdot \Big) \in B\Big(L^2(\Omega), D(a^{\frac{1}{2}}) \Big)$$

and calculate $(\tilde{\Lambda}_\pm + \lambda) R$.

$$\begin{aligned} (\tilde{\Lambda}_\pm + \lambda) R u = & \sum_{i,j} T_\alpha^* \circ \Big((T_{\alpha^*} \circ \varphi_i) \Big(\sigma_\alpha(y, D) \Lambda_{\pm\alpha} + \lambda \Big) \\ & \times T_{\alpha^*} \circ \varphi_j \Big(\sum_h T_h^* \circ \Big((T_{h^*} \circ \Phi_h) \Big(\sigma_h(y_h, D) \Lambda_{\pm h} + \lambda \Big)^{-1} T_{h^*} \circ \Phi_h u \Big) \Big) \end{aligned}$$

for $u \in L^2(\Omega)$.

We may assume that $\operatorname{Supp} \varphi_i \cap \operatorname{Supp} \varphi_j \neq \phi$, $\operatorname{Supp} \varphi_j \cap \operatorname{Supp} \Phi_h \neq \phi$. Then from the above convention we can choose α, h the same index. Hence

$$\begin{aligned} (\tilde{\Lambda}_\pm + \lambda) R u = & \sum_{i,j} \sum_h T_h^* \circ (T_{h^*} \circ \varphi_i) \Big(\sigma_h(y, D) \Lambda_{\pm h} + \lambda \Big) T_{h^*} \circ \varphi_j \\ & \times \Big((T_h^* \circ \Big(T_{h^*} \circ \Phi_h) \Big(\sigma_h(y_h, D) \Lambda_{\pm h} + \lambda \Big)^{-1} T_{h^*} \circ \Phi_h u \Big) \Big) + B u \\ = & \sum_{i,h} T_h^* \circ \Big((T_{h^*} \circ \varphi_i) \Big(\sigma_h(y, D) \Lambda_{\pm h} + \lambda \Big) (T_{h^*} \circ \Phi_h) \\ & \Big(\sigma_h(y_h, D) \Lambda_{\pm h} + \lambda \Big)^{-1} (T_{h^*} \circ \Phi_h u) \Big) + B u , \end{aligned}$$

here $\|Bu\|_{L^2(\Omega)} \leqslant \dfrac{C}{\lambda-\rho}\|u\|_{L^2(\Omega)}$ because of Lemmas 12, 14 and 15. Furthermore we find that

$$\begin{aligned}
&\left(\sigma_h(y, D)\Lambda_{\pm h}+\lambda\right)(T_{h^*}\circ\Phi_h)\left(\sigma_h(y_h, D)\Lambda_{\pm h}+\lambda\right)^{-1}(T_{h^*}\circ\Phi_h u) \\
&\quad=\left[\left(\sigma_h(y, D)\Lambda_{\pm h}+\lambda\right), (T_{h^*}\circ\Phi_h)\right]\left(\sigma_h(y_h, D)\Lambda_{\pm h}+\lambda\right)^{-1}(T_{h^*}\circ\Phi_h u) \\
&\qquad+(T_{h^*}\circ\Phi_h)\left(\left(\sigma_h(y, D)-\sigma_h(y_h, D)\right)\Lambda_{\pm h}\right)\left(\sigma_h(y_h, D)\Lambda_{\pm h}+\lambda\right)^{-1}(T_{h^*}\circ\Phi_h u) \\
&\qquad+(T_{h^*}\circ\Phi_h)\left(\sigma_h(y_h, D)\Lambda_{\pm h}+\lambda\right)\left(\sigma_h(y_h, D)\Lambda_{\pm h}+\lambda\right)^{-1}(T_{h^*}\circ\Phi_h u).
\end{aligned}$$

From the choosing of Φ_h, $\|(T_{h^*}\circ\Phi_h)(\sigma_h(y, D)-\sigma_h(y_h, D))\Lambda_\pm u\|_{y_n>0} \leqslant (\varepsilon_1+\varepsilon)\times \|\Lambda_\pm u\|_{y_n>0}+c(\varepsilon)\|u\|_{y_n>0}$ for $u\in D(\Lambda_\pm)$ similarly to Lemma 12. Moreover from Lemma 12 we find that

$$\begin{aligned}
&\left\|\left(\sigma_h(y_h, D)\Lambda_{\pm h}+\lambda\right)^{-1}u\right\|_{y_n>0} \leqslant (\lambda-\rho)^{-1}\|u\|_{y_n>0}, \\
&\left\|\Lambda_\pm\left(\sigma_h(y_h, D)\Lambda_{\pm h}+\lambda\right)^{-1}u\right\|_{y_n>0} \leqslant (\delta^{\frac{1}{2}}-\varepsilon')^{-1}\|u\|_{y_n>0}
\end{aligned}$$

for $u\in L^2(\boldsymbol{R}^n_+)$, here δ is a constant determined by means of the ellipticity of $a(x, D)$.

We remark here that in the case where $\operatorname{Supp}\Phi_h\cap\Gamma=\phi$ we use the theory of the ordinary singular integrals.

From this and Theorem 2 we find that

$$\begin{aligned}
&\Big\|\sum_{i,h} T_h^*\circ\left((T_{h^*}\circ\varphi_i)\left(\sigma_h(y, D)\Lambda_{\pm h}+\lambda\right)(T_{h^*}\circ\Phi_h)\left(\sigma_h(y_h, D)\Lambda_{\pm h}+\lambda\right)^{-1}(T_{h^*}\circ\Phi_h u)\right) \\
&\qquad+Bu-u\Big\|_{L^2(\Omega)} \\
&\leqslant \frac{C'}{\lambda-\rho}\|u\|_{L^2(\Omega)}+\frac{\varepsilon_1+\varepsilon}{\delta^{\frac{1}{2}}-\varepsilon'}\|u\|_{L^2(\Omega)}+\frac{c(\varepsilon)}{\lambda-\rho}\|u\|_{L^2(\Omega)}+\frac{C}{\lambda-\rho}\|u\|_{L^2(\Omega)} \\
&=\left(\frac{C'}{\lambda-\rho}+\frac{\varepsilon_1+\varepsilon}{\delta^{\frac{1}{2}}-\varepsilon'}+\frac{c(\varepsilon)}{\lambda-\rho}+\frac{C}{\lambda-\rho}\right)\|u\|_{L^2(\Omega)}.
\end{aligned}$$

We choose ε_1 so small that $\dfrac{\varepsilon_1}{\delta^{\frac{1}{2}}-\varepsilon'}<1$ and also ε small such that $\dfrac{\varepsilon_1+\varepsilon}{\delta^{\frac{1}{2}}-\varepsilon'}<1$, and then λ so large that $\dfrac{C'}{\lambda-\rho}+\dfrac{\varepsilon_1+\varepsilon}{\delta^{\frac{1}{2}}-\varepsilon'}+\dfrac{c(\varepsilon)}{\lambda-\rho}+\dfrac{C}{\lambda-\rho}<1$. Then $(I-(I-(\tilde{\Lambda}_\pm+\lambda)R))^{-1}=((\tilde{\Lambda}_\pm+\lambda)R)^{-1}$ exists and is bounded from $L^2(\Omega)$ into itself.

From $(\tilde{\Lambda}_\pm+\lambda)R\left((\tilde{\Lambda}_\pm+\lambda)R\right)^{-1}=I$, we find that

$$(\tilde{\Lambda}_\pm+\lambda)^{-1}=R\left((\tilde{\Lambda}_\pm+\lambda)R\right)^{-1}.$$

Moreover from Lemma 16 we find that

$$\|(\tilde{\Lambda}_{\pm}+\lambda)^{-1}u\|_{L^2(\Omega)} \leqslant (\lambda-\rho)^{-1}\|u\|_{L^2(\Omega)}. \qquad \text{q. e. d.}$$

Lemma 20. *We find that* $u \in D(a)$ *if and only if* $u \in D(\tilde{\Lambda}_{\pm}^{*}\tilde{\Lambda}_{\pm})$.

Proof. Let $u \in D(a)$. We set $a_1(x, D) = -\sum_{ij} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + k_1$, where k_1 is a positive constant, and a_1 is assumed a positive definite selfadjoint operator with the domain $D(a_1) = D(a)$.

Hence using the same consideration in Lemma 16 for $\varphi \in D(a^{\frac{1}{2}})$

$$(25) \qquad \left(a_1(x, D)u, \varphi\right)_{L^2(\Omega)} = (\tilde{\Lambda}_{\pm}u, \tilde{\Lambda}_{\pm}\varphi)_{L^2(\Omega)} + B_1(u, \varphi)_{L^2(\Omega)},$$

where

$$|B_1(u, \varphi)_{L^2(\Omega)}| \leqslant C_1\|u\|_{1, L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}.$$

Then

$$|(\tilde{\Lambda}_{\pm}u, \tilde{\Lambda}_{\pm}\varphi)_{L^2(\Omega)}| \leqslant \left(\|a_1 u\|_{L^2(\Omega)} + C_1\|u\|_{1, L^2(\Omega)}\right)\|\varphi\|_{L^2(\Omega)}.$$

Consequently, from the fact that $D(a^{\frac{1}{2}})$ is dense in $L^2(\Omega)$, we find that $\tilde{\Lambda}_{\pm}u \in D(\tilde{\Lambda}_{\pm}^{*})$, that is, $u \in D(\tilde{\Lambda}_{\pm}^{*}\tilde{\Lambda}_{\pm})$.

The fact that if $u \in D(\tilde{\Lambda}_{\pm}^{*}\tilde{\Lambda}_{\pm})$ then $u \in D(a)$ is shown by virtue of the similar method stated above.

Lemma 21. $D(\tilde{\Lambda}_{\pm}) = D(\tilde{\Lambda}_{\pm}^{*})$ and $D(a) = D(\tilde{\Lambda}_{\pm}^{2})$.

Proof. From Lemma 19, we find that

$$(\tilde{\Lambda}_{\pm}^{*}+\lambda)u = \left(I + (\tilde{\Lambda}_{\pm}^{*} - \tilde{\Lambda}_{\pm})(\tilde{\Lambda}_{\pm}+\lambda)^{-1}\right)(\tilde{\Lambda}_{\pm}+\lambda)u$$

for $u \in D(\tilde{\Lambda}_{\pm})$.

Moreover from Lemma 18 and 19 we find that

$$\left\|(\tilde{\Lambda}_{\pm}^{*} - \tilde{\Lambda}_{\pm})(\tilde{\Lambda}_{\pm}+\lambda)^{-1}\right\|_{L^2(\Omega)\to L^2(\Omega)} \leqslant \frac{C}{\lambda-\rho} < 1$$

for large λ.

From this it follows that $(I + (\tilde{\Lambda}_{\pm}^{*} - \tilde{\Lambda}_{\pm})(\tilde{\Lambda}_{\pm}+\lambda)^{-1})(\tilde{\Lambda}_{\pm}+\lambda)$ has a bounded inverse from $L^2(\Omega)$ onto $D(\tilde{\Lambda}_{\pm})$, from which it imples this lemma because $D(\tilde{\Lambda}_{\pm}) \subset D(\tilde{\Lambda}_{\pm}^{*})$ and Lemma 19. q. e. d.

Hereafter we set $\tilde{a}(x, D) = a(x, D) + \lambda$, $\tilde{\Lambda}_{\pm,\lambda} = \tilde{\Lambda}_{\pm} + \lambda$.

Corollary. *We find that for* $u \in D(a)$

$$\left\|\left(a(x, D) - \tilde{\Lambda}_{\pm}^{2}\right)u\right\|_{L^2(\Omega)} \leqslant C\|\tilde{\Lambda}_{\pm,\lambda}u\|_{L^2(\Omega)}.$$

Proof. From (25), Lemma 16 and 18 it implies this corollary.

Lemma 22. *Let* $\sigma(x,\xi)\in\mathfrak{A}_{\Omega}$ *with* $\sigma(x,\xi)\geqslant d>0$ *for* $x\in\bar{\Omega}$ *and* $|\xi|=1$. *Then for sufficiently large* $k(>0)$

$$\left\|\left((\sigma(x,D)\tilde{\Lambda}_{\pm}+k\right)u\right\|^2_{L^2(\Omega)}\geqslant(d-\varepsilon)^2\|\tilde{\Lambda}_{\pm,\lambda}u\|^2_{L^2(\Omega)}+(k-\rho)^2\|u\|^2_{L^2(\Omega)}$$

for $u\in D(\tilde{\Lambda}_{\pm})$, *here* ε *is an arbitrary small positive number*, $k>\lambda$, $k>\rho$ *and* ρ *depends on* n, σ *and* ε.

Proof. We can prove this lemma like as Lemma 12 by virtue of Lemma 16 and Theorem 3.

Lemma 23. *Let* $\sigma(x,\xi)$ *be as in Lemma* 22. *Then for sufficiently large* $k>0$ $\sigma(x,D)\tilde{\Lambda}_{\pm}+k$ *has a bounded inverse from* $L^2(\Omega)$ *into itself such that*

$$\left\|\left(\sigma(x,D)\tilde{\Lambda}_{\pm}+k\right)^{-1}u\right\|_{L^2(\Omega)}\leqslant(k-\rho)^{-1}\|u\|_{L^2(\Omega)}\qquad \textit{for}\ \ u\in L^2(\Omega).$$

Proof. $\sigma(x,\xi)\sqrt{a_0(x,\xi)}>d\sqrt{\delta}>0$ for $x\in\bar{\Omega}$ and $|\xi|=1$, which implies this lemma from Lemmas 14, 15, 22 and Theorem 3 similarly to Lemma 19.

q. e. d.

3. Reductions to the theory of semi group

We may consider $\rho(x)=0$ on Γ in the Neumann case. For, we set

$$\psi(x)=\varphi(x)\frac{|\mathrm{grad}\,\varphi(x)|}{\sum_{i,j}a^{(0)}_{ij}(x)\dfrac{\partial\varphi}{\partial x_i}\dfrac{\partial\varphi}{\partial x_j}}\tilde{\rho}(x)\quad\text{on}\quad U(\Omega),$$

where $\tilde{\rho}(x)$ is an extended function of $\rho(x)$ defined on $U(\Omega)$ such that $\tilde{\rho}(x)\in C^\infty(U(\Omega))$ and $\tilde{\rho}(x)=0$ on $\Omega_{\varepsilon_1}=\{x\in\Omega\,;\,\mathrm{dist}\,(\Gamma,x)\geqslant\varepsilon_1\}$, ε_1 is an arbitrary small but a fixed number. We furthermore set

$$u(x)=e^{-\psi(x)}v(x).$$

Then we see that

$$\left(\frac{\partial}{\partial n}+\rho(x)\right)u\Big|_{\Gamma}=\frac{\partial}{\partial n}v\Big|_{\Gamma}\qquad\text{and}\qquad\left(\frac{\partial}{\partial n}+\rho(x)\right)a^l u\Big|_{\Gamma}=\frac{\partial}{\partial n}a'^l v\Big|_{\Gamma},$$

where $a'=e^{\psi(x)}ae^{-\psi(x)}$, $l=1,2,\cdots$.

Here we remark that the principal part of $a'(x,D)$ is the same as one of $a(x,D)$. Hereafter we assume that $\rho(x)=0$ and set $\tilde{a}(x,D)=a(x,D)+\lambda$ for a sufficiently large, but fixed constant λ.

We state the following lemma to reduce the equation (1′) to the system.

Lemma 24. *If* $a_k(x,D)$ *satisfies the assumption stated in* §1, *then we find that*

$$a_{2k}(x, D)u = \alpha_{2k}(x, D)\tilde{a}^k u + \beta_{2k} u\,,$$

$$a_{2k+1}(x, D)u = \alpha_{2k+1}(x, D)\tilde{\Lambda}_{\pm,\lambda}\tilde{a}^k u + \beta_{2k+1} u$$

for $u \in D(a^k)$ *or* $u \in D(a^{k+\frac{1}{2}})$ $(k \geqslant 0)$, respectively, where $\alpha_k(x, \xi) = \dfrac{a_k(x, \xi)}{(a_0(x, \xi))^{\frac{k}{2}}} \in \bar{\mathfrak{A}}_\Omega$, β_k is a linear operator such that $\beta_{2k}\tilde{a}^{-(k-1)}\tilde{\Lambda}^{-1}_{\pm,\lambda}$ *and* $\beta_{2k+1}\tilde{a}^{-k} \in B(L^2(\Omega), L^2(\Omega))$.

Proof. From Lemma 15 the statement that $\alpha_k(x, \xi) = \dfrac{a_k(x, \xi)}{(a_0(x, \xi))^{\frac{k}{2}}} \in \bar{\mathfrak{A}}_\Omega$ is shown from corollary in [15] (page 244) and the first two expressions in this lemma. Here we remark that $\left(a_0\left(x, \dfrac{\xi}{|\xi|}\right)\right)^{-\frac{1}{2}} \in \bar{\mathfrak{A}}_\Omega$.

Hence we will show the first two expressions.

We at first consider for $a_{2k}(x, D)$. Let $u \in D(a^k)$. Using the partition of unity described next to Lemma 13, we have

$$a_{2k}(x, D)u = \sum_j a_{2k}(x, D)\varphi_j u + \sum_j [\varphi_j, a_{2k}]u$$

and that $\sum_j [\varphi_j, a_{2k}]$ is a differential operator of order at most $(2k-1)$ and $\sum_j [\varphi_j, a_{2k}]\tilde{a}^{-(k-1)}\tilde{\Lambda}^{-1}_{\pm,\lambda} \in B(L^2(\Omega), L^2(\Omega))$.

If $\mathrm{Supp}\,\varphi_j \cap \Gamma = \phi$, then $\varphi_j u \in D(a^k)$ and

$$a_{2k}(x, D)\varphi_j u = \alpha'_{2k}(x, D)a^k\varphi_j u + \beta'_{2k} u = \alpha'_{2k}(x, D)\varphi_j\tilde{a}^k u + \beta''_{2k} u\,,$$

where $\alpha'_{2k}(x, \xi) = \dfrac{a_{2k}(x, \xi)}{(a_0(x, \xi))_k}\varphi_j(x) \in \bar{\mathfrak{A}}_\Omega$ and $\beta''_{2k}\tilde{a}^{-(k-1)}\tilde{\Lambda}^{-1}_{\pm,\lambda} \in B\big(L^2(\Omega), L^2(\Omega)\big)$.

Let $\mathrm{Supp}\,\varphi_j \cap \Gamma \neq \phi$. Using coordinate transformation T_α such that $U_\alpha \supset \mathrm{Supp}\,\varphi_j$ we replace $\dfrac{\partial^2}{\partial y_n^2}$ in the principal part of $a_{2k}\big(T_\alpha^{-1}(y), (dT_\alpha^*)(D_y)\big)$ by $\dfrac{\partial^2}{\partial y_n^2} = b_{nn}^{(\alpha)-1}\left(a\big(T_\alpha^{-1}(y), (dT_\alpha^*)(D_y)\big) - \sum_{i,j}^{n-1} b_{ij}^{(\alpha)}\dfrac{\partial^2}{\partial y_i \partial y_j} - \sum_{i=1}^{n-1} b_{in}^{(\alpha)}\dfrac{\partial^2}{\partial y_i \partial y_n}\right) +$ (*lower order terms*).

Then from the assumption with respect to a_{2k} stated in §1,

$$\begin{aligned} &a_{2k}\big(T_\alpha^{-1}(y), (dT_\alpha^*)(D_y)\big)T_{\alpha^*}\circ\varphi_j u \\ &\quad = \sum_{|\eta|+|\gamma|+h=k} f_{\eta\gamma h}(y)X_{i_1j_1}^{\eta_1}\cdots X_{i_pj_p}^{\eta_p}Y_{l_1}^{\gamma_1}\cdots Y_{l_q}^{\gamma_q}\Big(a\big(T_\alpha^{-1}(y), (dT_\alpha^*)(D_y)\big)\Big)^h T_{\alpha^*}\circ\varphi_j u \\ &\qquad + (\textit{lower order terms})\,u\,, \end{aligned}$$

where $X_{ij} = D_iD_j$, $Y_l = y_nc_l(y)D_lD_n$ $(i, j, l \neq n)$, $\eta = (\eta_1, \cdots, \eta_p)$, $\gamma = (\gamma_1, \cdots, \gamma_q)$, $|\eta| = \eta_1 + \cdots + \eta_p$ and $|\gamma| = \gamma_1 + \cdots + \gamma_q$.

From this we see that

$a_{2k}(x, D)\varphi_j u = \sum A_{\eta\gamma h}(x, D)\left(a(x, D)\right)^h \varphi_j u + (\textit{lower order terms})u$ using inverse coordinate transformation T_α^{-1}, where $A_{\eta\gamma h}(x, D) = T_\alpha^{-1}\left(f_{\eta\gamma h}(x)X_{i_1j_1}^{\eta_1}\cdots X_{i_pj_p}^{\eta_p}Y_{i_1}^{\gamma_1}\cdots Y_{i_q}^{\gamma_q}\right)$. Furthermore using $\varphi_j u\in D(a)$ for $u\in D(a)$, $a\varphi_j u = \varphi_j au +$ (*lower order terms*) u.

Hence

$a_{2k}(x, D)\varphi_j u = \sum A_{\eta\gamma h}(x, D)\left(a(x, D)\right)^{h-1}\varphi_j au + (\textit{lower order terms})u$. Repeating this method for $a_{2k}(x, D)\varphi_j u$, we find that

$a_{2k}(x, D)\varphi_j u = \sum A_{\eta\gamma h}(x, D)\varphi_j a^h u + (\textit{lower order terms})\,u$ for $u\in D(a^k)$.

Using coordinate transformation T_α again, this goes over into

$$\begin{aligned}&a_{2k}\left(T_\alpha^{-1}(y), (dT_\alpha^*)(D_y)\right)T_{\alpha^*}\circ\varphi_j u\\ &= \sum f_{\eta\gamma h}(y)X_{i_1j_1}^{\eta_1}\cdots X_{i_pj_p}^{\eta_p}Y_{i_1}^{\gamma_1}\cdots Y_{i_q}^{\gamma_q}T_{\alpha^*}\circ\varphi_j(a^h u) + (\textit{lower order terms})\,u.\end{aligned}$$

Next we consider for $a_{2k+1}(x, D)$. Let $u\in D(a^{k+\frac{1}{2}})$. Similarly to the case for $a_{2k}(x, D)$ we can show that after using the partition of unity

$$\sum_j [\varphi_j, a_{2k+1}]\tilde{a}^{-k}\in B\left(L^2(\Omega), L^2(\Omega)\right),$$

$a_{2k+1}(x, D)\varphi_j u = \alpha'_{2k+1}(x, D)\varphi_j\tilde{\Lambda}_{\pm,\lambda}\tilde{a}^k u + \beta''_{2k+1}u$ if $\operatorname{Supp}\varphi_j\cap\Gamma = \phi$, where $\alpha'_{2k+1}(x, \xi) = \dfrac{a_{2k+1}(x, \xi)}{(a_0(x, \xi))^{k+\frac{1}{2}}}\varphi_j(x)\in\mathfrak{A}_\Omega$ and $\beta''_{2k+1}\tilde{a}^{-k}\in B\left(L^2(\Omega), (L^2(\Omega)\right)$, and if $\operatorname{Supp}\varphi_j\cap\Gamma\neq\phi$,

$$\begin{aligned}&a_{2k+1}\left(T_\alpha^{-1}(y), (dT_\alpha^*)(D_y)\right)T_{\alpha^*}\circ\varphi_j u\\ &= \sum_{|\eta|+|\gamma|+h=k} f_{\eta\gamma h}^1\left(y, \frac{\partial}{\partial y}\right)X_{i_1j_1}^{\eta_1}\cdots X_{i_pj_p}^{\eta_p}Y_{i_1}^{\gamma_1}\cdots Y_{i_q}^{\gamma_q}T_{\alpha^*}\circ\varphi_j(a^h u)\\ &\quad + (\textit{lower order terms})\,u,\end{aligned}$$

where $f_{\eta\gamma h}^1\left(y, \dfrac{\partial}{\partial y}\right)$ is a differential operator of order one. Furthermore from the assumption with respect to the coefficients stated in §1, the coefficients of $\dfrac{\partial}{\partial y_n}$ are zero at $y_n=0$. Hence we may prove only the following

Lemma 25. *Let X_{ij} and Y_i be as described before. Then every term* $\sum_{|\eta|+|\gamma|=k} X_{i_1j_1}^{\eta_1}\cdots X_{i_pj_p}^{\eta_p}Y_{i_1}^{\gamma_1}\cdots Y_{i_q}^{\gamma_q}T_{\alpha^*}\circ\varphi_j u$ *(Supp $\varphi_j\subset U_\alpha$) is represented as $\alpha(x, D)a^k u + \beta_k u$ for $u\in D(a^k)$ considering it in Ω, where $\alpha(x, \xi)\in\mathfrak{A}_\Omega$ and* $\beta_k\tilde{a}^{-(k-1)}\tilde{\Lambda}_{\pm,\lambda}^{-1}\in$

$B(L^2(\Omega), L^2(\Omega))$.

Proof. We prove this lemma by the methods of induction with respect to k.

At first we consider the case $k=1$. Let $u\in D(a)$. Then $u\in D(X_{ij})$ and $u\in D(Y_i)$ in $\boldsymbol{R}_y^n$, where $D(X_{ij})=D(Y_i)=H^2(\boldsymbol{R}_+^n)$, X_{ij} and Y_i are being considered as only differential operators.

We see that

$$\xi_i\xi_j=\alpha_{ij}(y, \xi)a_0\left(y, \frac{\xi}{|\xi|}\right)|\xi|^2,\ y_n c_{in}(y)\xi_i\xi_n=\alpha_i(y, \xi)a_0\left(y, \frac{\xi}{|\xi|}\right)|\xi|^2, \text{ and } \alpha_j(y, \xi),$$

$\alpha_i(y, \xi)\in\mathfrak{A}$.

Hence from Theorem 2, Lemma 21 and definition of $\eta(D)$ we find that for $u\in D(a)=D(a_0)$

$$\begin{aligned} T_\alpha^*\circ(X_{ij}T_{\alpha^*}\circ\varphi_l u) &= T_\alpha^*\left(\left(\sigma_{ij}(y, \xi)a_0\left(y, \frac{\xi}{|\xi|}\right)\right)\Big|_{\xi=D} \Lambda_\pm^2 T_{\alpha^*}\circ\varphi_l u\right) \\ &= T_\alpha^*\circ\left(\sigma_{ij}(y, D)a\left(y, \frac{\xi}{|\xi|}\right)\Big|_{\xi=D} \Lambda_\pm^2 T_{\alpha^*}\circ\varphi_l u\right)+\beta'_{ij}u \\ &= \sigma_{ij}(x, D)a\varphi_l u+\beta''_{ij}u=\sigma_{ij}(x, D)\varphi_l au+\beta_{ij}u \end{aligned}$$

and $T_\alpha^*(Y_iT_{\alpha^*}\circ\varphi_l u)=\sigma_i(x, D)\varphi_l au+\beta_i u$, where $\beta_{ij}\tilde{\Lambda}_{\pm,\lambda}^{-1}$ and $\beta_i\tilde{\Lambda}_{\pm,\lambda}^{-1}\in B\left((L^2(\Omega), L^2(\Omega)\right)$.

Next we will show that our assertion holds for the case k supposing that it holds for the case $(k-1)$. To this end it suffices to prove this for the one of k-times product of X_{ij} and Y_i. For example we consider $X^{k-1}Y\equiv X_{i_1j_1}\cdots X_{i_{k-1}j_{k-1}}Y_{i_k}$. Furthermore we set $X^{k-1}Y=X_{i_1j_1}X^{k-2}Y$. By virtue of the case $k=1$ we find that

$$\begin{aligned} T_\alpha^*\circ(X^{k-1}YT_{\alpha^*}\circ\varphi_l u) &= T_\alpha^*\circ(X_{i_1j_1}X^{k-2}YT_{\alpha^*}\circ\varphi_l u) \\ &= \alpha_{i_1j_1}(x, D)aT_\alpha^*\circ(X^{k-2}Y)\varphi_l u+\beta_{i_1j_1}u, \end{aligned}$$

where $\beta_{i_1j_1}\tilde{a}^{-(k-1)}\tilde{\Lambda}_{\pm,\lambda}^{-1}\in B\left(L^2(\Omega), L^2(\Omega)\right)$.

Furthermore

$$aT_\alpha^*\circ(X^{k-2}Y)\varphi_l u=T_\alpha^*\circ(X^{k-2}Y)\varphi_l au+\left[a, T_\alpha^*\circ(X^{k-2}Y)\varphi_l\right]u.$$

From $u\in D(a^k)$, we see that $au\in D(a^{k-1})$. Hence by methods of induction

$$\begin{aligned} T_\alpha^*\circ(X^{k-2}Y)\varphi_l au &= \alpha_{k-1}(x, D)\varphi_l a^{k-1}au+\beta_{k-1}au \\ &= \alpha_{k-1}(x, D)\varphi_l a^k u+\beta_{k-1}au\,. \end{aligned}$$

Since $[a, T_\alpha^*\circ(X^{k-2}Y)\varphi_l]$ is a differential operator of order at most $(2m-1)$, we can apply it to $u(\in D(a^k))$, and $[a, T_\alpha^* \circ (X^{k-2}Y)\varphi_l]\tilde{a}^{-(k-1)}\tilde{\Lambda}_{\pm,\lambda}^{-1} \in B(L^2(\Omega), L^2(\Omega))$, which complete the proof of this lemma. q. e. d.

By virtue of Lemma 24, the equation $(1')$ is rewritten as

$$Lu = \left(\frac{\partial^{2m}}{\partial t^{2m}} + \alpha_1(x, D)\tilde{\Lambda}_{\pm,\lambda}\frac{\partial^{2m-1}}{\partial t^{2m-1}} + \alpha_2(x, D)\tilde{a}\frac{\partial^{2m-2}}{\partial t^{2m-2}} + \alpha_{2m-1}(x, D)\tilde{\Lambda}_{\pm,\lambda}\tilde{a}^{m-1}\frac{\partial}{\partial t} + \alpha_{2m}(x, D)\tilde{a}^m\right)u + \left(\beta_1\frac{\partial^{2m-1}}{\partial t^{2m-1}} + \cdots + \beta_{2m}\right)u = f \tag{26}$$

for $\left(u, \frac{\partial u}{\partial t}, \cdots, \frac{\partial^{2m-1}u}{\partial t^{2m-1}}\right) \in D(a^m) \times D(a^{m-\frac{1}{2}}) \times \cdots \times D(a^{\frac{1}{2}})$, where β_j is a linear operator such that $\beta_{2i}\tilde{a}^{-(i-1)}\tilde{\Lambda}_{\pm,\lambda}^{-1}$, $\beta_{2i-1}\tilde{a}^{-(i-1)} \in B\left(L^2(\Omega), L^2(\Omega)\right)$.

Hence setting

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 0 & 1 \\ -\alpha_{2m}\tilde{a}^m & -\alpha_{2m-1}\tilde{\Lambda}_{\pm,\lambda}\tilde{a}^{m-1} & \cdot\ \cdot\ \cdot & & -\alpha_2\tilde{a} & -\alpha_1\tilde{\Lambda}_{\pm,\lambda} \end{pmatrix},$$

$$\boldsymbol{Q} = \begin{pmatrix} & & 0 & & \\ & & & & \\ -\beta_{2m} & -\beta_{2m-1} & \cdot\ \cdot\ \cdot & & -\beta_1 \end{pmatrix},$$

$$\boldsymbol{U} = {}^t\left(u, \frac{\partial u}{\partial t}, \cdots, \frac{\partial^{2m-1}u}{\partial t^{2m-1}}\right), \quad \boldsymbol{F} = {}^t(0, \cdots, 0, f),$$

we can rewrite (26) as

$$\boldsymbol{E}\frac{d\boldsymbol{U}}{dt} = (\boldsymbol{A} + \boldsymbol{Q})\boldsymbol{U} + \boldsymbol{F}, \qquad \boldsymbol{E}: \text{ unit matrix.} \tag{27}$$

From now on we consider only Neumann problem as we can consider Dirichlet problem similar to Neumann one.

Here $D(a) = \left\{u \in H^2(\Omega)\,;\ \frac{\partial}{\partial n}u\Big|_\Gamma = 0\right\}$.

Then the basic space is $\mathcal{E}(\tilde{\Lambda}_+) = D(a^{m-\frac{1}{2}}) \times D(a^{m-1}) \times \cdots \times D(a^{\frac{1}{2}}) \times L^2(\Omega)$, and the domain of $\boldsymbol{A}$ is $D(a^m) \times D(a^{m-\frac{1}{2}}) \times \cdots \times D(a^{\frac{1}{2}})$. Furthermore we define the norms: $\|u\|_{D(a^{j+\frac{1}{2}})}=\|\tilde{\Lambda}_{+,\lambda}\tilde{a}^{j}u\|_{L^2(\Omega)}$ for $u\in D(a^{j+\frac{1}{2}})$, $\|u\|_{D(a^j)}=\|\tilde{a}^j u\|_{L^2(\Omega)}$ for $u\in D(a^j)$.

Similarly to Leray's method, for the roots $\tau_j(x,\xi)$ $(j=1,\cdots,2m)$ of the equation $\tau^{2m}+\frac{a_1(x,\xi)}{a_0(x,\xi))^{\frac{1}{2}}}\tau^{2m-1}+\cdots+\frac{a_{2m}(x,\xi)}{(a_0(x,\xi))^m)}=0$, we set $\tau_j(x,\xi)=\sqrt{-1}\times\mu_j(x,\xi)$. From the assumption that $\tau_j(x,\xi)$ are pure imaginary, we see that $\mu_j(x,\xi)$ are real. Now we set

$$\boldsymbol{R}(x,\xi)=\begin{pmatrix} 1 & \cdots\cdots\cdots\cdots & 1 \\ \mu_1(x,\xi) & \cdots\cdots\cdots\cdots & \mu_{2m}(x,\xi) \\ & \cdots\cdots\cdots\cdots & \\ (\mu_1(x,\xi))^{2m-1} & \cdots\cdots & (\mu_{2m}(x,\xi))^{2m-1}\end{pmatrix},\quad \boldsymbol{S}=\boldsymbol{R}\boldsymbol{R}^t,$$

and $\boldsymbol{B}(x,\xi)=\boldsymbol{S}^{-1}\,det\,\boldsymbol{S}$, whose components are polynomials of the coefficients of the above equation.

Here we give an equivalent norm in $\mathcal{E}(\tilde{\Lambda}_+)$ with the aid of $\boldsymbol{B}(x,\xi)$, which will be needed to get the a priori estimate of (27). To this end we state the following Lemmas.

Lemma 26. (Leray)

Each component of $\boldsymbol{B}(x,\xi)$ belongs to $\overline{\mathfrak{A}}_\Omega$ (From now on we write it as $\boldsymbol{B}(x,\xi)\in\overline{\mathfrak{A}}_\Omega$).

Lemma 27. *Let $G(x,\xi)$ be a $(2m,2m)$ matrix such that $G(x,\xi)\in\overline{\mathfrak{A}}_\Omega$ and $G(x,\xi)\gg\delta_1\boldsymbol{E}>0$. Then $(G(x,\xi))^{\frac{1}{2}}\in\overline{\mathfrak{A}}_\Omega$.*

Proof. It is seen from

$$\left(G(x,\xi)\right)^{\frac{1}{2}}=\frac{1}{2\pi i}\int_\gamma \lambda^{\frac{1}{2}}(\lambda\boldsymbol{E}-G)^{-1}d\lambda,$$

where γ is a sufficiently large circle whose radius is larger than the absolute value of all eigenvalues with respect to $G(x,\xi)$ and $M_s(G)$ (see (2)).

Lemma 28. *Let $G(x,\xi)$ be as in Lemma 27. Then for an arbitrary small $\varepsilon(>0)$ there exists a constant $c(\varepsilon)$ such that*

$$Re\left(G(x,D)\tilde{\Lambda}_{+,\lambda}U,\ \tilde{\Lambda}_{+,\lambda}U\right)_{L^2(\Omega)}\geqslant(\delta_1-\varepsilon)\|\tilde{\Lambda}_{+,\lambda}U\|^2_{L^2(\Omega)}-c(\varepsilon)\|U\|^2_{L^2(\Omega)}$$

for all $U\in(D(\tilde{\Lambda}_+))^{2m}$, here $(U,V)_{L^2(\Omega)}=\sum_{i=1}^{2m}(u_i,v_i)_{L^2(\Omega)}$ for $U,V\in(L^2(\Omega))^{2m}$.

Proof. From Lemma 27 we see that $(G(x,\xi))^{\frac{1}{2}}\in\overline{\mathfrak{A}}_\Omega$. And $G^{\frac{1}{2}}(x,D)=G^{\frac{1}{2}\#}(x,D)$, from which we can prove this lemma by virtue of Theorem 3 and interpolation (see the proof of Lemma 12).

Now we will give an equivalent norm in $\mathcal{E}(\tilde{\Lambda}_+)$. We set

$$\boldsymbol{B}_1 = \boldsymbol{E}(i)^{-1}\frac{\boldsymbol{B}+\boldsymbol{B}^*}{2}\boldsymbol{E}(i) + \frac{\beta}{2}(\tilde{\Lambda}_{+,\lambda}^{-1} + \tilde{\Lambda}_{+,\lambda}^{-1*})$$

and set

$$|||U|||^2 = \left(\boldsymbol{B}_1\boldsymbol{E}(\tilde{\Lambda}_+)U,\ \boldsymbol{E}(\tilde{\Lambda}_+)\right)U)_{L^2(\Omega)}$$

for $U \in \mathcal{E}(\tilde{\Lambda}_+)$, where

$$\boldsymbol{E}(\tilde{\Lambda}_+) = \begin{pmatrix} \tilde{\Lambda}_{+,\lambda}\tilde{a}^{m-1} & & & & & \\ & \tilde{a}^{m-1} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot\ \tilde{a}\ \ \tilde{\Lambda}_{+,\lambda} & \\ & & & & & I \end{pmatrix}, \quad \boldsymbol{E}(i) = \begin{pmatrix} i^{2m-1} & & & \\ & i^{2m-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

$i = \sqrt{-1}$, β: a sufficiently large constant.

Lemma 29. $C_1\left\|\boldsymbol{E}(\tilde{\Lambda}_+)U\right\|^2_{L^2(\Omega)} \leqslant |||U|||^2 \leqslant C_2\left\|\boldsymbol{E}(\tilde{\Lambda}_+)U\right\|^2_{L^2(\Omega)}$ *for* $U \in \mathcal{E}(\tilde{\Lambda}_+)$, *here* C_1 *and* C_2 *are independent of* U.

Proof. $\boldsymbol{S}$ is real symmetric and $\boldsymbol{S} \gg \delta_2\boldsymbol{E} > 0$, from which it follows that $\boldsymbol{B}(x, \xi)$ is also real symmetric and $\boldsymbol{B} \gg \delta_3\boldsymbol{E} > 0$.

Then by Lemma 28 for $U \in (D(\tilde{\Lambda}_+))^{2m}$ and for sufficiently small $\varepsilon(>0)$ we obtain

$$\begin{aligned}&(\boldsymbol{B}_1\tilde{\Lambda}_{+,\lambda}U,\ \tilde{\Lambda}_{+,\lambda}U)_{L^2(\Omega)} \\ &\quad = Re\left(\boldsymbol{E}(i)\boldsymbol{B}\boldsymbol{E}(i)\tilde{\Lambda}_{+,\lambda}U,\ \tilde{\Lambda}_{+,\lambda}U\right)_{L^2(\Omega)} + \beta Re(\tilde{\Lambda}_{+,\lambda}^{-1}\tilde{\Lambda}_{+,\lambda}U,\ \tilde{\Lambda}_{+,\lambda}U)_{L^2(\Omega)} \\ &\quad \geqslant (\delta_3 - \varepsilon)\|\tilde{\Lambda}_{+,\lambda}U\|^2_{L^2(\Omega)} - c(\varepsilon)\|U\|^2_{L^2(\Omega)} + \beta\, Re\,(\tilde{\Lambda}_{+,\lambda}U,\ U)_{L^2(\Omega)}\,.\end{aligned}$$

Moreover from (24) it follows that

$$Re\,(\tilde{\Lambda}_{+,\lambda}U,\ U)_{L^2(\Omega)} \geqslant C_1\|U\|^2_{L^2(\Omega)} \quad \text{where} \quad C_1 > 0\,.$$

Therefore for sufficiently large β

$$(\boldsymbol{B}_1\tilde{\Lambda}_{+,\lambda}U,\ \tilde{\Lambda}_{+,\lambda}U)_{L^2(\Omega)} \geqslant (\delta_3 - \varepsilon)\|\tilde{\Lambda}_{+,\lambda}U\|^2_{L^2(\Omega)}\,.$$

Next for $V \in \mathcal{E}(\tilde{\Lambda}_+)$, we see that $\tilde{\Lambda}_{+,\lambda}^{-1}E(\tilde{\Lambda}_+)V \in (D(\tilde{\Lambda}_+))^{2m}$, from which it follows that

$$\left(\boldsymbol{B}_1\boldsymbol{E}(\tilde{\Lambda}_+)V,\ \boldsymbol{E}(\tilde{\Lambda}_+)V\right)_{L^2(\Omega)} \geqslant (\delta_3 - \varepsilon)\left\|\boldsymbol{E}(\tilde{\Lambda}_+)V\right\|^2_{L^2(\Omega)}\,.$$

Moreover it is easily seen that

$$\left(\boldsymbol{B}_1\boldsymbol{E}(\tilde{\Lambda}_+)V,\ \boldsymbol{E}(\tilde{\Lambda}_+)V\right)_{L^2(\Omega)} \leqslant C_2\left\|\boldsymbol{E}(\tilde{\Lambda}_+)V\right\|^2_{L^2(\Omega)}. \qquad \text{q. e. d.}$$

Lemma 30. *$\boldsymbol{B}_1$ is a bounded selfadjoint operator from $(L^2(\Omega))^{2m}$ onto itself and also has a bounded inverse.*

Proof. It is seen from the expression of $\boldsymbol{B}_1$ and Lemma 29.

Lemma 31. (a priori estimate)
For $U \in D(\boldsymbol{A})$ there exists a constant $\tau_0(>0)$ such that

$$\left\|\left(\tau\boldsymbol{E}-(\boldsymbol{A}+\boldsymbol{Q})\right)U\right\| \geqslant (\tau-\tau_0)\|U\| \qquad \textit{for} \quad \tau>\tau_0.$$

Proof. We see that

$$\left(\left((\boldsymbol{A}+\boldsymbol{Q})U,\ U\right)\right) \equiv \left(\boldsymbol{B}_1\boldsymbol{E}(\tilde{\Lambda}_+)(\boldsymbol{A}+\boldsymbol{Q})U,\ \boldsymbol{E}(\tilde{\Lambda}_+)U\right)_{L^2(\Omega)},$$

where

$$\boldsymbol{E}(\tilde{\Lambda}_+)\boldsymbol{A} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & 1 & \\ & & & & 0 & 1 \\ -\alpha_{2m} & -\alpha_{2m-1} & \cdot & \cdot & -\alpha_2 & -\alpha_1 \end{pmatrix} \tilde{\Lambda}_{+,\lambda}\boldsymbol{E}(\tilde{\Lambda}_+) +$$

$$+\begin{pmatrix} 0 & 0 & & & & & \\ & 0 & K\tilde{\Lambda}^{-1}_{+,\lambda} & & & & \\ & & 0 & 0 & & & \\ & & & \ddots & \ddots & & \\ & & & & & K\tilde{\Lambda}^{-1}_{+,\lambda} & \\ & & & & & 0 & 0 \\ -\alpha_{2m}K\tilde{\Lambda}^{-1}_{+,\lambda} & 0 & -\alpha_{2m-2}K\tilde{\Lambda}^{-1}_{+,\lambda} & 0 & \cdots & -\alpha_2K\tilde{\Lambda}^{-1}_{+,\lambda} & 0 \end{pmatrix} \boldsymbol{E}(\tilde{\Lambda}_+)$$

$$\equiv \boldsymbol{P}\tilde{\Lambda}_{+,\lambda}\boldsymbol{E}(\tilde{\Lambda}_+)+\boldsymbol{Q}_1\boldsymbol{E}(\tilde{\Lambda}_+), \quad K=\tilde{a}-\tilde{\Lambda}^2_{+,\lambda}.$$

From Corollary stated above Lemma 22 $\boldsymbol{Q}_1$ is a bounded operator from $\left(L^2(\Omega)\right)^{2m}$ into itself.

Hence we find that

$$\text{(28)} \qquad \left(\left((\boldsymbol{A}+\boldsymbol{Q})U,\ U\right)\right) = \left(\boldsymbol{B}_1\boldsymbol{P}\tilde{\Lambda}_{+,\lambda}\boldsymbol{E}(\tilde{\Lambda}_+)U,\ \boldsymbol{E}(\tilde{\Lambda}_+)U\right)_{L^2(\Omega)}$$
$$+\left(\boldsymbol{B}_1\boldsymbol{Q}_1\boldsymbol{E}(\tilde{\Lambda}_+)U,\ \boldsymbol{E}(\tilde{\Lambda}_+)U\right)_{L^2(\Omega)}+\left(\boldsymbol{B}_1\boldsymbol{Q}_2\boldsymbol{E}(\tilde{\Lambda}_+)U,\ \boldsymbol{E}(\tilde{\Lambda}_+)U\right)_{L^2(\Omega)},$$

where $\quad \boldsymbol{E}(\tilde{\Lambda}_{+})\boldsymbol{Q}=\boldsymbol{Q}_2\boldsymbol{E}(\tilde{\Lambda}_{+})$ and $\quad \boldsymbol{Q}_2\in B\Big(\big(L^2(\Omega)\big)^{2m}, \big(L^2(\Omega)\big)^{2m}\Big)$.

To estimate the first term on the right of (28) we set $\boldsymbol{P}=i\boldsymbol{E}(i)^{-1}\boldsymbol{P}_1\,\boldsymbol{E}(i)$. Then we find that the symbol of the principal part of $\boldsymbol{B}_1\boldsymbol{P}$ is $i\boldsymbol{E}(i)^{-1}\times \boldsymbol{B}(x,\xi)\boldsymbol{P}_1(x,\xi)\boldsymbol{E}(i)$ which is anti symmetric, that is, $Re\,(i\boldsymbol{E}(i)^{-1}\boldsymbol{B}(x,\xi)\times \boldsymbol{P}_1(x,\xi)\boldsymbol{E}(i))=0$ (see [11]),
hence

$$\left|Re\Big(\boldsymbol{B}_1\boldsymbol{P}\tilde{\Lambda}_{+,\lambda}\boldsymbol{E}(\tilde{\Lambda}_{+})U,\,\boldsymbol{E}(\tilde{\Lambda}_{+})U\Big)_{L^2(\Omega)}\right|\leqslant C\Big\|\boldsymbol{E}(\tilde{\Lambda}_{+})U\Big\|^2_{L^2(\Omega)}\leqslant C|\!|\!|U|\!|\!|^2.$$

The other terms on the right of (28) are also bounded by $C|\!|\!|U|\!|\!|^2$. Therefore

$$\left|Re\Big(\big((\boldsymbol{A}+\boldsymbol{Q})U,\,U\big)\Big)\right|\leq C|\!|\!|U|\!|\!|^2. \tag{29}$$

It follows from this that for $U\in D(\boldsymbol{A})$ there exists a constant $\tau_0(>0)$ such that

$$\Big|\!\Big|\!\Big|\Big(\tau E-(\boldsymbol{A}+\boldsymbol{Q})\Big)U\Big|\!\Big|\!\Big|\geqslant(\tau-\tau_0)|\!|\!|U|\!|\!| \qquad \textit{for}\ \ \tau>\tau_0.$$

Theorem 4. (Existence of resolvent)

For sufficiently large constant $\tau(>\tau_0)$, $\tau\boldsymbol{E}-(\boldsymbol{A}+\boldsymbol{Q})$ *has a bounded inverse* $R(\tau,\boldsymbol{A}+\boldsymbol{Q})$ *from* $\mathcal{E}(\tilde{\Lambda}_{+})$ *into itself such that*

$$\Big|\!\Big|\!\Big|R(\tau,\,\boldsymbol{A}+\boldsymbol{Q})\Big|\!\Big|\!\Big|\leqslant\frac{1}{\tau-\tau_0}.$$

Proof. From Lemma 31, we may prove that for any $\boldsymbol{F}\in(L^2(\Omega))^{2m}$ there exists a solution $U\in(D(\tilde{\Lambda}_{+}))^{2m}$ of the equation

$$\Big(\tau\boldsymbol{E}-\boldsymbol{P}\tilde{\Lambda}_{+,\lambda}-(\boldsymbol{Q}_1+\boldsymbol{Q}_2)\Big)U=F.$$

This can be shown by the same consideration in Lemma 19 indicating the operator R. To this end, let x_0 be a fixed point in Γ. Then we may show that $(\tau\boldsymbol{E}-\boldsymbol{P}(T_\alpha^{-1}(y_0),(dT_\alpha^*)(D_y))(\sigma_\alpha(y_0,D)\Lambda_{+}+\lambda)U$ has a similar estimate in Lemma 16, that is, for a sufficiently large τ

$$\begin{aligned}&\Big\|\Big(\tau\boldsymbol{E}-\boldsymbol{P}\Big(T_\alpha^{-1}y_0),\,(dT_\alpha^*)(D_y)\Big(\sigma_\alpha(y_0,D)\Lambda_{+}+\lambda\Big)U\Big\|_{y_n>0}\\&\geqslant C\Big(\|\Lambda_{+}U\|_{y_n>0}+(\tau-\rho)\|U\|_{y_n>0}\Big)\end{aligned}$$

for $U\in(D(\Lambda_{+}))^{2m}$, and also has a bounded inverse, here $x_0\in U_\alpha$, $y_0=T_\alpha(x_0)$.

Set

$$\begin{pmatrix} 1 & \cdots\cdots & 1 \\ \tau_1(T_\alpha^{-1}(y_0), (dT_\alpha^*)(\xi)) & \cdots\cdots & \tau_{2m}(T_\alpha^{-1}(y_0), (dT_\alpha^*)(\xi)) \\ \vdots & \vdots\ \vdots\ \vdots & \vdots \\ (\tau_1(T_\alpha^{-1}(y_0), (dT_\alpha^*)(\xi)))^{2m-1} & \cdots & (\tau_{2m}(T_\alpha^{-1}(y_0), (dT_\alpha^*)(\xi)))^{2m-1} \end{pmatrix}$$

$$= \boldsymbol{N}'(y_0, \xi),\ \boldsymbol{N}(y_0, \xi) = (\boldsymbol{N}'(y_0, \xi))^{-1},$$

$$\begin{pmatrix} \mu_1(T_\alpha^{-1}(y_0), (dT_\alpha^*)(\xi)) & & 0 \\ & \ddots & \\ 0 & & \mu_{2m}(T_\alpha^{-1}(y_0), (dT_\alpha^*)(\xi)) \end{pmatrix} = \boldsymbol{D}(y_0, \xi).$$

Then $\boldsymbol{D}(y_0, \xi) \in \mathfrak{A}$, $\boldsymbol{N}(y_0, \xi)\boldsymbol{P}(y_0, \xi) = i\boldsymbol{D}(y_0, \xi)\boldsymbol{N}(y_0, \xi)$ and from Theorem 2, $\boldsymbol{N}(\tau\boldsymbol{E} - \boldsymbol{P}(\sigma_\alpha\Lambda_+ + \lambda) - (\boldsymbol{Q}_1 + \boldsymbol{Q}_2))\boldsymbol{N}^{-1} = (\tau\boldsymbol{E} - i\boldsymbol{D}(\sigma_\alpha\Lambda_+ + \lambda) - \boldsymbol{Q}_3$, where $\boldsymbol{Q}_3$ is a bounded operator and $\boldsymbol{D}(y_0, D)(\sigma_\alpha(y_0, D)\Lambda_+ + \lambda)$ is a selfadjoint operator on $(L^2(\boldsymbol{R}_+^n))^{2m}$ with its domain $(D(\Lambda_+))^{2m}$. Thus our assertion is proved. q.e.d.

By virtue of Lemma 31 and Theorem 4 we will solve the equation (27) applying the theory of semi group.

For $U(0) = {}^t\Big(u(0, x), \cdots, \dfrac{\partial^{2m-1}u}{\partial t^{2m-1}}(0, x)\Big) \in D(\boldsymbol{A})$ and $\boldsymbol{F}(t) \in C^0\big([0, T];\ \mathcal{E}(\tilde{\Lambda}_+)\big)$, $(\boldsymbol{A} + \boldsymbol{Q})\boldsymbol{F}(t) \in C^0\big([0, T];\ \mathcal{E}(\tilde{\Lambda}_+)\big)$, there exists a unique solution $U(t)$ of the equation (27) and

$$|||U(t)||| \leqslant \exp(\tau_0 t)\Big(|||U(0)||| + \int_0^t \|f(s)\|_{L^2(\Omega)} ds\Big),$$

that is,

$$\text{(30)}\qquad \|u(t, x)\|_{D(a^{m-\frac{1}{2}})} + \Big\|\frac{\partial u}{\partial t}(t, x)\Big\|_{D(a^{m-1})} + \cdots + \Big\|\frac{\partial^{2m-1}u}{\partial t^{2m-1}}(t, x)\Big\|_{L^2(\Omega)}$$
$$\leqslant C\exp(\tau_0 t)\Big(\|u(0, x)\|_{D(a^{m-\frac{1}{2}})} + \Big\|\frac{\partial u}{\partial t}(0, x)\Big\|_{D(a^{m-1})} + \cdots$$
$$+ \Big\|\frac{\partial^{2m-1}u}{\partial t^{2m-1}}(0, x)\Big\|_{L^2(\Omega)} + \int_0^t \|f(s)\|_{L^2(\Omega)} ds\Big).$$

Moreover for $f(t) \in C^1([0, T]; L^2(\Omega))$, we choose $f_j(t)$ such that $f_j(t) \in C^0([0, T];\ D(\tilde{\Lambda}_+)) \cap C^1([0, T];\ L^2(\Omega))$ and passing to the limit we find the existence of solution $u(t)$. Furthermore using the equation itself we find that

$$(31)\qquad \left\|u(t,x)\right\|_{D(a^m)}+\left\|\frac{\partial u}{\partial t}(t,x)\right\|_{D(a^{m-\frac{1}{2}})}+\cdots+\left\|\frac{\partial^{2m}u}{\partial t^{2m}}(t,x)\right\|_{L^2(\Omega)}$$
$$\leqslant C\exp(\tau_0 t)\Big(\left\|u(0,x)\right\|_{D(a^m)}+\left\|\frac{\partial u}{\partial t}(0,x)\right\|_{D(a^{m-\frac{1}{2}})}+\cdots$$
$$+\left\|\frac{\partial^{2m}u}{\partial t^{2m}}(0,x)\right\|_{L^2(\Omega)}+\left\|f(0)\right\|_{L^2(\Omega)}+\int_0^t\left\|\frac{\partial f}{\partial s}(s,x)\right\|_{L^2(\Omega)}ds\Big).$$

Remark 1. As seen directly in our proofs of §2.1, in the statements of Theorems 1, 2, Ξ_4^∞ can be replaced by $\Xi_{1+\delta}^\infty(0<\delta<1)$ which is the set of $\sigma(x,\xi)$ such that the first order derivatives $\frac{\partial\sigma}{\partial x_i}(x,\xi)$ exist and uniformly Hörder continuous of order δ. Furthermore we can drop Hölder continuity δ by the similar treatment as one in [4].

Remark 2. For the system mentioned in Introduction we can also apply the same treatment as one in §3. Because setting $\frac{\partial^j u_i}{\partial t^j}=u_{i,j+1}$ we may consider ${}^t(u_{11},u_{12},\cdots,u_{1,m_1},u_{21},\cdots,u_{2,m_2},\cdots,u_{k,1},\cdots,u_{k,m_k})$ and the corresponding system. Then using the partition of unity with respect to $\bar{\Omega}\times\sum$ we can construct $R(x,\xi)$ locally and the norm same as one treating in Cauchy problem (see [14]).

Remark 3. By the method of reflection we can reduce our mixed problems where $\Omega=\boldsymbol{R}_+^n$ and $a(x,D)=\Delta$ into Cauchy problems over the whole space $\boldsymbol{R}^n$. Here we must remark that the coefficients of principal part of reduced equation have Lipshitz continuity. Therefore for the case of single equation we can apply the theory of Leray and Gårding. Furthermore for the case of system we can extend the theory of Calderón-Zygmund's singular integral operator to one for symbols which are introduced by the reduced equations (see Remark 1, Corollary of Lemma 5 and [4]). Thus in this case we can obtain the results in §3.

Remark 4. In the case for single equation it is easily seen that by coordinate transformation in Lemma 13 $a(x,D)$ is transformed to $b(y,D)$ with $b_{ni}(y',0)=0$ $(i=1,\cdots,n-1)$, therefore our boundary conditions (D) and (N) are reduced to $\left(\frac{\partial^i}{\partial n^i}+(\textit{lower order terms})\right)u=0$ on Γ, $i=2j$ $(j=0,1,\cdots,m-1)$, $i=2j+1$ $(j=0,1,\cdots,m-1)$, respectively. Therefore by the transformation such that $t'=t+\varepsilon\sum_{i=1}^{n-1}y_i^2$ the form of the boundary conditions are invariant, we can prove that local uniqueness theorem is valid, that is, the speed of propagation of waves is finite. Moreover by the same consideration this fact is also valid for the case of systems.

As mentioned Remark 2-4, we have various methods for solving our problems. To apply these ones we must treat the perturbation with respect to boundary operators by using the theory of elliptic operators for boundary value problems and gather local solutions. As long as we are concerned with the mixed problems mentioned in Introduction, we can avoid such troublesomeness by considering our singular integral operators. Furthermore by our method our system can be reduced into the first order system written in our singular integral operators, which are as invariant for certain change of variables as formulation of our problems. Moreover we obtain (29) directly by our method.

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Department of Mathematics,
Hokkaido University

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