ON ENERGY INEQUALITIES OF MIXED PROBLEMS FOR HYPERBOLIC EQUATIONS OF SECOND ORDER

By

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§ 1. Introduction

Let $\boldsymbol{R}_+^n$ be the open half space $\{x=(x', x_n)\in \boldsymbol{R}^n; x'\in \boldsymbol{R}^{n-1}, x_n>0\}$ and $\overline{\boldsymbol{R}}_+^n$ its closure. We consider a hyperbolic operator in $[0, T]\times \overline{\boldsymbol{R}}_+^n (T>0)$:

$$(1.1)\qquad P(t, x; D)=\frac{\partial^2}{\partial t^2}-2\sum_{j=1}^{n} a_j(t, x)\frac{\partial^2}{\partial t\partial x_j}-\sum_{j,k=1}^{n} a_{jk}(t, x)\frac{\partial^2}{\partial x_j\partial x_k}+(\textit{first order})$$

which satisfies

$$(1.2)\qquad \sum_{j,k=1}^{n}\left\{a_{jk}(t, x)+a_j(t, x)a_k(t, x)\right\}\xi_j\xi_k>0$$

$$(1.3)\qquad a_{nn}(t, x)>0$$

for any $(t, x)\in[0, T]\times \overline{\boldsymbol{R}}_+^n$ and any non-zero $\xi=(\xi_1, \cdots, \xi_n)\in \boldsymbol{R}^n$. The condition (1.2) means that $P(t, x; D)$ is strictly hyperbolic and (1.3) assures to impose *one* boundary condition on a mixed problem considered below (cf. §2). Here we assume that a_{jk} are symmetric. Moreover we consider the following boundary operator on the boundary $[0, T]\times(\overline{\boldsymbol{R}}_+^n-\boldsymbol{R}_+^n)$:

$$(1.4)\qquad B(t, x'; D)=\frac{\partial}{\partial x_n}-\sum_{j=1}^{n-1} b_j(t, x')\frac{\partial}{\partial x_j}-c(t, x')\frac{\partial}{\partial t}+h(t, x').$$

Here we assume that all coefficients in (1.1) and (1.4) are real valued, sufficiently smooth and constant except a compact set.

In the case of operators with constant coefficients, a necessary and sufficient condition for L^2-well-posedness[1)] of a mixed problem with homo-

1) The mixed problem (P, B) is L^2-well-posed if and only if there exist positive constants C, T and $T'(0<T'\leq T)$ satisfying the following property: For every $f\in H^1((0, T)\times \boldsymbol{R}_+^n)$ with $f=0$ $(t<0)$ the problem

$$Pu=f\ (t>0,\ x_n>0),\quad Bu=0\ (t>0,\ x_n=0),\quad u=\frac{\partial u}{\partial t}=0\ (t=0,\ x_n>0)$$

has a unique solution $u\in H^2((0, T')\times \boldsymbol{R}_+^n)$ such that

$$\int_0^{T'}|||u(t,\cdot)|||_1^2dt\leq C\int_0^T||f(t,\cdot)||_0^2dt.$$

geneous initial-boundary conditions is established in [1] (see also [10]). Let $P(t, x; D)$ and $B(t, x'; D)$ be the constant coefficient operators resulting from freezing the coefficients at (t, x). Then this condition is written by the terms of coefficients, that is,

(C_1) $a_{nn}(t, x)c(t, x')+a_n(t, x)\geqq 0$ and

(C_2) the following quadratic form $H(t, x; \sigma)$ in $\sigma=(\sigma_1, \cdots, \sigma_{n-1})\in \boldsymbol{R}^{n-1}$ is positive semi-definite,

where

$$H(t, x; \sigma)=(a_{nn}c+a_n)^2(a_{nn}\mathfrak{e}-\mathfrak{d}^2)-2(a_{nn}c+a_n)(a_{nn}\mathfrak{a}-a_n\mathfrak{d})(a_{nn}\mathfrak{b}+\mathfrak{d})$$
$$-(a_{nn}+a_n^2)(a_{nn}\mathfrak{b}+\mathfrak{d})^2,$$

$$\mathfrak{a}=\sum_{j=1}^{n-1} a_j(t, x)\sigma_j, \qquad \mathfrak{b}=\sum_{j=1}^{n-1} b_j(t, x')\sigma_j,$$

$$\mathfrak{d}=\sum_{j=1}^{n-1} a_{nj}(t, x)\sigma_j, \qquad \mathfrak{e}=\sum_{j,k=1}^{n-1} a_{jk}(t, x)\sigma_j\sigma_k.$$

When $a_{nn}c+a_n>0$ and H is positive definite on the boundary, it is so called the uniformly Lopatinskii condition. These facts are proved in §2.

The purpose of this paper is to prove the following energy inequality which is shown in §3.

Theorem. *Suppose that the conditions* (C_1) *and* (C_2) *are satisfied on the boundary* $[0, T]\times(\overline{\boldsymbol{R}}^n_+-\boldsymbol{R}^n_+)$. *Then there exists a positive constant* K *such that for every real* $u\in H^2((0, T)\times \boldsymbol{R}^n_+)$ *with* $Bu=0$ *on* $[0, T]\times(\overline{\boldsymbol{R}}^n_+-\boldsymbol{R}^n_+)$ *the following energy inequality holds: for any* t $(0\leq t\leq T)$

$$|||u(t, \cdot)|||_1^2 \leqq K\left\{\int_0^t \|(Pu)(s, \cdot)\|_0^2 ds+|||u(0, \cdot)|||_1^2\right\}, \tag{1.5}$$

where $\|u(\cdot)\|_0^2=\|u(\cdot)\|^2_{L^2(\boldsymbol{R}^n_+)}$ *and*

$$|||u(t, \cdot)|||_1^2=\|u(t, \cdot)\|_0^2+\left\|\frac{\partial u}{\partial t}(t, \cdot)\right\|_0^2+\sum_{j=1}^{n}\left\|\frac{\partial u}{\partial x_j}(t, \cdot)\right\|_0^2.$$

It seems to us that one of difficulities of our problem comes from the following fact: there is non zero vector (τ, σ) with $\boldsymbol{Re}\ \tau=0$ and $\sigma\in\boldsymbol{R}^{n-1}$ such that Lopatinskii determinant $R(\tau, \sigma)=0$ and the characteristic equation $P(\tau, \sigma, \lambda)=0$ has a pure imaginary double root with respect to λ[2]. However, we can avoid this difficulity by introducing the above algebraic conditions (C_1) and (C_2).

Combining the method of the proof of the theorem with a certain remark, it is shown that energy inequalities of higher order are valid. By

2) See §2 and refer also to [4], [7] and [8]. In particular consider only τ with $Re\ \tau\geqq 0$.

the method of approximation, we can show the existence and the reguarity of the solution of our problem if $\sum_{j,k=1}^{n} a_{jk}(t, x)\xi_j\xi_k$ is positive definite. Here we use the following facts: Cauchy-Kowalewsky theorem [9] with respect to mixed problem and that the conditions (C_1) and (C_2) are invariant under the Holmgren transformations[3].

Concerning already known results related this problems see [2] and [3].

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§ 2. Conditions (C_1) and (C_2)

In this section we show that, in the case of constant coefficients, conditions (C_1) and (C_2) are a necessary and sufficient condition for L^2-well-posedness. Throughout this section we assume that $P(D)$ and $B(D)$ are homogeneous and with constant coefficients. We use notations in [1].

First of all let us remark that the condition (1.2) is equivalent that

(2.1) $$a_{nn}+a_n^2>0 \quad \text{and}$$

(2.2) the quadratic form $D(\sigma)=(a_{nn}+a_n^2)(\mathfrak{a}^2+\mathfrak{e})-(a_n\mathfrak{a}+\mathfrak{b})^2$ is positive definite,

where, $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{e}$ are defined in §1.

Let $P(\tau, \sigma, \lambda)=\tau^2-2i\mathfrak{a}\tau-2ia_n\tau\lambda+a_{nn}\lambda^2+2\mathfrak{b}\lambda+\mathfrak{e}$ be the characteristic polynomial for $P(D)$ and let $\lambda^+(\tau, \sigma)(\lambda^-(\tau, \sigma))$ be a root in λ of $P(\tau, \lambda, \sigma)=0$ which has positive (negative) imaginary part for $\tau\in\boldsymbol{C}_+=\{\tau: \boldsymbol{Re}\,\tau>0\}$ respectively. Then by (1.3) they are written by the form in $\boldsymbol{C}_+$:

$$\lambda^{\pm}(\tau, \sigma)=a_{nn}^{-1}\left[(ia_n\tau-\mathfrak{b})\pm i\left\{(a_{nn}+a_n^2)\tau^2-2i(a_{nn}\mathfrak{a}-a_n\mathfrak{b})\tau+a_{nn}\mathfrak{e}-\mathfrak{b}^2\right\}^{\frac{1}{2}}\right]$$

where the square root $(\quad)^{\frac{1}{2}}$ is determined such that $Re\,(\quad)^{\frac{1}{2}}>0$ if $\boldsymbol{Re}\,\tau>0$. Moreover we consider $\lambda^{\pm}(\tau, \sigma)$ to be continuously extended to $\bar{\boldsymbol{C}}_+\times\boldsymbol{R}^{n-1}$ where $\bar{\boldsymbol{C}}_+$ is the closure of $\boldsymbol{C}_+$. By the choice of the square root it implies

(2.3) $$(\boldsymbol{Im}\{\quad\}^{\frac{1}{2}})\left\{(a_{nn}+a_n^2)(\boldsymbol{Im}\,\tau)-(a_{nn}\mathfrak{a}-a_n\mathfrak{b})\right\}\geq 0 \quad \text{in } \bar{\boldsymbol{C}}_+,$$

where the bracket $\{\ \}^{\frac{1}{2}}$ is the same one in $\lambda^+(\tau, \sigma)$.

To get a necessary and sufficient condition for L^2-well-posedness, zeros of Lopatinskii determinant $R(\tau, \sigma)$ in $\bar{\boldsymbol{C}}_+\times\boldsymbol{R}^{n-1}$ and the behavior of the reflection coefficient $C(\tau, \sigma)$ near zeros play an important role. In this case,

3) See the remark in the end of the paper.

$$(2.4)\qquad R(\tau,\sigma)=i\lambda^{+}(\tau,\sigma)-i\mathfrak{b}-c\tau$$
$$=-a_{nn}^{-1}\Big[\Big\{(a_{nn}c+a_n)\tau+i(a_{nn}\mathfrak{b}+\mathfrak{d})\Big\}+\Big\{(a_{nn}+a_n^2)\tau^2$$
$$-2i(a_{nn}\mathfrak{a}-a_n\mathfrak{d})\tau+a_{nn}\mathfrak{e}-\mathfrak{d}^2\Big\}^{\frac{1}{2}}\Big],$$
$$C(\tau,\sigma)=\Big\{i\lambda^{-}(\tau,\sigma)-i\mathfrak{b}-c\tau\Big\}\Big/R(\tau,\sigma).$$

Applying now the results in [1] to this case we obtain the following criteria for L^2-well-posedness.

Theorem.

(A) *If the mixed problem* (P, B) *is* L^2*-well-posed, then* $S(\tau)=\{\sigma\in \boldsymbol{R}^{n-1};\ R(\tau,\sigma)=0\}$ *is independent of* $\tau\in\boldsymbol{C}_+$.

(B) *Let* $S(\tau)$ $(=S)$ *be independent of* $\tau\in\boldsymbol{C}_+$. *Then the mixed problem* (P, B) *is* L^2*-well-posed if and only if the reflection coefficient* $C(\tau,\sigma)$ *is bounded in a neighbourhood of* $(\tau_0,\sigma_0)\neq 0$ *with* $\boldsymbol{Re}\,\tau_0=0$ *or* $(\tau_0,\sigma_0)\in\boldsymbol{C}_+\times S$.

As a special case of (B) we see that

(C) *if* (P, B) *satisfies the uniformly Lopatinskii condition* (*that is, if* $R(\tau,\sigma)\neq 0$ *for any non zero* $(\tau,\sigma)\in\bar{\boldsymbol{C}}_+\times\boldsymbol{R}^{n-1}$), *then the mixed problem* (P, B) *is* L^2*-well-posed.*

Using the theorem we shall show that the conditions (C_1) and (C_2) are a necessary and sufficient conditions for L^2-well-posedness. To prove this we need the following lemmas.

Lemma 2.1.

(i) *If* $a_{nn}c+a_n\neq-(a_{nn}+a_n^2)^{\frac{1}{2}}$, *then* $R(\tau,0)\neq 0$ *in* $\bar{C}_+-\{0\}$.

(ii) *If* $a_{nn}c+a_n=-(a_{nn}+a_n^2)^{\frac{1}{2}}$, *then* $R(\tau,0)=0$ *in* $\boldsymbol{C}_+$ *and* $C(\tau,\sigma)$ *is not bounded in any neighbourhood of* $(\tau_0, 0)$ $(\tau_0\in\boldsymbol{C}_+)$.

Proof. By the choise of the square root we get

$$R(\tau,0)=-a_{nn}^{-1}\Big\{a_{nn}c+a_n+(a_{nn}+a_n^2)^{\frac{1}{2}}\Big\}\tau,$$
$$C(\tau,0)=\Big\{a_{nn}c+a_n-(a_{nn}+a_n^2)^{\frac{1}{2}}\Big\}\Big/\Big\{a_{nn}c+a_n+(a_{nn}+a_n^2)^{\frac{1}{2}}\Big\},$$

which proves the lemma.

Hereafter we may assume that $\sigma\neq 0$.

Now it follows from (2.4) that $R(\tau,\sigma)=0$ implies

$$(2.5)\qquad F(\tau,\sigma)=\Big\{(a_{nn}c+a_n)^2-(a_{nn}+a_n^2)\Big\}\tau^2+2i\Big\{(a_{nn}\mathfrak{a}-a_n\mathfrak{d})$$
$$+(a_{nn}c+a_n)(a_{nn}\mathfrak{b}+\mathfrak{d})\Big\}\tau-(a_{nn}\mathfrak{b}+\mathfrak{d})^2-(a_{nn}\mathfrak{e}-\mathfrak{d}^2)=0,$$

where the first equality is a definition.

The roots in τ of $F(\tau, \sigma)=0$ are

$$(2.6)\qquad \frac{-i\{(a_{nn}c+a_n)(a_{nn}\mathfrak{b}+\mathfrak{d})+(a_{nn}\mathfrak{a}-a_n\mathfrak{d})\}\pm(H(\sigma)-a_{nn}D(\sigma))^{\frac{1}{2}}}{(a_{nn}c+a_n)^2-(a_{nn}+a_n^2)}$$

where $H(\sigma)$ is the quadratic form in (C_2). From (2.6) it is seen that $R(i\eta, \sigma)\neq 0$ for any real η if $H(\sigma)-a_{nn}D(\sigma)>0$ and $R(\tau, \sigma)\neq 0$ for any $\tau\in\boldsymbol{C}_+$ if $H(\sigma)-a_{nn}D(\sigma)<0$.

Now we first consider the case that $(a_{nn}c+a_n)^2\neq a_{nn}+a_n^2$ and investigate zeros of Lopatinskii determinant according to a sign of $H(\sigma)-a_{nn}D(\sigma)$.

Lemma 2.2. *Suppose that* $(a_{nn}c+a_n)^2\neq a_{nn}+a_n^2$ *and* $H(\sigma_0)-a_{nn}D(\sigma_0)>0$ *for some* $\sigma_0\neq 0$. *Then*

(i) $R(i\eta, \sigma)\neq 0$ *for any real* η,

(ii) *if* $a_{nn}c+a_n>0$, $R(\tau, \sigma_0)\neq 0$ *for any* $\tau\in\boldsymbol{C}_+$,

(iii) *if* $a_{nn}c+a_n<0$, $R(\tau_0, \sigma_0)=0$ *for some* $\tau_0\in\boldsymbol{C}_+$.

Remark 1) In this lemma we may assume that $a_{nn}c+a_n\neq 0$. In fact, if $a_{nn}c+a_n=0$, then $H(\sigma_0)=-(a_{nn}+a_n^2)(a_{nn}\mathfrak{b}_0+\mathfrak{d}_0)^2$ where $\mathfrak{b}_0=\mathfrak{b}(\sigma_0)$ and $\mathfrak{d}_0=\mathfrak{d}(\sigma_0)$. Hence by (1.3), (2.1) and (2.2) we have $H(\sigma_0)-a_{nn}D(\sigma_0)<0$. This contradicts the assumption.

2) If $a_{nn}c+a_n<0$ and $H(\sigma)-a_{nn}D(\sigma)$ is positive definite, then by the proof of this lemma $S(\tau)$ depends on $\tau\in\boldsymbol{C}_+$.

Proof of Lemma 2.2. From the remark mentioned before this lemma it sufficies to prove (ii) and (iii). It is obvious that $R(\tau, \sigma_0)\neq 0$ except roots in τ of $F(\tau, \sigma_0)=0$. Let τ_0 be a root of $F(\tau, \sigma_0)=0$. Then it follows from (2.4) (2.5) that

$$R(\tau_0, \sigma_0)=-a_{nn}^{-1}\Big[\Big\{(a_{nn}c+a_n)\tau_0+i(a_{nn}\mathfrak{b}_0+\mathfrak{d}_0)\Big\} + \Big\{\Big\{(a_{nn}c+a_n)\tau_0+i(a_{nn}\mathfrak{b}_0+\mathfrak{d}_0)\Big\}^2\Big\}^{\frac{1}{2}}\Big].$$

By the choice of the square root we obtain

$$R(\tau_0, \sigma_0)=\begin{cases}-2a_{nn}^{-1}\Big\{(a_{nn}c+a_n)\tau_0+i(a_{nn}\mathfrak{b}_0+\mathfrak{d}_0)\Big\} & \text{if } a_{nn}c+a_n>0, \\ 0 & \text{if } a_{nn}c+a_n<0,\end{cases}$$

from which our assertion follows directly. Remark 2) is proved by the above equality and (2.6).

Lemma 2.3. *Suppose that* $(a_{nn}c+a_n)^2\neq a_{nn}+a_n^2$ *and* $H(\sigma_0)-a_{nn}D(\sigma_0)\leq 0$ *for some* $\sigma_0\neq 0$. *Then*

(i) $R(\tau, \sigma_0)\neq 0$ *for any* $\tau\in\boldsymbol{C}_+$,

(ii) *if* $H(\sigma_0)>0$ *and* $a_{nn}c+a_n>0$, $R(i\eta,\sigma_0)\neq 0$ *for any real* η,

(iii) *if* $H(\sigma_0)=0$ *and* $a_{nn}c+a_n\geqq 0$, $R(i\eta_0,\sigma_0)=0$ *for some real* η_0, *but* $C(\tau,\sigma)$ *is bounded in a neighbourhood of* $(i\eta_0,\sigma_0)$,

(iv) *if* $H(\sigma_0)<0$ *or if* $H(\sigma_0)\geqq 0$ *and* $a_{nn}c+a_n<0$, *there exists a real* η_0 *such that* $R(i\eta_0,\sigma_0)=0$ *and* $C(\tau,\sigma)$ *is not bounded in any neighbourhood of* $(i\eta_0,\sigma_0)$.

Remark. In the case (ii) we may assume that $a_{nn}c+a_n\neq 0$. In fact, if $a_{nn}c+a_n=0$ then by (2.1) we have $H(\sigma_0)=-(a_{nn}+a_n^2)(a_{nn}\mathfrak{b}_0+\mathfrak{d}_0)^2\leqq 0$.

Proof of Lemma 2.3. (i) is proved before Lemma 2.2. By (2.6) and the assumption of the lemma the equation $F(i\eta,\sigma_0)=0$ in η has two real roots (counting the multiplicity). Clearly $R(i\eta,\sigma_0)\neq 0$ except roots in η of $F(i\eta,\sigma_0)=0$. Let η_0 be a root of $F(i\eta,\sigma_0)$. Then it follows from (2.3), (2.4) and (2.5) that

$$(2.7)\quad R(i\eta_0,\sigma_0)=\begin{cases}-2ia_{nn}^{-1}\{(a_{nn}c+a_n)\eta_0+a_{nn}\mathfrak{b}_0+\mathfrak{d}_0\} & \text{if } G(\sigma_0)>0\,,\\ 0 & \text{if } G(\sigma_0)\leq 0\,,\end{cases}$$

and

$$(2.8)\quad \begin{matrix}\text{the numerator}\\ \text{of } C(i\eta_0,\sigma_0)\end{matrix}=\begin{cases}0 & \text{if } G(\sigma_0)\geq 0\,,\\ -2ia_{nn}^{-1}\{(a_{nn}c+a_n)\eta_0+a_{nn}\mathfrak{b}_0+\mathfrak{d}_0\} & \text{if } G(\sigma_0)>0\,,\end{cases}$$

where $\mathfrak{a}_0=\mathfrak{a}(\sigma_0)$ and $G(\sigma_0)=\{(a_{nn}c+a_n)\eta_0+a_{nn}\mathfrak{b}_0+\mathfrak{d}_0\}\{(a_{nn}+a_n^2)\eta_0-(a_{nn}\mathfrak{a}_0-a_n\mathfrak{d}_0)\}$.

To determine a sign of $G(\sigma_0)$ we first remark the following facts. Substituting $-i(a_{nn}\mathfrak{b}+\mathfrak{d})/(a_{nn}c+a_n)$ and $i(a_{nn}\mathfrak{a}-a_nd)/(a_{nn}+a_n^2)$ as values of τ into $F(\tau,\sigma)$ we get directly the following relations:

$$(2.9)\qquad F\left(-\frac{a_{nn}\mathfrak{b}+\mathfrak{d}}{a_{nn}c+a_n}i,\sigma\right)=-\frac{H(\sigma)}{(a_{nn}c+a_n)^2}\,,$$

$$F\left(\frac{a_{nn}\mathfrak{a}-a_n\mathfrak{d}}{a_{nn}+a_n^2}i,\sigma\right)=-\frac{a_{nn}D(\sigma)}{a_{nn}+a_n^2}-\frac{I(\sigma)^2}{(a_{nn}+a_n^2)^2}\,,$$

where $I(\sigma)=(a_{nn}c+a_n)(a_{nn}\mathfrak{a}-a_n\mathfrak{d})+(a_{nn}+a_n^2)(a_{nn}\mathfrak{b}+\mathfrak{d})$. Hence it follows from (1.3), (2.1) and (2.2) that

$$(2.10)\qquad F\left(\frac{a_{nn}\mathfrak{a}-a_n\mathfrak{d}}{a_{nn}+a_n^2}i,\sigma\right)\ \text{is negative definite.}$$

Moreover by the definition of $I(\sigma)$ we have the following relations:

$$(2.11)\qquad \frac{a_{nn}\mathfrak{a}-a_n\mathfrak{d}}{a_{nn}+a_n^2}-\left(-\frac{a_{nn}\mathfrak{b}+\mathfrak{d}}{a_{nn}c+a_n}\right)=\frac{I(\sigma)}{(a_{nn}+a_n^2)(a_{nn}c+a_n)}\,,$$

(2.12)
$$\frac{-(a_{nn}c+a_n)(a_{nn}\mathfrak{b}+\mathfrak{d})-(a_{nn}\mathfrak{a}-a_n\mathfrak{d})}{(a_{nn}c+a_n)^2-(a_{nn}+a_n^2)}-\left(-\frac{a_{nn}\mathfrak{b}+\mathfrak{d}}{a_{nn}c+a_n}\right)$$
$$=\frac{-I(\sigma)}{(a_{nn}c+a_n)\{(a_{nn}c+a_n)^2-(a_{nn}+a_n^2)\}},$$

where the first term of the left hand of (2.12) is a value at which the function $F(i\eta, \sigma)$ in η take an extremum.

We consider only the case that $a_{nn}c+a_n>(a_{nn}+a_n^2)^{\frac{1}{2}}$, because for another case (ii)–(iv) are proved by the same method. In this case $F(i\eta, \sigma)$ is a concave function in η.

(ii) The case that $H(\sigma_0)>0$. It follows from (2.9)–(2.12) that $I(\sigma_0)\neq 0$,

$$\eta_0<-\frac{a_{nn}\mathfrak{b}_0+\mathfrak{d}_0}{a_{nn}c+a_n}<\frac{a_{nn}\mathfrak{a}_0-a_n\mathfrak{d}_0}{a_{nn}+a_n^2}\qquad \text{if}\quad I(\sigma_0)>0\,,$$
$$\eta_0>-\frac{a_{nn}\mathfrak{b}_0+\mathfrak{d}_0}{a_{nn}c+a_n}>\frac{a_{nn}\mathfrak{a}_0-a_n\mathfrak{d}_0}{a_{nn}+a_n^2}\qquad \text{if}\quad I(\sigma_0)<0\,.$$

This shows that $G(\sigma_0)>0$. Hence by (2.6) we have $R(i\eta_0, \sigma_0)\neq 0$.

(iii) The case that $H(\sigma_0)=0$. By (2.9) we see that $\eta_0=-(a_{nn}\mathfrak{b}_0+\mathfrak{d}_0)/(a_{nn}c+a_n)$ is a root in η of $F(i\eta, \sigma_0)=0$. Hence by the same as the case (ii) $R(i\eta, \sigma_0)\neq 0$ for any $\eta\neq\eta_0$. From (2.7) and (2.8) it is easily seen that $R(i\eta_0, \sigma)=0$ and the numerator of $C(i\eta_0, \sigma_0)$ is equal to zero. To prove that $C(\tau, \sigma)$ is bounded in a neighbourhood of $(i\eta_0, \sigma_0)$ we consider the expansion of $\lambda^{\pm}(\tau, \sigma)$ near $(i\eta_0, \sigma_0)$. (cf. [6]). Let $P(\tau, \sigma, \lambda)=(\tau-i\tau_1(\sigma, \lambda))(\tau-i\tau_2(\sigma, \lambda))$ where $\tau_j(\sigma, \lambda)$ $(j=1.2)$ are real analytic and distinct and moreover let λ_0 be a root in λ of $P(i\eta_0, \sigma_0, \lambda)=0$. Then without restriction we may asseme that $\eta_0=\tau_1(\sigma_0, \lambda_0)$ and $\lambda_0=-(a_n\eta_0+\mathfrak{d}_0)/a_{nn}$. Since λ_0 is double, we have $\frac{\partial\tau_1}{\partial\lambda}(\sigma_0, \lambda_0)=0$ and $\frac{\partial^2\tau_1}{\partial\lambda^2}(\sigma_0, \lambda_0)\neq 0$. Hence there exists a real analytic function $\lambda_0(\sigma)$ such that $\lambda_0(\sigma_0)=\lambda_0$ and $\frac{\partial\tau_1}{\partial\lambda}(\sigma, \lambda_0(\sigma))=0$ in a small neighbourhood of σ_0. Now let us set

$$i\tau-\eta_0=\tau_1(\sigma, \lambda)-\tau_1(\sigma_0, \lambda_0)$$
$$=\tau_1\big(\sigma, \lambda_0(\sigma)\big)-\tau_1(\sigma_0, \lambda_0)+\frac{1}{2}\frac{\partial^2\tau_1}{\partial\lambda^2}\big(\sigma, \lambda_0(\sigma)\big)\big(\lambda-\lambda_0(\sigma)\big)^2+\cdots.$$

Let ξ^+ (ξ^-) be the square root of $-(2i\tau+\tau_1\big(\sigma, \lambda_0(\sigma)\big)\Big/\frac{\partial^2\tau_1}{\partial\lambda^2}\big(\sigma, \lambda_0(\sigma)\big)$ with positive (negative) imaginary part respectively. Then for an arbitrarily fixed σ near σ_0 we obtain the expansion in $\xi^{\pm}$ of $\lambda^{\pm}(\tau, \sigma)$ near $i\eta_0$:

$$\lambda^{\pm}(\tau, \sigma) = \lambda(\sigma) + \xi^{\pm} + c_2(\sigma)(\xi^{\pm})^2 + \cdots,$$

where $c_j(\sigma)$ are real. Using $\lambda_0(\sigma_0) = \lambda_0 = -(a_n\eta_0 + \mathfrak{d}_0)/a_{nn}$,

$$C(\tau, \sigma_0) = \frac{\xi^{+} + 0((\xi^{+})^2)}{\xi^{-} + 0((\xi^{-})^2)} .$$

Since $|\xi^{+}/\xi^{-}| = 1$ and $\xi^{+}(\xi^{-}) \to 0$ when $\tau \to i\eta_0$, $C(\tau, \sigma)$ is bounded in a neighbourhood of $(i\eta_0, \sigma_0)$.

(iv) The case that $H(\sigma_0) < 0$. From (2.9) and (2.10) we see that there exists a root η_0 between $(a_{nn}\mathfrak{a}_0 - a_n\mathfrak{d}_0)/(a_{nn} + a_n^2)$ and $-(a_{nn}\mathfrak{b}_0 + \mathfrak{d}_0)/(a_{nn}c + a_{nn})$. Hence we have that $G(\sigma_0) < 0$. Consequently it follows from (2.7) and (2.8) that the conclusion of (iv) in the lemma is valid.

Next, we consider the case that $(a_{nn}c + a_n)^2 = a_{nn} + a_n^2$. In the case that $a_{nn}c + a_n = -(a_{nn} + a_n^2)^{\frac{1}{2}}$ we use only Lemma 2.1 for our aim.

Lemma 2.4. *Suppose that* $a_{nn}c + a_n = (a_{nn} + a_n)^{\frac{1}{2}}$. *Then*

(i) *if* $(a_{nn}c + a_n)(a_{nn}\mathfrak{b}_0 + \mathfrak{d}_0) + (a_{nn}\mathfrak{a}_0 - a_n\mathfrak{d}_0) = 0$ *for some* $\sigma_0 \neq 0$, $R(\tau, \sigma_0) \neq 0$ *in* $\bar{C}_{+}$,

(ii) *if* $(a_{nn}c + a_n)(a_{nn}\mathfrak{b}_0 + \mathfrak{d}_0) + (a_{nn}\mathfrak{a}_0 - a_n\mathfrak{d}_0) \neq 0$ *for some* $\sigma_0 \neq 0$, $R(\tau, \sigma) \neq 0$ *in* $\boldsymbol{C}_{+}$ *and the conclusions of (ii), (iii), (iv) in Lemma (2.3) are valid according to* $H(\sigma_0) > 0$, $H(\sigma_0) = 0$, $H(\sigma_0) < 0$ *respectively.*

Proof. We prove only (i) because (ii) is proved by same method in Lemma 2.3. In the case (i) $F(\tau, \sigma_0) = a_{nn}D(\sigma_0)/(a_{nn} + a_n^2)$. Hence the assertion follows immediately from (1.3), (2.1) and (2.2).

Using Theorem (A), (B), (C) and lemmas (2.1)–(2.4) we can show the following

Theorem 2.5. *The mixed problem* (P, B) *is* L^2*-well-posed if and only if* $a_{nn}c + a_n \geq 0$ *and the quadratic form* $H(\sigma)$ *is positive semi-definite.*

Proof. 1. Sufficiency of our condition. First cosider the case that $a_{nn}c + a_n > 0$ and $H(\sigma) - a_{nn}D(\sigma)$ is positive definite. By the remark 1 of Lemma 2.2 we may assume that $a_{nn}c + a_n > 0$. Note that $H(\sigma)$ is positive difinite and $(a_{nn}c + a_n)^2 \neq a_{nn} + a_n^2$. In fact, the first assertion follows immediately from (1.3) and (2.2). If $(a_{nn}c + a_n)^2 = a_{nn} + a_n^2$, then $H(\sigma) - a_{nn}D(\sigma) = -\{(a_{nn}c + a_n)(a_{nn}\mathfrak{b} + \mathfrak{d}) + (a_{nn}\mathfrak{a} - a_n\mathfrak{d})\}^2 \leq 0$. Hence it follows from Lemma 2.1, 2.2 and Theorem (C) that the mixed problem (P, B) satisfies the uniformly Lopatinskii condition and consequently it is L^2-well-posed. Next consider the case that $a_{nn}c + a_n \geq 0$ and $H(\sigma_0) - a_{nn}D(\sigma_0) \leq 0$ for some σ_0. If $H(\sigma_0)$ is always positive for such all σ_0, then by lemmas 2.1–2.4 the problem (P, B) satisfies the uniformly Lopatinskii condition. If $H(\sigma_0)$ is non negative for such all σ_0 and in fact $H(\sigma_0) = 0$ for some σ_0, then by lemmas 2.1–2.4 $S(\tau)$

$=\phi$ for any $\tau\in\boldsymbol{C}_+$ and the reflection coefficient $C(\tau\ \sigma)$ is bounded in a neighbourhood of a zero of Lopatinskii determinant. Therefore the problem is L^2-well-posed by Theorem B.

2. Necessity of our condition. First if $a_{nn}c+a_n\geq 0$ and $H(\sigma_0)<0$ for some σ_0, then by lemmas 2.3, 2.4 and Theorem (B) the problem is not L^2-well-posed. Secondly if $a_{nn}c+a_n<0$, $a_{nn}c+a_n\neq(a_{nn}+a_n^2)$ and $H(\sigma)-a_{nn}D(\sigma)$ is positive definite, then by the remark 2 of lemma 2.2 and Theorem (A) the problem is not L^2-well-posed. Thirdly if $a_{nn}c+a_n<0$, $a_{nn}c+a_n\neq(a_{nn}+a_n^2)$ and $H(\sigma_0)-a_{nn}D(\sigma_0)\leq 0$ for some $\sigma_0\neq 0$, then by Lemma 2.3 and Theorem (B) the problem is not L^2-well-posed. Finally if $a_{nn}c+a_n=-(a_{nn}+a_n^2)$, then by Lemma 2.1 and Theorem (B) the problem is not L^2-well-posed. In a final case note that $S(\tau)=\{0\}$. Thus the proof is complete.

Remark. From the proof of the theorem we see that (P, B) satisfies the uniformly Lopatinskii condition if and only if $a_{nn}c+a_n>0$ and $H(\sigma)$ is positive definite.

§ 3. Energy inequalities

In this section we prove Theorem stated in § 1.

First of all we may assume that $B(t, x'\colon D)$ is homogeneous. In fact, we take a real valued function $\phi\in C_0^\infty([0, T]\times\overline{\boldsymbol{R}}{}_+^n)$ with following properties:

$$\frac{1}{2}<\phi<1 \qquad \text{in} \quad [0, T]\times\overline{\boldsymbol{R}}{}_+^n\,,$$

$$\phi=1 \quad \text{and} \quad \frac{\partial\phi}{\partial x_n}+h=0 \quad \text{on} \quad [0, T]\times(\overline{\boldsymbol{R}}{}_+^n-\boldsymbol{R}_+^n)\,,$$

and set $u=\phi v$. Then u satisfies

$$B(t, x'\colon D)u=\frac{\partial v}{\partial x_n}-\sum_{j=1}^{n-1}b_j\frac{\partial v}{\partial x_j}-c\frac{\partial v}{\partial t} \quad \text{on} \quad [0, T]\times(\overline{\boldsymbol{R}}{}_+^n-\boldsymbol{R}_+^n).$$

Hence it follows from this and invariance of principal part that the energy inequality for u follows from that for v.

Denote the inner products in $L^2(\boldsymbol{R}_+^n)$ and $L^2(\overline{\boldsymbol{R}}{}_+^n-\boldsymbol{R}_+^n)$ by $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$ respectively. Furthermore set $\|u\|^2=(u, u)$ and $\langle\!\langle u\rangle\!\rangle^2=\langle u, u\rangle$. Hereafter we may assume that $u\in C_0^\infty([0, T]\times\overline{\boldsymbol{R}}{}_+^n)$ with $B(t,\colon x'\ D)u=0$ on the boundary $[0, T]\times(\overline{\boldsymbol{R}}{}_+^n-\boldsymbol{R}_+^n)$.

Using the integration by parts we obtain that for any $t(0<t<T)$

$$(3.1)\qquad 2\int_0^t\left((Pu)(s, \cdot), \frac{\partial u}{\partial t}(s, \cdot)\right)ds$$

$$= \left\{ \left\| \frac{\partial u}{\partial t}(s, \cdot) \right\|^2 + \sum_{j,k=1}^{n} \left(a_{jk}(s, \cdot) \frac{\partial u}{\partial x_j}(s, \cdot), \frac{\partial u}{\partial x_k}(s, \cdot) \right) \right\} \Bigg|_0^t$$

$$+ 2\int_0^t < a_n(s, \cdot, 0) \frac{\partial u}{\partial t}(s, \cdot, 0) + \sum_{j=1}^{n} a_{nj}(s, \cdot, 0) \frac{\partial u}{\partial x_j}(s, \cdot, 0),$$

$$\frac{\partial u}{\partial t}(s, \cdot, 0) > ds + (\textit{lower order term}),$$

(3.2)
$$2\int_0^t \left((Pu)(s, \cdot), \frac{\partial u}{\partial x_k}(s, \cdot) \right) ds$$

$$= 2\left(\frac{\partial u}{\partial t}(s, \cdot) - \sum_{j=1}^{n} a_j(s, \cdot) \frac{\partial u}{\partial x_j}(s, \cdot), \frac{\partial u}{\partial x_k}(s, \cdot) \right) \Bigg|_0^t$$

$$+ 2\int_0^t < a_n(s, \cdot, 0) \frac{\partial u}{\partial t}(s, \cdot, 0) + \sum_{j=1}^{n} a_{nj}(s, \cdot, 0) \frac{\partial u}{\partial x_j}(s, \cdot, 0),$$

$$\frac{\partial u}{\partial x_k}(s, \cdot, 0) > ds + (\textit{lower order term}) \qquad (k = 1, 2, \cdots, n-1),$$

(3.3)
$$2\int_0^t \left((Pu)(s, \cdot), \frac{\partial u}{\partial x_n}(s, \cdot) \right) ds$$

$$= 2\left(\frac{\partial u}{\partial t}(s, \cdot) - \sum_{j=1}^{n} a_j(s, \cdot) \frac{\partial u}{\partial x_j}(s, \cdot), \frac{\partial u}{\partial x_n}(s, \cdot) \right) \Bigg|_0^t$$

$$+ \int_0^t \Bigg\{ \ll \frac{\partial u}{\partial t}(s, \cdot, 0) \gg^2 + < a_{nn}(s, \cdot, 0) \frac{\partial u}{\partial x_n}(s, \cdot, 0), \frac{\partial u}{\partial x_n}(s, \cdot, 0) >$$

$$- 2 < \frac{\partial u}{\partial t}(s, \cdot, 0), \sum_{j=1}^{n-1} a_j(s, \cdot, 0) \frac{\partial u}{\partial x_j}(s, \cdot, 0) >$$

$$- \sum_{j,k=1}^{n-1} < a_{jk}(s, \cdot, 0) \frac{\partial u}{\partial x_j}(s, \cdot, 0), \frac{\partial u}{\partial x_k}(s, \cdot, 0) > \Bigg\} ds$$

$$+ (\textit{lower order term}).$$

Here "*lower order term*" means an integral of following type:

$$\int_0^t \left(\text{a bilinear form in } u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_j} \text{ whose coefficients are at most first derivatives of those of } P \text{ and } B\right) ds.$$

Consider the following integral:

(3.4)
$$2\int_0^t \left((Pu)(s, \cdot), A(s, \cdot) \frac{\partial u}{\partial t}(s, \cdot) + \sum_{j=1}^{n} A_j(s, \cdot) \frac{\partial u}{\partial x_j}(s, \cdot) \right) ds,$$

where $A(t, x)$ and $A_j(t, x)$ are real valued and determined later on.

By the same calculation as (3.1), (3.2) and (3.3) the part $(\cdot, \cdot)|_0^t$ of the integral (3.4) becomes

$$(3.5)\quad \Big[\Big(A(s,\cdot)\frac{\partial u}{\partial t}(s,\cdot),\frac{\partial u}{\partial t}(s,\cdot)\Big)+2\Big(\frac{\partial u}{\partial t}(s,\cdot),\sum_{j=1}^{n}A_j(s,\cdot)\frac{\partial u}{\partial x_j}(s,\cdot)\Big)$$
$$+\sum_{j,k=1}^{n}\Big(\big\{A(s,\cdot)a_{jk}(s,\cdot)-2a_j(s,\cdot)A_k(s,\cdot)\big\}\frac{\partial u}{\partial x_j}(s,\cdot),\frac{\partial u}{\partial x_k}(s,\cdot)\Big)\Big]\Big|_0^t.$$

Using the relation $(Bu)(t, x', 0)=0$ and denoting $f|_{x_n=0}=f(t, x', 0)$ the boundary part of the integral (3.4) becomes

$$(3.6)\quad \int_0^t <\big\{2(a_{nn}c+a_n)A+(a_{nn}c^2+1)A_n\big\}\frac{\partial u}{\partial t}\Big|_{x_n=0},\ \frac{\partial u}{\partial t}\Big|_{x_n=0}>ds$$
$$+2\int_0^t <\frac{\partial u}{\partial t}\Big|_{x_n=0},\ \sum_{j=1}^{n-1}\big\{(a_{nn}b_j+a_{nj})A+(a_{nn}cb_j-a_j)A_n$$
$$+(a_{nn}c+a_n)A_j\big\}\frac{\partial u}{\partial x_j}\Big|_{x_n=0}>ds$$
$$+\int_0^t\sum_{j,k=1}^{n-1}<\big\{2(a_{nn}b_j+a_{nj})A_k+(a_{nn}b_jb_k-a_{jk})A_n\big\}\frac{\partial u}{\partial x_j}\Big|_{x_n=0},$$
$$\frac{\partial u}{\partial x_k}\Big|_{x_n=0}>ds.$$

First we see that the quadratic form corresponding to (3.5) is positive definite if and only if $A>0$ and $\sum_{j,k=1}^{n}(A^2a_{jk}-2a_jA_kA-A_jA_k)\xi_j\xi_k>0$ for any non-zero $\xi\in R^n$. Put $\mathfrak{B}=\sum_{j=1}^{n-1}A_j(t, x)\sigma_j$ where $\xi=(\sigma, \xi_n)$. Then by rewriting this condition we get

$$(3.7)\qquad A>0,$$

$$(3.8)\qquad a_{nn}A^2-2a_nAA_n-A_n^2>0,$$

(3.9) the following quadratic form $L(t, x: \sigma)$ in σ is positive definite,

where

$$L(t, x; \sigma)=(a_{nn}\mathfrak{e}-\mathfrak{d}^2)A^2-2\big\{(a_{nn}\mathfrak{a}-a_n\mathfrak{d})\mathfrak{B}+(a_n\mathfrak{e}-\mathfrak{a}\mathfrak{d})A_n\big\}A$$
$$-(a_{nn}+a_n^2)\mathfrak{B}^2+2(\mathfrak{d}+a_n\mathfrak{a})A_n\mathfrak{B}-(\mathfrak{a}^2+\mathfrak{e})A_n^2,$$

and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{d}$ and $\mathfrak{e}$ are defined in §1.

Next we see that the quadratic form corresponding to (3.6) is positive semi-definite if and only if on the boundary

$$(3.10)\qquad 2(a_{nn}c+a_n)A+(a_{nn}c^2+1)A_n\geqq 0,$$

(3.11) the following quadratic form $J(t, x'; \sigma)$ in σ is negative semi-definite, where

$$J(t, x'; \sigma) = \{(a_{nn}\mathfrak{b}+\mathfrak{d})A-(a_{nn}c+a_n)\mathfrak{B}\}^2 + 2AA_n\{(a_{nn}\mathfrak{b}+\mathfrak{d})(a_{nn}c\mathfrak{b}-\mathfrak{a}) - (a_{nn}c+a_n)(a_{nn}\mathfrak{b}^2-\mathfrak{e})\} + 2A_n\mathfrak{B}\{(a_{nn}c+a_n)(a_{nn}c\mathfrak{b}-\mathfrak{a}) - (a_{nn}c^2+1)(a_{nn}\mathfrak{b}+\mathfrak{d})\} + A_n^2\{(a_{nn}c\mathfrak{b}-\mathfrak{a})^2-(a_{nn}c^2+1)(a_{nn}\mathfrak{b}^2-\mathfrak{e})\}.$$

Multiplying $J(t, x'; \sigma)$ by $(a_{nn}c+a_n)^2$ a simple calculation gives

$$(3.12) \quad (a_{nn}c+a_n)^2 J(\sigma) = \Big[(a_{nn}c+a_n)^2\mathfrak{B}-(a_{nn}c+a_n)(a_{nn}\mathfrak{b}+\mathfrak{d})A + \{(a_{nn}c+a_n)(a_{nn}c\mathfrak{b}-\mathfrak{a})-(a_{nn}c^2+1)(a_{nn}\mathfrak{b}+\mathfrak{d})\}A_n\Big]^2 + A_n\{2(a_{nn}c+a_n)A+(a_{nn}c^2+1)A_n\}\{(2(a_{nn}c+a_n)(a_{nn}\mathfrak{b}+\mathfrak{d})(a_{nn}c\mathfrak{b}-\mathfrak{a}) - (a_{nn}c+a_n)^2(a_{nn}\mathfrak{b}^2-\mathfrak{e})-(a_{nn}c^2+1)(a_{nn}\mathfrak{b}+\mathfrak{d})^2\}.$$

Consider the last factor of second term in the right hand of (3.12). Compare coefficients of powers in c of this factor with those of $H(\sigma)$. Then we see that this factor is equal to $H(\sigma)/a_{nn}$. Consequently

$$(3.13) \quad (a_{nn}c+a_n)^2 J(\sigma) = \Big[(a_{nn}c+a_n)^2\mathfrak{B}-(a_{nn}c+a_n)(a_{nn}\mathfrak{b}+\mathfrak{d})A + \{(a_{nn}c+a_n)(a_{nn}c\mathfrak{b}-\mathfrak{a})-(a_{nn}c^2+1)(a_{nn}\mathfrak{b}+\mathfrak{d})\}A_n\Big]^2 + \frac{A_n H(\sigma)}{a_{nn}}\{2(a_{nn}c+a_n)A+(a_{nn}c^2+1)A_n\}.$$

Now let us set

$$(3.14) \quad \begin{aligned} A &= (a_{nn}c+a_n)^2-a_n(a_{nn}c+a_n)+a_{nn}+a_n^2, \\ A_j &= a_{nn}b_j(a_{nn}c+a_n)-(a_{nn}a_j-a_n a_{nj}) \qquad (j=1,\cdots,n-1), \\ A_n &= -a_{nn}(a_{nn}c+a_n), \end{aligned}$$

Here we extend the functions $c(t, x')$ and $b_j(t, x')$ $(j=1,\cdots,n-1)$ defined on the boundary to $[0, T]\times \boldsymbol{R}_+^n$ as follows: $c(t, x)=c(t, x')$ and $b_j(t, x)=b_j(t, x')$. Then we shall show that conditions (3.7) and (3.8) hold in $[0, T]\times \overline{\boldsymbol{R}}_+^n$ and conditions (3.9), (3.10) and (3.11) hold on the boundary $[0, T]\times(\overline{\boldsymbol{R}}_+^n-\boldsymbol{R}_+^n)$.

First by (C_1), (1.3) and (2.1) we see that relations

$$\begin{aligned} A &= (a_{nn}^2c^2+a_{nn}a_nc+a_n^2)+a_{nn}, \\ a_{nn}&A^2-2a_nA_nA-A_n^2 \\ &= a_{nn}\{(a_{nn}c+a_n)^4+(a_{nn}c+a_n)^2(a_{nn}+a_n^2)+(a_{nn}+a_n^2)^2\}, \end{aligned}$$

$$(3.15)\quad 2(a_{nn}c+a_n)A+(a_{nn}c^2+1)A_n=(a_{nn}c+a_n)\left\{(a_{nn}c+a_n)^2+(a_{nn}+a_n^2)\right\}$$

imply the assertions on (3.7), (3.8) and (3.10).
Next, after a little complicated culculation, we arrange $L(\sigma)$ with respect to powers in $a_{nn}c+a_n$. Thus we see that

$$(3.16)\quad L(\sigma)=(a_{nn}c+a_n)^2H(\sigma)+a_{nn}D(\sigma)\left\{(a_{nn}c+a_n)^2+(a_{nn}+a_n^2)\right\},$$

where $D(\sigma)(=D(t,x;\sigma))$ is defined by (2.2). By (C_2), (1.3), (2.1) and (2.2) the condition (3.9) holds on the boundary. Finally by a short calculation the part [ ] of (3.13) vanishes. Hence it follows from (3.13) and (3.15) that

$$(3.17)\quad J(\sigma)=-H(\sigma)\left\{(a_{nn}c+a_n)^2+(a_{nn}+a_n^2)\right\}.$$

Consequently by (1.3) and (C_2) the condition (3.11) holds on the boundary.

Suppose that the quadratic form $L(t,x;\sigma)$ is positive definite in $[0,T]\times\overline{\boldsymbol{R}}^n_+$.[4] Then we shall show the energy inequality (1.5). The foregoing consideration gives that for any t $(0<t\leq T)$

$$\begin{aligned}(3.18)\quad &K_1\left(\left\|\frac{\partial u}{\partial t}(t,\cdot)\right\|^2+\sum_{j=1}^n\left\|\frac{\partial u}{\partial x_j}(t,\cdot)\right\|^2\right)-K_1'\left(\left\|\frac{\partial u}{\partial t}(0,\cdot)\right\|^2+\sum_{j=1}^n\left\|\frac{\partial u}{\partial x_j}(0,\cdot)\right\|^2\right)\\ &\leqq\left|\int_0^t\left((Pu)(s,\cdot),\,A(s,\cdot)\frac{\partial u}{\partial t}(s,\cdot)+\sum_{j=1}^nA_j(s,\cdot)\frac{\partial u}{\partial x_j}(s,\cdot)\right)ds\right|\\ &\quad+|(lower\ order\ term)|,\end{aligned}$$

where A and A_j are defined by (3.14) and constants K_1 and K_1' are independent of u. Note that

$$(3.19)\quad \left(\|u(s,\cdot)\|^2\right)\Big|_0^t\leqq2\left(\int_0^t\|u(s,\cdot)\|^2ds+\int_0^t\left\|\frac{\partial u}{\partial t}(s,\cdot)\right\|^2\right)ds.$$

Using Schwarz inequality it follows from (3.18) and (3.19) that for any t

$$|||u(t,\cdot)|||_1^2\leqq K_2\left(\int_0^t|||u(s,\cdot)|||_1^2dt+\int_0^t\|(Pu)(s,\cdot)\|^2ds+|||u(0,\cdot)|||_1^2\right).$$

Since $\int_0^t\|(Pu)(s,\cdot)\|^2ds+|||u(0,\cdot)|||_1^2$ is increasing in t, we obtain

$$(3.20)\quad |||u(t,\cdot)|||_1^2\leqq K_2e^{K_2t}\left(\int_0^t\|(Pu)(s,\cdot)\|^2ds+|||u(0,\cdot)|||_1^2\right).$$

Set $K=K_2e^{K_2T}$. Then (3.20) leads immediately to the energy inequality (1.5).

Now we shall remove the above assumption. Note that by (3.16) $L(t,x;\sigma)$ is positive difinite on the boundary. Then by the assumption and

4) This assumption may be removed after.

continuity of coefficients there exists a small positive constant δ such that $L(t, x; \sigma)$ is positive definite in $[0, T]\times \boldsymbol{R}^{n-1}\times[0, \delta]$. From the foregoing paragraph this shows that the energy inequality (1.5) holds for u with its support in $[0, T]\times \boldsymbol{R}^{n-1}\times[0, \delta]$. Let us take a real valued function $\psi(x_n)\in C_0^\infty(\overline{\boldsymbol{R}}_+^1)$ such that $\psi(x_n)=1$ in $0\leq x_n\leq\delta/2$ and $\psi(x_n)=0$ in $x_n\geq\delta$ and set $u=\psi u+(1-\psi)u$. Clearly the support of ψu is contained in $[0, T]\times \boldsymbol{R}^{n-1}\times[0, \delta]$ and $B(\psi u)=Bu$ on the boundary. Hence ψu satisfies the energy inequality (1.5). By regarding $(1-\psi)u$ as a solution of Cauchy problem for $P(D)$ the proof of Theorem is complete.

Finally we remark on the case of the uniformly Lopatinskii condition. The operators P and B satisfy the uniformly Lopatinskii condition if and only if on the boundary $a_{nn}c+a_n>0$ and the quadratic form $H(\sigma)$ is positive definite (See the remark of Theorem 2.5). Then the following corollary follows immediately from the proof of Theorem (especially (3.15) and (3.17)).

Corollary. *Suppose that P and B satisfy the uniformly Lopatinskii condition. Then there exists a positive constant K such that for every real $u\in H^2((0, T)\times \boldsymbol{R}_+^n)$ the following energy inequality holds for any t $(0<t\leq T)$:*

$$(3.20)\qquad |||u(t, \cdot)|||_1^2+\int_0^t \langle\langle u(s, \cdot, 0)\rangle\rangle_1^2 ds \leq K\Big(\int_0^t \|(Pu)(s, \cdot)\|^2 ds+\int_0^t \langle\langle (Bu)(s, \cdot, 0)\rangle\rangle_1^2 ds+|||u(0, \cdot)|||_1^2\Big),$$

where

$$\langle\langle u(t, \cdot, 0)\rangle\rangle_1^2=\langle\langle u(t, \cdot, 0)\rangle\rangle^2+\langle\langle \frac{\partial u}{\partial t}(t, \cdot, 0)\rangle\rangle^2+\sum_{j=1}^n \langle\langle \frac{\partial u}{\partial x_j}(t, \cdot, 0)\rangle\rangle^2 .$$

Remark. We first prove the following energy inequality of higher order.

For every $u\in H^{m+3}((0, T)\times R_+^n)$ $(m\geqq 0:$ integer$)$ with $Bu=0$ on the boundary the energy inequality holds: for any $t\in(0, T)$

$$(1.5)'\qquad |||u(t, \cdot)|||_{m+2}^2\leqq K\Big(|||u(0, \cdot)|||_{m+2}^2+|||(Pu)(0, \cdot)|||_m^2 +\int_0^t |||(Pu)(s, \cdot)|||_{m+1}^2 ds\Big),$$

where

$$|||u(s, \cdot)|||_m^2=\sum_{j=0}^m \Big|\Big|\Big|\frac{\partial u}{\partial t}(s, \cdot)\Big|\Big|\Big|_{m-j}^2 .$$

It suffices to prove the case $m=0$. The difference between the proof of

(1.5) and one of (1.5)' is as follows. Replacing u by $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_k}$ in (3.4) and moreover $Bu=0$ by $\frac{\partial}{\partial t}(Bu)=0$, $\frac{\partial}{\partial x_k}(Bu)=0$ $(k=1,\cdots,n-1)$ respectively, the following remaining term arises from (3. 1), (3. 2) and (3. 3):

$$\int_0^t <(Q_1 u)(s, x', 0),\ (Q_2 u)(s, x', 0)> ds,$$

where $Q_j (j=1, 2)$ is a j-th order differential oparator in $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x_k}$ $(k<n)$.

Using the trace inequality, the above integral is estimated by

$$K\left\{\varepsilon\left\|u(t,\cdot)\right\|_2^2+C(\varepsilon)\left\|u(t,\cdot)\right\|_1^2+\left\|u(0,\cdot)\right\|_2^2+\int_0^t\left\|u(s,\cdot)\right\|_2^2 ds\right\},$$

where ε is a arbitrary positive number. To estimate $\frac{\partial^2 u}{\partial x_n^2}$ we use the equation (1. 1). Combining these facts with our method in §3 we obtain (1.5)'.

Secondly to prove the existence and the regularity of the solution of our problem we use the method of approximation. In fact, by the assumption that $a_{nn}>0$ and $a_{nn}\mathfrak{c}-\mathfrak{b}^2$ is positive definite, the condition (C_1) and (C_2) are equivalent to $a_{nn}c+a_n\geqq 0$ and $a_{nn}c+a_n\geqq$ the positive root of $H=0$ with respect to $a_{nn}c+a_n$. Hence P and B are approximated by P and B_ε which satisfy the uniformly Lopatinskii condition, where $B_\varepsilon=\frac{\partial}{\partial x_n}-\sum_{j=1}^{n-1}b_j\cdot$ $\cdot\frac{\partial}{\partial x_j}-(c+\varepsilon)\frac{\partial}{\partial t}+h$.

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