Generalized Minkowski formulas for closed hypersurfaces in a Riemannian manifold

Dedicated to Professor Yoshie Katsurada on her Sixtieth Birthday

By Takao MURAMORI

Introduction.

Our starting point is the following well known formula for an ovaloid F in an Euclidean space E^3 of three dimensions:

$$\iint_F (Kp+H)\,dA = 0, \tag{0.1}$$

where H and K are the mean curvature and the Gauss curvature at a point P of F, p denotes the oriented distance from a fixed point 0 in E^3 to the tangent space of F at P and dA is the area element of F at P. For convex hypersurfaces, these formulas have been obtained by H. Minkowski for $m=2$ **[11]**[1] and by T. Kubota for a general m **[9]** (cf. also **[2]**, p. 64).

As a generalization of this formula for a closed orientable hypersurface, C. C. Hsiung derived the following integral formulas of Minkowski type which are valid in an Euclidean space E^{m+1} **[4]**.

THEOREM A (C. C. Hsiung) *Let V^m be a closed orientable hypersurface twice differentiably imbedded in an Euclidean space E^{m+1} of $m+1$ ($\geqq 3$) dimensions, then*

$$\int_{V^m} H_{\nu+1}p\,dA + \int_{V^m} H_\nu\,dA = 0 \ \textit{for}\ \nu = 0, 1, \cdots, m-1, \tag{0.2}$$

where $H_0=1$, $H_\nu (1\leqq\nu\leqq m)$ be the ν-th mean curvature of V^m at P, p denotes the oriented distance from a fixed point 0 in E^{m+1} to the tangent hyperplane of V^m at P, and dA be the area element of V^m at P.

Extension of this formulas in a Riemannian manifold R^{m+1} has been established by Y. Katsurada **[5]** **[6]**. Main result for a hypersurface V^m in a Riemannian manifold is as follows:

THEOREM B (*Y. Katsurada*) *Let V^m be a closed orientable hypersurface of class C^3 imbedded in an $(m+1)$-dimensional Riemannian manifold R^{m+1} which admits an infinitesimal point transformation, then*

---

1) Numbers in brackets refer to the references at the end of the paper.

$$(0.3)\qquad \int_{V^m} H_1 p\,dA + \frac{1}{2m}\int_{V^m} g^{*ij} \underset{\xi}{\mathscr{L}}\, g_{ij}\,dA = 0,$$

and if the manifold R^{m+1} assumes of constant curvature which includes an Euclidean space,

$$(0.4)\qquad \int_{V^m} H_{\nu+1} p\,dA + \frac{1}{2m}\int_{V^m} H_\nu^{\alpha\beta} B_\alpha^i B_\beta^j \underset{\xi}{\mathscr{L}}\, g_{ij}\,dA = 0 \qquad (1 \leqq \nu \leqq m-1).$$

We use integral formulas of Minkowski type to generalize the Liebmann theorem [**10**] : the only ovaloids with constant mean curvature H in an Euclidean space E^3 are spheres. Extension of this theorem to a convex hypersurface V^m in an $(m+1)$-dimensional Euclidean space E^{m+1} has been given by W. Süss [**17**] (cf. also [**2**], p. 118). The same problem for closed hypersurfaces in an $(m+1)$-dimensional Euclidean space E^{m+1} has been investigated by C. C. Hsiung [**4**].

THEOREM C (*C. C. Hsiung*) *Let V^m be a hypersurface satisfying the condition of Theorem A. Suppose that there exist a point* 0 *in E^{m+1} and an integer s, $1 \leqq s \leqq m$, such that at all points of V^m the support function p is of the same sign, $H_i > 0$, for $i = 1, 2, \cdots, s$, and H_s is constant. Then V^m is a hypersphere.*

Extension of this theorem in a Riemannian manifold has been established by Y. Katsurada [**5**] as follows :

THEOREM D (*Y. Katsurada*) *Let R^{m+1} be a space of constant curvature, V^m a closed orientable hypersurface in R^{m+1}. If there exists a one-parameter group G of conformal transformations of R^{m+1} such that the scalar product $n_i\,\xi^i$ of the normal vector n of V^m and the generating vector ξ of G does not change the sign on V^m and is not identically zero, and if the principal curvatures $k_1, k_2, \cdots, k_m$ at each point of the hypersurface V^m are positive and H_ν is constant for any $\nu\,(1 \leqq \nu \leqq m-1)$, then every point of V^m is umbilic.*

For $\nu = 1$, Y. Katsurada [**6**] obtained the following interesting

THEOREM E (*Y. Katsurada*) *Let R^{m+1} be an Einstein space, V^m a closed orientable hypersurface in R^{m+1}. If there exists a one-parameter group G of conformal transformations of R^{m+1} such that the scalas product $n_i\,\xi^i$ of the normal vector n of V^m and the generating vector ξ of G does not change the sign on V^m and is not identically zero, and if H_1 is constant, then every point of V^m is umbilic.*

The analogous problems for a closed orientable hypersurface V^m in a Riemannian manifold R^{m+1} have been discussed by A. D. Alexandrov [**1**], T. Koyanagi [**8**], M. Okumura [12], [**13**], T. Ōtsuki [**14**], R. C. Reilly [**15**]. M. Tani [**18**], K. Yano [**20**], [**21**], [**22**] and K. Yano and M. Tani [**23**].

The purpose of this paper is to give certain generalization of formulas (0. 3) and (0. 4) of Katsurada, and to obtain some properties of a closed orientable hypersurface in a Riemannian manifold. Notations and general formulas on hypersurfaces are given in §1. In §2, we derive generalized Minkowski formulas. As a special case of §2, the later sections §3 and §4 are devoted to establish several integral formulas of Minkowski type. In §5, we give some properties of a closed orientable hypersurface in a Riemannian manifold R^{m+1}.

The present author wishes to express his sincere thanks to Professor Dr. Yoshie Katsurada for her constant guidance and criticism, and also to Dr. Tamao Nagai for his kind help.

§1. Notations and general formulas on hypersurfaces.

Let R^{m+1} be an $(m+1)$-dimensional orientable Riemannian manifold of class $C^r\,(r\geqq 3)$, and x^i, g_{ij}, "$;i$", R^h_{ijk}, $R_{ij}=R^h_{ijh}$ and R be local coordinates, a Riemannian metric, the operator of covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop ij} \right\}$ formed with the metric g_{ij}, the curvature tensor, the Ricci tensor, and the curvature scalar of R^{m+1} respectively.

We now consider a closed orientable hypersurface V^m of class C^3 imbedded in a Riemannian manifold R^{m+1} whose local parametric expression is

$$x^i = x^i(u^\alpha),$$

where u^α are local coordinates in V^m. Throughout this paper we will agree on the following ranges of indices unless otherwise stated:

$$1 \leqq h, i, j, \cdots \leqq m+1,$$
$$1 \leqq \alpha, \beta, \gamma, \cdots \leqq m,$$
$$0 \leqq \lambda, \mu, \nu, \cdots \leqq m-1.$$

We use the convention that repeated indices imply summation.

If we put

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha},$$

then B^i_α are m linearly independent vectors tangent to V^m. The first fundamental tensor $g_{\alpha\beta}$ of V^m is given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta \tag{1.1}$$

and $g^{\alpha\beta}$ is defined by $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$, where δ^α_γ means the Kronecker deltas. We assume that m vectors $B^i_1, B^i_2, \cdots, B^i_m$ give the positive orientation on V^m and we denote by n^i the uniquely determined unit normal vector of V^m such that $B_1^i, B_2^i, \cdots, B_m^i, n^i$ give the positive orientation in R^{m+1}. Denoting by "$;\alpha$" the operation of van der Waerden-Bortolotti covariant differentiation along the hypersurface V^m, we have the equations of Gauss and Weingarten:

$$B^i_{\alpha;\beta} = b_{\alpha\beta} n^i, \tag{1. 2}$$

$$n^i_{;\alpha} = -b^\beta_\alpha B^i_\beta, \tag{1. 3}$$

where $b_{\alpha\beta} = b_{\beta\alpha}$ are components of the second fundamental tensor of V^m and $b^\beta_\alpha = b_{\alpha\gamma} g^{\gamma\beta}$, $b^{\alpha\beta} = b^\beta_\gamma g^{\alpha\gamma}$. We also have the equations of Gauss and Codazzi:

$$R_{hijk} B^h_\alpha B^i_\beta B^j_\gamma B^k_\delta = R_{\alpha\beta\gamma\delta} - (b_{\alpha\gamma} b_{\beta\delta} - b_{\beta\gamma} b_{\alpha\delta}), \tag{1. 4}$$

$$R_{hijk} B^h_\alpha n^i B^j_\beta B^k_\gamma = -(b_{\alpha\beta;\gamma} - b_{\alpha\gamma;\beta}) = -2 b_{\alpha[\beta;\gamma]}, \tag{1. 5}$$

where $R_{\alpha\beta\gamma\delta} = g_{\alpha\varepsilon} R^\varepsilon_{\beta\gamma\delta}$ is the curvature tensor of the hypersurface V^m, and the symbol [ ] means alternating in 2([**16**], p. 14). Contracting (1. 5) by the contravariant tensor $g^{\gamma\alpha}$ and using

$$g^{\alpha\beta} B^i_\alpha B^j_\beta = g^{ij} - n^i n^j, \tag{1. 6}$$

we have

$$R_{ij} n^i B^j_\beta = -(b^\gamma_{\beta;\gamma} - b^\gamma_{\gamma;\beta}) = -2 b^\gamma_{[\beta;\gamma]}. \tag{1. 7}$$

If we denote by $k_1, k_2, \cdots, k_m$ the principal curvature of V^m, that is, the roots of the characteristic equation

$$\left| b_{\alpha\beta} - k g_{\alpha\beta} \right| = 0, \tag{1. 8}$$

then the ν-th mean curvature H_ν is given by

$$\binom{m}{\nu} H_\nu = \sum_{\alpha_1 < \cdots < \alpha_\nu} k_{\alpha_1} \cdots k_{\alpha_\nu} = \sum_{\alpha_1, \cdots, \alpha_\nu} b^{\alpha_1}_{[\alpha_1} \cdots b^{\alpha_\nu}_{\alpha_\nu]}, \tag{1. 9}$$

and $H_0 = 1$. From equations (1. 8) and (1. 9) it follows immeadiately

$$m H_1 = b^\alpha_\alpha \qquad H_m = \frac{b}{g'}, \tag{1. 10}$$

where b and g' are determinants of $b_{\alpha\beta}$ and $g_{\alpha\beta}$ respectively. Moreover we have

$$\binom{m}{2} H_2 = \frac{1}{2} \left(b^\alpha_\alpha b^\beta_\beta - b^\alpha_\beta b^\beta_\alpha \right), \tag{1. 11}$$

$$b^\alpha_\beta b^\beta_\alpha = m^2 H_1^2 - m(m-1) H_2 = m \left\{ m H_1^2 - (m-1) H_2 \right\}. \tag{1. 12}$$

We note here that

(1.13)
$$H_1^2-H_2=\frac{1}{m(m-1)}\left(b_\beta^\alpha b_\alpha^\beta-\frac{1}{m}b_\alpha^\alpha b_\beta^\beta\right)$$
$$=\frac{1}{m^2(m-1)}\sum_{\beta<\alpha}(k_\beta-k_\alpha)^2\geqq 0$$

and consequently, if

(1.14)
$$H_1^2-H_2=0,$$

then

$$k_1=k_2=\cdots=k_m=k,$$

that is

$$b_{\alpha\beta}=kg_{\alpha\beta}.$$

A point of a hypersurface V^m, at which all principal curvatures are equal, is called an umbilical point. Furthermore we have

(1.15)
$$H_1H_\nu-H_{\nu+1}=\frac{\nu!(m-\nu-1)!}{mm!}\sum_{\alpha_1<\cdots<\alpha_{\nu+1}}k_{\alpha_1}\cdots k_{\alpha_{\nu-1}}(k_{\alpha_\nu}-k_{\alpha_{\nu+1}})^2.$$

If $H_1, H_2, \cdots, H_\nu$, $1\leqq\nu\leqq m$ are positive, then

(1.16)
$$H_1\geqq H_2^{\frac{1}{2}}\geqq\cdots\geqq H_\nu^{\frac{1}{\nu}},$$

where the equality at all stage implies $k_1=k_2=\cdots=k_m$. The proof of the formula (1.16) will be omitted here, but can be found in [3], p. 52.

For any ν, if we put

(1.17)
$$H_{(\nu)}^{\alpha\beta}=\frac{1}{(m-1)!}\varepsilon^{\alpha}_{\alpha_1\cdots\alpha_\nu\beta_{\nu+1}\cdots\beta_{m-1}}\varepsilon^{\beta\beta_1\cdots\beta_{m-1}}b_{\beta_1}^{\alpha_1}\cdots b_{\beta_\nu}^{\alpha_\nu},$$

(1.18)
$$H_{(\nu)\beta}=\frac{1}{m!}\varepsilon^{\alpha_1\cdots\alpha_{\nu+1}\gamma_{\nu+2}\cdots\gamma_m}\varepsilon_{\beta\beta_2\cdots\beta_{\nu+1}\gamma_{\nu+2}\cdots\gamma_m}b_{\alpha_1;\alpha_2}^{\beta_2}b_{\alpha_3}^{\beta_3}\cdots b_{\alpha_{\nu+1}}^{\beta_{\nu+1}}$$
$$=\frac{1}{\binom{m}{\nu+1}}b_{[\beta;\alpha_1}^{\alpha_1}b_{\alpha_2}^{\alpha_2}\cdots b_{\alpha_\nu]}^{\alpha_\nu},$$

then we have the following relations

(1.19)
$$g_{\alpha\beta}H_{(\nu)}^{\alpha\beta}=mH_\nu,\quad b_{\alpha\beta}H_{(\nu)}^{\alpha\beta}=mH_{\nu+1},$$

and

(1.20)
$$H_{(\nu);\alpha}^{\alpha\beta}=-\nu mH_{(\nu)\alpha}g^{\alpha\beta},$$

where $\varepsilon_{\alpha_1\cdots\alpha_m}$ denotes the ε-symbol of V^m and the symbol [ ] means alternating in $\nu+1$. In particular

(1. 21) $$H^{\alpha\beta}_{(0)}=g^{\alpha\beta},\quad H_{(0)\gamma}=0,$$

(1. 22) $$H_{(1)\alpha}=\frac{1}{\binom{m}{2}}b^{\beta}_{[\alpha;\beta]}=-\frac{(m-2)!}{m!}R_{ij}n^{i}B^{j}_{\alpha},$$

and in virtue of (1. 11) we have

(1. 23) $$H_{(2)\alpha}=\frac{1}{\binom{m}{3}}b^{\beta}_{[\alpha;\beta}b^{\gamma}_{\gamma]}=\frac{(m-3)!}{m!}\left(C^{\beta}_{\alpha;\beta}-\frac{m(m-1)}{2}H_{2;\alpha}\right),$$

where we put

$$C^{\beta}_{\alpha}=b^{\beta}_{\alpha}b^{\gamma}_{\gamma}-b^{\beta}_{\gamma}b^{\gamma}_{\alpha}$$

(see [8], p. 118).

§ 2. On a generalization of Minkowski Formulas.

We suppose that R^{m+1} admits an one-parameter continuous group G of transformations generated by an infinitesimal transformation

(2. 1) $$\bar{x}^{i}=x^{i}+\xi^{i}\delta\tau,$$

where ξ^{i} are the components of a contravariant vector and $\delta\tau$ is an infinitesimal. In R^{m+1}, we consider a domain U. If the domain U is simply covered by the orbits of transformations generated by ξ^{i}, and ξ^{i} is everywhere of class C^{3} and $\neq 0$ in U, then we call U a regular domain with respect to the vector field (cf. [7], p. 448). If ξ^{i} is a Killing vector, a homothetic Killing vector, a conformal Killing vector, then the group G is called isometric, homothetic and conformal respectively. The vector field ξ^{i} is said to be conformal, homothetic, or Killing when it satisfies

(2. 2) $$\begin{aligned}\underset{\xi}{\mathcal{L}}\, g_{ij}&=\xi_{i;j}+\xi_{j;i}=2\phi(x)g_{ij},\\ \underset{\xi}{\mathcal{L}}\, g_{ij}&=2c\,g_{ij},\\ \underset{\xi}{\mathcal{L}}\, g_{ij}&=0\end{aligned}$$

respectively, where $\underset{\xi}{\mathcal{L}}\, g_{ij}$ denotes the Lie derivative of g_{ij} with respect to the infinitesimal transformation (2. 1), $\phi(x)$ is a scalar function, c is a constant and $\xi_{i}=g_{ij}\xi^{j}$ [19]. When ξ^{i} is a conformal Killing vector, it satisfies

(2. 3) $$\begin{aligned}\underset{\xi}{\mathcal{L}}\left\{{h \atop ij}\right\}&=\xi^{h}_{;i;j}+R^{h}_{ijk}\xi^{k}\\ &=\delta^{h}_{i}\phi_{j}+\delta^{h}_{j}\phi_{i}-\phi^{h}g_{ij},\end{aligned}$$

where $\phi_{i}=\phi_{;i},\ \phi^{h}=\phi_{i}g^{ih}.$

On the hypersurface V^m we can put

(2. 4) $$\xi^i = B^i_\alpha \xi^\alpha + pn^i,$$

where $$p = n_i \xi^i.$$

Hereafter we denote by V^m an m-dimensional closed orientable hypersurface of class C^3 imbedded in a regular domain U with respect to the vector ξ^i. We assume that at any point P on V^m, the vector ξ^i is not on its tangent space.

Let us consider a differential form of $(m-1)$-degree at a point P of the hypersurface V^m, defined by

(2. 5)
$$\begin{aligned}
&((n, f\xi, \underbrace{\delta n, \cdots, \delta n}_{\nu}, \underbrace{dx, \cdots, dx}_{m-\nu-1}))\\
&= \sqrt{g}\,(n, f\xi, \delta n, \cdots, \delta n, dx, \cdots, dx)\\
&= \sqrt{g}\left(n, f\xi, n_{;\alpha_1}, \cdots, n_{;\alpha_\nu}, \frac{\partial x}{\partial u^{\alpha_{\nu+1}}}, \cdots, \frac{\partial x}{\partial u^{\alpha_{m-1}}}\right) du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_{m-1}},
\end{aligned}$$

where the symbol ( ) means a determinant of order $m+1$ whose columns are the components of respective vectors or vector-valued differential forms, and let dx^i be a displacement along the hypersurface V^m, i. e., $dx^i = B^i_\alpha du^\alpha$, g the determinant of a metric tensor g_{ij} of R^{m+1} and f a differentiable scalar function on V^m.

Differentiating exteriorly, we have

(2. 6)
$$\begin{aligned}
&d((n, f\xi, \delta n, \cdots, \delta n, dx, \cdots, dx))\\
&= ((\delta n, f\xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) + ((n,\\
&\quad df\xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) + ((n, f\delta\xi,\\
&\quad \delta n, \cdots, \delta n, dx, \cdots, dx)) + \nu((n, f\xi, \delta(\delta n),\\
&\quad \delta n, \cdots, \delta n, dx, \cdots, dx)).
\end{aligned}$$

On substituting (1. 3) into the first term of the right-hand member of (2. 6), we obtain

(2. 7)
$$\begin{aligned}
&((\delta n, f\xi, \delta n, \cdots, \delta n, dx, \cdots, dx))\\
&= (-1)^{m-\nu}\, m!\; H_{\nu+1}\, fp\,dA,
\end{aligned}$$

where dA denotes the volume element of V^m.

Since the vector $n \times \underbrace{\delta n \times \cdots \times \delta n}_{\nu} \times \underbrace{dx \times \cdots \times dx}_{m-\nu-1}$ is orthogonal to the normal vector n and $\delta n^i = -b_\alpha^\beta B_\beta^i du^\alpha$, the second and the third terms of the right-hand member of (2. 6) become as follows :

$$(2.8)\qquad ((n, df\xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) = (-1)^{m-\nu}\frac{m!}{m} H_{(\nu)}^{\alpha\beta}\xi_\alpha f_\beta dA,$$

$$(2.9)\qquad ((n, f\delta\xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) = (-1)^{m-\nu}\frac{m!}{2m} f H_{(\nu)}^{\alpha\beta} B_\alpha^i B_\beta^j \underset{\xi}{\mathcal{L}}\, g_{ij} dA,$$

where
$$f_\alpha = \frac{\partial f}{\partial u^\alpha}, \quad \xi_\alpha = B_\alpha^i \xi_i.$$

Since we have

$$(2.10)\qquad \delta(\delta n^i) = (b_{\alpha;\beta}^\gamma B_\gamma^i + b_\alpha^\gamma b_{\gamma\beta} n^i) du^\alpha \wedge du^\beta,$$

the fourth term of the right-hand member of (2. 6) becomes

$$(2.11)\qquad ((n, f\xi, \delta(\delta n), \delta n, \cdots, \delta n, dx, \cdots, dx)) = (-1)^{m-\nu-1}\, m!\ f\xi^\alpha H_{(\nu)\alpha} dA.$$

Accordingly by means of (2. 7), (2. 8), (2. 9) and (2. 11) it follows that

$$(2.12)\qquad \begin{aligned} &\frac{1}{m!} d((n, f\xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) \\ &= (-1)^{m-\nu}\Big\{(H_{\nu+1} p\, dA + \frac{1}{2m} H_{(\nu)}^{\alpha\beta} B_\alpha^i B_\beta^j \underset{\xi}{\mathcal{L}}\, g_{ij} dA \\ &\quad -\nu\xi^\alpha H_{(\nu)\alpha} dA) f + \frac{1}{m} H_{(\nu)}^{\alpha\beta}\xi_\alpha f_\beta dA\Big\}. \end{aligned}$$

Integrating both members of (2. 12) over the whole hypersurface V^m and applying the Stokes' theorem, we have

$$(2.13)\qquad \begin{aligned} &\frac{1}{m!}\int_{\partial V^m} ((n, f\xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) \\ &= (-1)^{m-\nu}\int_{V^m}\Big\{(H_{\nu+1} p\, dA + \frac{1}{2m} H_{(\nu)}^{\alpha\beta} B_\alpha^i B_\beta^j \underset{\xi}{\mathcal{L}}\, g_{ij} dA \\ &\quad -\nu\xi^\alpha H_{(\nu)\alpha} dA) f + \frac{1}{m} H_{(\nu)}^{\alpha\beta}\xi_\alpha f_\beta dA\Big\}, \end{aligned}$$

where ∂V^m means the boundary of V^m. Since the hypersurface V^m is closed, it follows that

$$\text{(I)}\qquad \begin{aligned}&\int_{V^m} fH_{\nu+1}p\,dA+\frac{1}{2m}\int_{V^m} fH^{\alpha\beta}_{(\nu)}B^i_\alpha B^j_\beta \underset{\xi}{\mathcal{L}}\, g_{ij}\,dA\\ &-\nu\int_{V^m} f\xi^\alpha H_{(\nu)\alpha}dA+\frac{1}{m}\int_{V^m} H^{\alpha\beta}_{(\nu)}\xi_\alpha f_\beta\,dA=0\,.\end{aligned}$$

This formula is nothing but the generalization of (0. 3) and (0. 4).

§ 3. Minkowski formulas concerning a conformal transformation.

In this section we shall discuss the formula (I) for a conformal infinitesimal point transformation.

Let G be a group of conformal transformations, then from equations (1. 1), (1. 19) and (2. 2) we obtain

$$\text{(3. 1)}\qquad H^{\alpha\beta}_{(\nu)}B^i_\alpha B^j_\beta \underset{\xi}{\mathcal{L}}\, g_{ij}=2m\phi H_\nu\,.$$

Therefore (I) is rewritten in the following form:

$$\text{(3. 2)}\qquad \int_{V^m}\left\{(H_{\nu+1}p+H_\nu\phi-\nu\xi^\alpha H_{(\nu)\alpha})f+\frac{1}{\mathrm{m}}H^{\alpha\beta}_{(\nu)}\xi_\alpha f_\beta\right\}dA=0\,.$$

On substituting $f=\text{const.}$ into the formula (3. 2), we obtain

$$\text{(I)}_c\qquad \int_{V^m}(H_{\nu+1}p+H_\nu\phi-\nu\xi^\alpha H_{(\nu)\alpha})\,dA=0\,.$$

For $\nu=0$, we have

$$\text{(II)}_c\qquad \int_{V^m}(H_1p+\phi)\,dA=0\,.$$

Formula $(\text{II})_c$ is due to Y. Katsurada ([**5**], p. 288).

Especially if our manifold R^{m+1} is an Einstein space, that is,

$$\text{(3. 3)}\qquad R_{ij}=\frac{R}{\mathrm{m}+1}g_{ij}\,,$$

we have $H_{(1)\alpha}=0$ in virtue of (1. 22) and (3. 3), and consequently, for $\nu=1$ from $(\text{I})_c$ we get

$$\text{(3. 4)}\qquad \int_{V^m}(H_2p+H_1\phi)\,dA=0\,.$$

Furthermore, if we assume that R^{m+1} is a space of constant Riemann curvature, that is,

$$\text{(3. 5)}\qquad R_{hijk}=\kappa(g_{hj}g_{ik}-g_{hk}g_{ij})\,,$$

we obtain $H_{(\nu)\alpha}=0$ from (1. 5), (1. 18) and (3. 5), and consequently from $(\text{I})_c$ we have

(3. 6) $$\int_{V^m}(H_{\nu+1}p+H_\nu\phi)\,dA=0\,.$$

This formula (3. 6) is due to Y. Katsurata ([**5**], p. 288).

In the case where R^{m+1} is a Euclidean space E^{m+1} and ξ is the homothetic Killing vector field on E^{m+1} with components $\xi^i=x^i$, x^i being rectangular coordinates with a point in the interior of V^m as origin in the space E^{m+1}, then the orbits of the transformations generated by ξ are the straight lines through the origin and we have

$$\underset{\xi}{\mathcal{L}}\, g_{ij}=2g_{ij}\,.$$

Consequently, from (3. 2) and (3. 6) we obtain

(3. 7) $$\int_{V^m}\left\{(H_{\nu+1}p+H_\nu)f+\frac{1}{m}H^{\alpha\beta}_{(\nu)}x_\alpha f_\beta\right\}dA=0\,,$$

(3. 8) $$\int_{V^m}(H_{\nu+1}p+H_\nu)\,dA=0\,,$$

where $p=n_ix^i$, $x_\alpha=\dfrac{\partial x}{\partial u^\alpha}$. The formula (3. 7) is a generalization of (0. 2) (also [**5**], p. 290).

Now, let us consider a differential form of $(m-1)$-degree at a point of the hypersurface V^m, defined by

$$((n,\ \xi_{;i}n^i,\ \underbrace{dx,\ \cdots,\ dx}_{m-1}))\,.$$

Differentiating exteriorly, and applying the Stokes' theorem, we have

$$\frac{1}{(\mathrm{m}-1)!}\int_{\partial V^m}((n,\ \xi_{;i}n^i,\ dx,\ \cdots,\ dx))$$
$$=(-1)^m\int_{V^m}(R_{ij}n^i\xi^j+mq)\,dA\,,$$

by virtue of (2. 3), where $q=n^i\phi_i$.

On making use of that the hypersurface V^m is closed, we have

(3. 9) $$\int_{V^m}(R_{ij}n^i\xi^j+mq)\,dA=0\,.$$

Let G be the group of homothetic transformations, that is, $\phi\equiv$const., then we have

(3. 10) $$\int_{V^m}R_{ij}n^i\xi^j\,dA=0\,.$$

Formulas (3. 9) and (3. 10) are due to K. Yano ([20], p. 337), who derived these formulas by using the Green's theorem.

§ 4. Integral formulas in R^{m+1} admitting a scalar field such that $\rho_{;i;j}=h(\rho)g_{ij}$.

In this section we assume that the Riemannian manifold R^{m+1} admits a non-constant scalar field ρ such that

$$\rho_{;i;j}=h(\rho)g_{ij}, \quad \rho_i=\rho_{;i}, \tag{4. 1}$$

where $h(\rho)$ is a differentiable function of ρ, and we put

$$\rho^i=B^i_\alpha\rho^\alpha+\alpha n^i \tag{4. 2}$$

on the hypersurface V^m.

We consider a differential form of $(m-1)$-degree at a point P of the hypersurface V^m defined by

$$((n, f\Phi, \underbrace{\delta n, \cdots, \delta n}_{\nu}, \underbrace{dx, \cdots, dx}_{m-\nu-1})),$$

where $\Phi=\rho^i\dfrac{\partial}{\partial x^i}$, $\rho^i=g^{ij}\rho_{;j}$. Differentiating exteriorly and making use of calculations analogous to these in §2, we have the following integral formula:

$$\int_{V^m}\left\{(H_{\nu+1}\alpha+H_\nu h-\nu\rho^\alpha H_{(\nu)\alpha})f+\frac{1}{m}H^{\alpha\beta}_{(\nu)}\rho_\alpha f_\beta\right\}dA=0. \tag{4. 3}$$

where $\alpha=n^i\rho_{;i}$, $\rho_\alpha=\rho_{;i}B^i_\alpha$. On substituting $f=$const. into the formula (4. 3), we obtain

$$\int_{V^m}(H_{\nu+1}\alpha+H_\nu h-\nu\rho^\alpha H_{(\nu)\alpha})dA=0, \tag{I'}$$

in particular for $\nu=0$ we have also

$$\int_{V^m}(H_1\alpha+h)dA=0. \tag{II'}$$

§ 5. Some properties of a closed orientable hypersurface.

In this section we shall show the follwing seven theorems for a closed orientable hypersurface V^m in a Riemannian manifold R^{m+1}.

THEOREM 5. 1. *Let R^{m+1} be a Riemannian manifold which admits a continuous one-parameter group G of conformal transformations and V^m a closed orientable hypersurface such that*

(i) $H_\nu = const.$ *and* $\xi^\alpha H_{(\nu)\alpha} = 0$ *for any* ν

$(1 \leqq \nu \leqq m-1)$,

(ii) $k_1 > 0, k_2 > 0, \cdots, k_m > 0$ *for any* ν

$(2 \leqq \nu \leqq m-1)$,

(iii) *inner product* $p = n_i \xi^i$ *does not change the sign on* V_m.

Then every point of V^m *is umbilic.*

PROOF. On substituting the assumption $\xi^\alpha H_{(\nu)\alpha} = 0$ into the formula $(\mathrm{I})_c$ in §3, we obtain

$$\int_{V^m} (H_{\nu+1} p + H_\nu \phi) dA = 0. \tag{III$_c$}$$

From $(\mathrm{III})_c$ and $(\mathrm{II})_c$ in §3, we obtain

$$\int_{V^m} (H_{\nu+1} p + H_\nu \phi) dA = 0,$$

$$\int_{V^m} (H_1 H_\nu p + H_\nu \phi) dA = 0$$

because of H_ν=constant. Therefore we have

$$\int_{V^m} (H_1 H_\nu - H_{\nu+1}) p dA = 0. \tag{5.1}$$

Due to (1.15) and the assumptions (ii) and (iii), the integrand on the left side of equation (5.1) is non negative, and therefore

$$H_1 H_\nu - H_{\nu+1} = 0,$$

which implies that

$$k_1 = k_2 = \cdots = k_m$$

at all points of the hypersurface V^m. Accordingly every point of V^m is umbilic.

Theorem 5.1 is due to T. Ōtsuki ([**14**], p. 339) for the case where $\nu = 1$ and due to T. Koyanagi ([**8**], p. 121) for the case where $\nu = 2$. In the case where R^{m+1} admits a group G of proper homothetic transformations, Theorem 5.1 has been obtained by K. Yano ([**20**], p. 340) for $\nu = 1$. Especially in the case that R^{m+1} is an Einstein space or a space of constant curvature, Theorem 5.1 becomes Katsurada's ones, i.e., Theorem D and Theorem E stated in the introduction.

THEOREM 5.2. *Let R^{m+1} be a Riemannian manifold which admits a non zero scalar field ρ such that $\rho_{;i;j}=h(\rho)g_{ij}$ and V^m a closed orientable hypersurface such that*

(i) $H_\nu=const.$ *and* $\rho^\alpha H_{(\nu)\alpha}=0$ *for any* ν

$(1\leqq\nu\leqq m-1)$,

(ii) $k_1>0, k_2>0, \cdots, k_m>0$ *for any* ν

$(2\leqq\nu\leqq m-1)$,

(iii) *inner product* $\alpha=n^i\rho_i$ *does not change the sign on* V^m.

Then every point of V^m is umbilic.

PROOF. On substituting the assumption $\rho^\alpha H_{(\nu)\alpha}=0$ into the formula (I′) in § 4, we have

$$\int_{V^m}(H_{\nu+1}\alpha+H_\nu h)\,dA=0. \tag{III′}$$

From (III′) and (II′) in § 4, we obtain

$$\int_{V^m}(H_{\nu+1}\alpha+H_\nu h)\,dA=0,$$

$$\int_{V^m}(H_1H_\nu\alpha+H_\nu h)\,dA=0$$

because of H_ν=constant. Therefore we have

$$\int_{V^m}(H_1H_\nu-H_{\nu+1})\alpha dA=0. \tag{5.2}$$

Due to (1.15) and the assumptions (ii) and (iii), the integrand on the left side of equation (5.2) is non negative, and therefore

$$H_1H_\nu-H_{\nu+1}=0,$$

which implies that

$$k_1=k_2=\cdots=k_m$$

at all points of the hypersurface V^m. Accordingly every point of V^m is umbilic.

For $\nu=1$, Theorem 5.2 becomes Yano's one ([**21**], p. 440).

THEOREM 5.3. *Let R^{m+1} be a Riemannian manifold which admits a continuous one-parameter group G of conformal transformations and V^m a closed orientable hypersurface such that*

(i) $H_1 p+\phi \leqq 0$ (or $\geqq 0$) *and* $\xi^{\alpha} H_{(\nu)\alpha}=0$ *for any* ν $(1 \leqq \nu \leqq m-1)$,

(ii) $k_1>0, k_2>0, \cdots, k_m>0$ *for any* ν $(2 \leqq \nu \leqq m-1)$,

(iii) *inner product* $p=n_i \xi^i$ *does not change the sign on* V^m.

Then every point of V^m *is umbilic.*

PROOF. From our assumption (i) and $(\mathrm{II})_{\mathrm{c}}$ in §3 we have the relation

$$H_1 p=-\phi . \tag{5.3}$$

Substituting (5.3) into the formula $(\mathrm{III})_{\mathrm{c}}$, we obtain

$$\int_{V^m}(H_1 H_\nu-H_{\nu+1}) p d A=0,$$

which hold if and only if

$$H_1 H_\nu-H_{\nu+1}=0 .$$

Then we obtain the conclusion.

Theorem 5.3 for $\nu=1$ is due to Y. Katsurada ([**5**], p. 292).

THEOREM 5.4. *Let* R^{m+1} *be a Riemannian manifold which admits a continuous one-parameter group* G *of conformal transformations and* V^m *a closed orientable hypersurface such that*

(i) $H_{\nu+1} p+H_\nu \phi \leqq 0$ (or $\geqq 0$) *and* $\xi^{\alpha} H_{(\nu)\alpha}=0$ *for any* ν $(1 \leqq \nu \leqq m-1)$,

(ii) $k_1>0, k_2>0, \cdots, k_m>0$,

(iii) *inner product* $p=n_i \xi^i$ *does not change the sign on* V^m.

Then every point of V^m *is umbilic.*

PROOF. If we express the formula $(\mathrm{III})_{\mathrm{c}}$ as follows

$$\int_{V^m}(H_{\nu+1} p+H_\nu \phi) d A=0,$$

then from our assumption (i) we have the relation

$$H_{\nu+1} p=-H_\nu \phi . \tag{5.4}$$

Substituting (5.4) into the formula $(\mathrm{II})_{\mathrm{c}}$ in §3, we obtain

$$\int_{V^m} \frac{1}{H_\nu}(H_1 H_\nu-H_{\nu+1}) p d A=0 . \tag{5.5}$$

Due to (1. 15) and the assumptions (ii) and (iii), the integrand on the left side of equation (5. 5) is non negative and therefore

$$H_1H_\nu - H_{\nu+1} = 0,$$

which implies that

$$k_1 = k_2 = \cdots = k_m$$

at all points of the hypersurface V^m. Accordingly every point of V^m is umbilic.

THEOREM 5. 5. *Let R^{m+1} be a Riemannian manifold which admits a continuous one-parameter group G of conformal transformations and V^m a closed orientable hypersurface such that*

(i) $-\dfrac{\phi}{H_1} \geqq p$ *(or* $\leqq p$*) and* $\xi^\alpha H_{(\nu)\alpha} = 0$ *for any* ν $(1 \leqq \nu \leqq m-1)$,

(ii) $k_1>0, k_2>0, \cdots, k_m>0$ *for any* ν $(2 \leqq \nu \leqq m-1)$ *and* $H_1>0$ *(or* <0*) for* $\nu=1$,

(iii) *inner product* $p = n_i\xi^i$ *does not change the sign on* V^m.

Then every point of V^m is umbilic.

PROOF. By virtue of our assumptions and $(\mathrm{II})_c$ in §3, we obtain the following relation

$$p = -\frac{\phi}{H_1}. \tag{5. 6}$$

Substituting (5. 6) into $(\mathrm{III})_c$ in §3, we obtain

$$\int_{V^m} (H_1H_\nu - H_{\nu+1})p\,dA = 0,$$

which hold if and only if

$$H_1H_\nu - H_{\nu+1} = 0.$$

Then we obtain the conclusion.

Theorem 5. 5 for $\nu=1$ is due to Y. Katsurada ([**5**], p. 293).

THEOREM 5. 6. *Let R^{m+1} be a Riemannian manifold which admits a continuous one-parameter group G of conformal transformations and V^m a closed orientable hypersurface such that*

(i) $-\dfrac{H_\nu}{H_{\nu+1}}\phi \geqq p$ *(or* $\leqq p$*) and* $\xi^\alpha H_{(\nu)\alpha}=0$ *for any* ν

$(1 \leqq \nu \leqq m-1)$,

(ii) $k_1>0, k_2>0, \cdots, k_m>0$,

(iii) *inner product* $p=n_i\xi^i$ *does not change the sign on* V^m.

Then every point of V^m *is umbilic.*

PROOF. The formula $(\mathrm{III})_c$ is rewritten as follows

$$\int_{V^m} H_{\nu+1}\left(p+\frac{H_\nu}{H_{\nu+1}}\phi\right)dA=0.$$

By virtue of our assumptions, we have the following relation

$$p=-\frac{H_\nu}{H_{\nu+1}}\phi. \tag{5.7}$$

Substituting (5.7) into $(\mathrm{II})_c$ in §3, we obtain

$$\int_{V^m}\frac{1}{H_\nu}(H_1H_\nu-H_{\nu+1})p\,dA=0,$$

which holds if and only if

$$H_1H_\nu-H_{\nu+1}=0.$$

Then we obtain the conclusion.

THEOREM 5.7. *Let* R^{m+1} *be a Riemannian manifold which admits a continuous one-parameter group* G *of conformal transformations and* V^m *a closed orientable hypersurface such that*

(i) $H_\nu^{\frac{1}{\nu}}p=-\phi$ *for any* ν $(2\leqq\nu\leqq m-1)$,

(ii) $H_1>0, H_2>0, \cdots, H_\nu>0$,

(iii) *inner product* $p=n_i\xi^i$ *does not change the sign on* V^m.

Then every point of V^m *is umbilic.*

PROOF. On substituting the assumption (i) into the formula $(\mathrm{II})_c$ in §3, we obtain

$$\int_{V^m}(H_1-H_\nu^{\frac{1}{\nu}})p\,dA=0. \tag{5.8}$$

Due to the inequality (1.16) the integrand in the left side of equation (5.8) is non negative, and therefore

$$H_1=H_\nu^{\frac{1}{\nu}},$$

which implies that

$$k_1=k_2=\cdots=k_m$$

at all points of the hypersurface V^m. Then we obtain the conclusion.

The method of calculation referring to a differential form is learned much from the paper [5] of Y. Katsurada.

Department of Mathematics,
Hokkaido University

References

[1] A. D. ALEXANDROV: A characteristic property of spheres, Ann. di Mat. p. appl., 58 (1962), 303–315.

[2] T. BONNESEN und W. FENCHEL: Theorie der konvexen Körper, (Springer, Berlin 1934).

[3] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA: Inequalities, (Cambridge, 1934).

[4] C. C. HSIUNG: Some integral formulas for closed hypersurfaces, Math. Scand., 2 (1954), 286–294.

[5] Y. KATSURADA: Generalized Minkowski formulas for closed hypersurfaces in Riemann space, Ann. di Mat. p. appl., 57 (1962), 283–293.

[6] Y. KATSURADA: On a certain property of closed hypersurfaces in an Einstein space, Comment. Math. Helv., 38 (1964), 165–171.

[7] Y. KATSURADA: On a piece of hypersurface in a Riemannian manifold with mean curvature bounded away from zero, Trans. Am. Math. Soc. 129 (1967). 447–457.

[8] T. KOYANAGI: On a certain property of a closed hypersurface in a Riemann space, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 115–121.

[9] T. KUBOTA: Über die konvex-geschlossenen Mannigfaltigteiten im n-dimensionalen Raume, Sci. Rep. Tôhoku Univ., 14 (1925), 85–99.

[10] H. LIEBMANN: Ueber die Verbiegung der geschlossenen Flächen positiver Krümmung, Math. Ann., 53 (1900), 81–112.

[11] H. MINKOWSKI: Volumen und Oberfläche, Math. Ann., 57 (1903), 447–495.

[12] M. OKUMURA: Certain hypersurfaces of an odd dimensional sphere, Tôhoku Math. Jour., 19 (1967), 381–395.

[13] M. OKUMURA: Sur l'hypersurface localement convexe dans une variété riemannienne à courbure constante, C.R. Acad. Sci. Paris 270 (1970), 541–542.

[14] T. ŌTSUKI: Integral formulas for hypersurfaces in a Riemannian manifold and their applications, Tôhoku Math. Jour., 17 (1965), 335–348.

[15] R. C. REILLY: Extrinsic rigidity theorems for compact submanifolds of the sphere, Jour. of Diff. Geom., 4 (1970), 487–497.

[16] J. S. SCHOUTEN: Ricci-Calculus, (Second edition) (Sprineger, Berlin 1954).

[17] W. SÜSS: Zur relativen Differentialgeometrie V: Über Eihyperflächen im R^{n+1}, Tôhoku Math. Jour., 30 (1929), 202–209.

[18] M. TANI: On hypersurfaces with constant k-th mean curvature, Kôdai Math. Sem. Rep., 20 (1968), 94–102.

[19] K. YANO: The theory of Lie derivatives and its applications, (Amsterdam, 1957).

[20] K. YANO: Closed hypersurfaces with constant mean curvature in a Riemannian manifold, Jour, Math. Soc. Japan, 17 (1965), 333–340.

[21] K. YANO: Notes on hypersurfaces in a Riemannian manifold, Canadian Jour. Math., 19 (1967), 439–446.

[22] K. YANO: Integral formulas in Riemannian Geometry, (Dekker, 1970).

[23] K. YANO and M. TANI: Integral formulas for closed hypersurfaces, Kôdai Math. Sem. Rep., 21 (1969), 335–349.

(Received, May 6, 1971)