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Characteristic function of Cayley projective plane as a harmonic manifold

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Abstract. Any locally rank one Riemannian symmetric space is a harmonic manifold. We give the characteristic function of a Cayley projective plane as a harmonic manifold. The aim of this work is to show the explicit form of the characteristic function of the Cayley projective plane.

Key words: Cayley projective plane, harmonic manifold, characteristic function.

1. Introduction

Let $M = (M, g)$ be an m-dimensional Riemannian manifold and $\theta_p(q) = \sqrt{\det(g_{ij}(q))}$ (resp. $\Theta_p(q) = r_p(q)^{m-1}\theta_p(q)$) be the volume density function (resp. the density function of the geodesic sphere $S(p, r_p(q))$) in a normal coordinate neighborhood $U_p(x^1, \ldots, x^m)$ centered at $p \in M$, where $r_p(q) = d(p, q)$ is the geodesic distance from p to q in U_p.

Definition 1 A Riemannian manifold $M = (M, g)$ is said to be locally harmonic if the volume density function θ_p is a radial function (correspondingly, the density function Θ_p of the geodesic sphere $S(p, r_p(q))$ is also a radial function).

In the sequel, we call a locally harmonic manifold briefly a harmonic manifold. Let $M = (M, g)$ be a harmonic manifold. Then, it is shown that the density function Θ_p does not depend on the choice of p. A rank one symmetric space is a harmonic manifold. There are several equivalent definitions for harmonic manifolds ([1, pp. 156]). One of them is as follows:

Theorem 2 *A Riemannian manifold $M = (M, g)$ is a harmonic manifold if and only if the equality*

$$\triangle\Omega = f_p(\Omega_p) \qquad \left(\Omega_p = \frac{1}{2}r_p^2\right)$$

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holds for a certain smooth function f_p on $[0, \varepsilon(p))$, where $\varepsilon(p)$ is the injectivity radius at $p \in M$.

We note that the function f_p in Theorem 2 does not depend on the choice of $p \in M$ ([1, Proposition 6.16]) then the function $f = f_p$ $(p \in M)$ is called the characteristic function of a harmonic manifold $M = (M, g)$. The characteristic function plays an important role in the geometry of harmonic manifolds and its applications [4], [6], [7], [9]. The characteristic functions of rank one symmetric spaces have been obtained except for Cayley projective plane $\mathfrak{C}\mathrm{P}^2$ and its non-compact dual $\mathfrak{C}\mathrm{H}^2$ (Cayley hyperbolic plane) [6], [7], [9]. So it seems natural to determine the characteristic functions for Cayley projective plane $\mathfrak{C}\mathrm{P}^2$ and Cayley hyperbolic plane $\mathfrak{C}\mathrm{H}^2$ in order to complete the table of the characteristic functions of rank one symmetric spaces. In this article, we shall prove the following theorems 3 and 4.

Theorem 3 *Let $\mathfrak{C}P^2$ be a Cayley projective plane. Then, the characteristic function as a harmonic manifold is given by*

$$f(\Omega) = 1 + \sqrt{\frac{\Omega}{2}}\left\{15 \cot \sqrt{\frac{\Omega}{2}} - 7 \tan \sqrt{\frac{\Omega}{2}}\right\}. \tag{1.1}$$

Theorem 4 *Let $\mathfrak{C}H^2$ be a Cayley hyperbolic plane. Then, the characteristic function as a harmonic manifold is given by*

$$^*f(\Omega) = 1 + \sqrt{\frac{\Omega}{2}}\left\{15 \coth \sqrt{\frac{\Omega}{2}} + 7 \tanh \sqrt{\frac{\Omega}{2}}\right\}. \tag{1.2}$$

Our arguments in this paper are much indebted to the article by R. Brown and A. Gray [2] and I. Yokota [10]. We aimed our paper to be self-contained as much as possible. The authors thank to the referee for the kind suggestions.

2. Preliminaries

In this section, we prepare a brief review on on algebraic background which plays a basic role in the geometry of Cayley projective plane $\mathfrak{C}\mathrm{P}^2$. Let $\mathfrak{C}$ be the Cayley division normed algebra with the multiplicative unity 1 and positive definite bilinear form $\langle\, ,\, \rangle$ where associated norm $\|\cdot\|$ satisfies $\| ab \| = \| a \| \cdot \| b \|$ for $a, b \in \mathfrak{C}$. Every element $a \in \mathfrak{C}$ is written as $a = \alpha + a_0$, where α is a real number and $\langle a_0, 1 \rangle = 0$, where a_0 is said to be purely imaginary. We denote by $\bar{a}$ the conjugate of $a = \alpha + a_0$ defined by $\bar{a} = \alpha - a_0$. we may easily check that $a\bar{a} = \bar{a}a = \langle a, a \rangle = 1 = \| a \|^2$ holds for any $a \in \mathfrak{C}$ and further, by linearizing the equality $a\bar{a} = \langle a, a \rangle 1$, we have

$$a\bar{b} + b\bar{a} = \bar{a}b + \bar{b}a = 2\langle a, b\rangle 1 \tag{2.1}$$

for any a, $b \in \mathfrak{C}$. A canonical basis of $\mathfrak{C}$ is defined as a basis of the form $\{1, e_1, \ldots, e_7\}$ for which $\langle e_i, e_j \rangle = \delta_{ij}$, $e_i^2 = -1$, $e_ie_j + e_je_i = 0$ $(1 \leq i \neq j \leq 7)$ satisfying the following multiplicative operations given by the following figure:

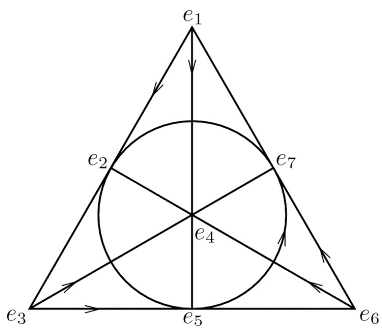

Figure 1.

We denote by $\mathbf{D}_4$ the Lie algebra consisting of linear maps $A : \mathfrak{C} \to \mathfrak{C}$ such that $\langle Aa, b \rangle = -\langle a, Ab \rangle$ for a, $b \in \mathfrak{C}$. It is well-known that $\mathbf{D}_4$ is the compact simple Lie algebra over real number $\mathbb{R}$ with an outer automorphism $Aut(\mathbf{D}_4)/Inn(\mathbf{D}_4)$ of order 3. $Aut(\mathbf{D}_4)/Inn(\mathbf{D}_4)$ is isomorphic to the symmetric group on 3 letters $\mathfrak{S}_3$. Namely, there exist κ, $\lambda \in Aut(\mathbf{D}_4)$ which generate $Aut(\mathbf{D}_4)/Inn(\mathbf{D}_4)$ and satisfy the relations $\lambda^3 = 1$, $\kappa^2 = 1$, $\kappa\lambda\kappa = \lambda^2$. Here, we may choose κ and λ as follows. Let $\{e_i\} = \{e_0 = 1, e_1, \ldots, e_7\}$ be a canonical orthonormal basis of $\mathfrak{C} = (\mathfrak{C}, \langle\,,\,\rangle)$ and $\mathbf{D}_4$ be the real Lie algebra of skew-symmetric endomorphisms of $\mathfrak{C} = (\mathfrak{C}, \langle\,,\,\rangle)$. Now, we define $G_{ij} \in \mathbf{D}_4$ and $F_{ij} \in \mathbf{D}_4$ $(i \neq j,\ i,\ j = 0, 1, \ldots, 7)$ be the linear endomorphisms of $\mathfrak{C}$ defined respectively by

$$G_{ij}e_j = e_i, \quad G_{ij}e_i = -e_j, \quad G_{ij}e_k = 0 \quad (k \neq i, j) \tag{2.2}$$

and

$$F_{ij}e_i = \frac{1}{2}e_ie_j, \quad F_{ij}e_j = -\frac{1}{2}e_ie_j \quad (i \geq 1, 0 \leq j \leq 7), \tag{2.3}$$

and

$$F_{ij}a = \frac{1}{2}e_j(e_ia) \quad (i \neq 0, j \neq 0, i \neq j)$$

for any $a \in \mathfrak{C}$. Then, we may easily check that $\{G_{ij}\}$ (resp. $\{F_{ij}\}$) $(i < j)$ is a basis of $\mathbf{D}_4$. We here define linear endomorphisms κ, π and λ on $\mathbf{D}_4$ respectively by

$$\begin{gathered} \kappa(G_{ij}) = G_{ij} \quad (i, j \geq 1), \quad \kappa(G_{0i}) = -G_{0i} \quad (i \geq 1), \\ \pi(G_{ij}) = F_{ij} \quad (i \neq j) \quad \text{and} \quad \lambda = \pi\kappa. \end{gathered} \tag{2.4}$$

Then, we see that κ and λ satisfy the required relations and further, the following identity

$$(\lambda(A)a)b + a(\lambda^2(A)b) = \kappa(A)(ab) \tag{2.5}$$

holds for any $A \in \mathbf{D}_4$ and any a, $b \in \mathfrak{C}$ [10]. The identity (2.5) is called the principle of triality of $\mathbf{D}_4$.

Now, for a, b, $c \in \mathfrak{C}$, we define $T(a,b)$, $G(a,b)$, $D(a,b) \in \mathbf{D}_4$ as follows:

$$\begin{aligned} T(a,b)c &= 4\langle a, c\rangle b - 4\langle b, c\rangle a, \\ G(a,b)c &= \bar{a}(bc) - \bar{b}(ac), \\ D(a,b)c &= (cb)\bar{a} - (ca)\bar{b}. \end{aligned} \tag{2.6}$$

Then, they satisfy

$$\begin{gathered} \lambda(T(a,b)) = -G(a,b), \quad \lambda^2(T(a,b)) = -D(a,b), \\ \kappa(T(a,b)) = T(\bar{a},\bar{b}), \quad \kappa(G(a,b)) = D(\bar{a},\bar{b}), \end{gathered} \tag{2.7}$$

and further

$$\begin{aligned} \langle T(a,b)c, d\rangle &= 4\big(\langle a, c\rangle\langle b, d\rangle - \langle a, d\rangle\langle b, c\rangle\big) \\ \langle G(a,b)c, d\rangle &= \langle ad, bc\rangle - \langle ac, bd\rangle, \\ \langle D(a,b)c, d\rangle &= \langle da, cb\rangle - \langle ca, db\rangle \end{aligned} \tag{2.8}$$

for any a, b, c, $d \in \mathfrak{C}$.

We denote by $\mathbf{B}_4$ the real Lie algebra consisting of 9×9 skew-symmetric matrices. Now we shall define a 16-dimensional representation of the Lie algebra $\mathbf{B}_4$ on the real vector space $V = V_2 = \mathfrak{C} \oplus \mathfrak{C}$. First, we regard each $X \in \mathbf{B}_4$ as a 9×9 skew-symmetric matrix and the last column vector as an element $a \in \mathfrak{C}$. Further, considering the ordinary inclusion of $\mathbf{D}_4$ in $\mathbf{B}_4$ we may write as follows:

$$X = A + M_a, \tag{2.9}$$

where $A \in \mathbf{D}_4$ and $M_a = \left(\begin{smallmatrix} 0 & 2a \\ -2a & 0 \end{smallmatrix}\right)$. Now, we define an action of $\mathbf{B}_4$ on V by

$$A(b,c) = (\lambda(A)b, \lambda^2(A)c) \tag{2.10}$$

for $A \in \mathbf{D}_4$ and

$$M_a(b,c) = (a\bar{c}, -\bar{b}a) \tag{2.11}$$

for $(b,c) \in \mathfrak{C} \oplus \mathfrak{C}$. Then, we may check that the above action of $\mathbf{B}_4$ on $\mathfrak{C} \oplus \mathfrak{C}$ defines a representation of the real Lie algebra $\mathbf{B}_4$ on $\mathfrak{C} \oplus \mathfrak{C}$ ([2, pp. 46]). The vector space $V = \mathfrak{C} \oplus \mathfrak{C}$ has a positive definite symmetric bilinear form $\langle\, ,\, \rangle$ given by $\langle (a,c),(b,d)\rangle = \langle a,b\rangle + \langle c,d\rangle$ for a, b, c, $d \in \mathfrak{C}$. Then each element of $\mathbf{B}_4$ is skew-symmetric with respect to the bilinear form $\langle\, ,\, \rangle$.

3. The curvature tensor of the Cayley projective plane

Let $\mathfrak{C}\mathrm{P}^2 = (F_4/spin(9), g)$ be Cayley projective plane equipped with a Riemannian metric g defined by a bi-invariant Riemannian metric on the compact Lie group F_4. Then, it is well known that $\mathfrak{C}\mathrm{P}^2$ is a compact rank one symmetric space and further the holonomy group is isomorpic to Spin(9) ([2, Examples]). It is easily checked that the corresponding Cartan decomposition is given by

$$\mathbf{F}_4 = \mathbf{B}_4 \oplus \mathfrak{m}, \tag{3.1}$$

where $\mathfrak{m} = \{(a,b) \in \mathfrak{C} \times \mathfrak{C}\} \cong \mathfrak{C} \oplus \mathfrak{C}$, which can be identified with the tangent space $T_o(\mathfrak{C}\mathrm{P}^2)$ at the origin $o = \mathrm{Spin}(9)$. Further, we may also see that the linear isotropy representation of the isotropy group Spin(9) on $\mathfrak{m} \cong \mathfrak{C} \oplus \mathfrak{C}$ is equivalent to the representation of the group Spin(9) on $V = \mathfrak{C}\oplus\mathfrak{C}$ defined by (2.10) and (2.11) in §2. From the above observation identifying the tangent space $(T_o(\mathfrak{C}\mathrm{P}^2), g_o)$ with $(\mathfrak{C} \oplus \mathfrak{C}, \langle\ ,\ \rangle)$, we see that the curvature tensor R of the Cayley projective plane $\mathfrak{C}\mathrm{P}^2$ at the origin o is given algebraically by the following formula:

$$R(\,(a,b) \wedge (c,d)\,) = \frac{1}{4}\{D(a,c) + G(b,d) + M_{ad-cb}\} \tag{3.2}$$

for a, b, c, $d \in \mathfrak{C}$ and a positive real number μ ([2, Example 4, pp. 52]). Here, we assume that the curvature tensor R is defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{3.3}$$

for any smooth vector fields X, Y on $\mathfrak{C}\mathrm{P}^2$ where ∇ denotes the Levi-Civita connection of the Riemannian metric g. Now, we shall rewrite (3.2) to the more explicit form. From (3.2) with (3.3), taking account of (2.6)∼(2.8), we have

$$\begin{aligned} &R\big(\,(a,b),(c,d)\,\big)(u,v)\\ &\quad = \frac{1}{4}\big\{D(a,c)(u,v) + G(b,d)(u,v) + M_{ad-cb}(u,v)\big\} \end{aligned} \tag{3.4}$$

for a, b, c, d, u, $v \in \mathfrak{C}$. Here, from (2.6)∼(2.8), (2.10) and (2.11), we get

$$\begin{aligned} D(a,c)(u,v) &= \big(\,\lambda(D(a,c))u, \lambda^2(D(a,c))v\big)\\ &= \big(-T(a,c)u, G(a,c)v\big)\\ &= \big(-4\langle a,u\rangle c + 4\langle c,u\rangle a, \bar{a}(cv) - \bar{c}(av)\big), \end{aligned} \tag{3.5}$$

$$\begin{aligned} G(b,d)(u,v) &= \big(\lambda(G(b,d)u, \lambda^2(G(b,d)v)\big)\\ &= (-\lambda^2(T(b,d)u, -T(b,d)v\big)\\ &= (D(b,d)u, -T(b,d)v\big)\\ &= \big(\,(ud)\bar{b} - (ub)\bar{d}, -4\langle b,v\rangle d + 4\langle d,v\rangle b\,\big), \end{aligned} \tag{3.6}$$

$$M_{ad-cb}(u,v) = \big(\,(ad-cb)\bar{v}, -\bar{u}(ad-cb)\,\big). \tag{3.7}$$

Thus, from (3.4), taking account of (3.5)∼(3.7), we have

$$
\begin{aligned}
&R\big((a,b),(c,d) \big)(u,v)\\
&\quad = \frac{1}{4}\big(-4\langle a,u\rangle c + 4\langle c,u\rangle a + (ud)\bar{b} - (ub)\bar{d} + (ad-cb)\bar{v},\\
&\qquad\qquad -4\langle b,v\rangle d + 4\langle d,v\rangle b + \bar{a}(cv) - \bar{c}(av) - \bar{u}(ad-cb)\big)
\end{aligned}
\tag{3.8}
$$

for a, b, c, d, u, $v \in \mathfrak{C}$ ([3, (1.7), pp. 269]). Now, let $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be a canonical basis of $\mathfrak{C}$ and we set

$$
\begin{aligned}
&y_0 = (1,0), && y_1 = (e_1,0), && y_2 = (e_2,0), && y_3 = (e_3,0),\\
&y_4 = (e_4,0), && y_5 = (e_5,0), && y_6 = (e_6,0), && y_7 = (e_7,0),\\
&y_{\bar{0}} = (0,1), && y_{\bar{1}} = (0,e_1), && y_{\bar{2}} = (0,e_2), && y_{\bar{3}} = (0,e_3),\\
&y_{\bar{4}} = (0,e_4), && y_{\bar{5}} = (0,e_5), && y_{\bar{6}} = (0,e_6), && y_{\bar{7}} = (0,e_7).
\end{aligned}
\tag{3.9}
$$

Then, $\{y_0, y_1, \ldots, y_7, y_{\bar{0}}, \ldots, y_{\bar{7}}\}$ is regarded as an orthonormal basis of $(T_0(\mathfrak{C}P^2), g_0)$ and hence, from the formula (3.8), taking account of (3.9) and Figure 1, we have

$$
\begin{aligned}
&R(y_i, y_j)y_i = -y_j, \qquad (i \neq j)\\
&R(y_i, y_{\bar{j}})y_i = -\frac{1}{4}y_{\bar{j}}
\end{aligned}
\tag{3.10}
$$

and further,

$$R(y_i, y_j)y_k = 0, \qquad (k \neq i, j) \tag{3.11}$$

$$R(y_{\bar{i}}, y_j)y_{\bar{i}} = -\frac{1}{4}y_j, \tag{3.12}$$

$$R(y_{\bar{i}}, y_{\bar{j}})y_{\bar{i}} = -y_{\bar{j}}, \tag{3.13}$$

$$R(y_{\bar{i}}, y_{\bar{j}})y_{\bar{k}} = 0 \qquad (k \neq i, j) \tag{3.14}$$

for $0 \leq i$, j, $k \leq 7$.

4. Proofs of Theorems 3 and 4

First, let $\{e_0 = 1, e_1, \ldots, e_7\}$ be a canonical basis of $\mathfrak{C}$ and $\{y_0, y_1, \ldots, y_7, y_{\bar{0}}, y_{\bar{1}}, \ldots, y_{\bar{7}}\}$ be the basis of the real vector space $T_o(\mathfrak{C}\mathrm{P}^2)$ of the Cayley projective plane $\mathfrak{C}\mathrm{P}^2 = (F_4/\mathrm{Spin}(9), g)$ at the origin $o = \mathrm{Spin}(9)$

can be identified with $V = \mathfrak{C} \oplus \mathfrak{C}$ with the canonical positive definite symmetric bilinear form $\langle\, ,\, \rangle$ defined in Section 2. Then, it follows that $\{y_0, y_1, \ldots, y_7, y_{\bar{0}}, y_{\bar{1}}, \ldots, y_{\bar{7}}\}$ is an orthonormal basis of $(V, \langle\, ,\, \rangle)$. We now identify $(T_o(\mathfrak{C}\mathrm{P}^2), g_o)$ with the vector space $(V, \langle\, ,\, \rangle)$ by the above identification.

Now, we denote by $\gamma = \gamma(s)$ the normal geodesic in $(\mathfrak{C}\mathrm{P}^2, g)$ through the origin $o = \gamma(0)$ with the initial direction $\gamma'(0) = y_0$. Further, we set $y_0(s) = \gamma'(s)$ and assume that the vector fields, $y_1(s)$, $\ldots$, $y_7(s)$, $y_{\bar{0}}(s)$, $\ldots$, $y_{\bar{7}}(s)$ are parallel along γ satisfying

$$y_i(0) = y_i \quad (1 \leq i \leq 7) \text{ and } y_{\bar{k}}(0) = y_{\bar{k}} \quad (0 \leq k \leq 7). \tag{4.1}$$

Then, we can check that $\{y_0(s), y_1(s), \ldots, y_7(s), y_{\bar{0}}(s), y_{\bar{1}}(s), \ldots, y_{\bar{7}}(s)\}$ is an orthonormal frame field along γ. Now, let $Y_i(s)$ $(1 \leq i \leq 7)$ and $Y_{\bar{k}}(s)$ $(0 \leq k \leq 7)$ be the Jacobi vector fields along γ satisfying the following conditions

$$\begin{gathered} Y_i(0) = 0,\ Y_{\bar{k}}(0) = 0 \quad \text{and} \\ Y_i'(0) = (\nabla_{\gamma'} Y_i)(0) = y_i, \quad Y_{\bar{k}}'(0) = (\nabla_{\gamma'} Y_{\bar{k}})(0) = y_{\bar{k}}, \end{gathered} \tag{4.2}$$

for $1 \leq i \leq 7$, $0 \leq k \leq 7$. Then, we set as follows along γ:

$$\begin{aligned} Y_i(s) &= \sum_{j=1}^{7} a_{ji}(s) y_j(s) + \sum_{l=0}^{7} a_{\bar{l}i} y_{\bar{l}}(s), \\ Y_{\bar{k}}(s) &= \sum_{j=1}^{7} a_{j\bar{k}}(s) y_j(s) + \sum_{l=0}^{7} a_{\bar{l}\bar{k}}(s) y_{\bar{l}}(s), \end{aligned} \tag{4.3}$$

for $1 \leq i \leq 7$, $0 \leq k \leq 7$ and

$$\begin{aligned} R(\gamma'(s), y_i(s))\gamma'(s) &= \sum_{j=1}^{7} K_{ij}(s) y_j(s) + \sum_{l=0}^{7} K_{i\bar{l}}(s) y_{\bar{l}}(s), \\ R(\gamma'(s), y_{\bar{k}}(s))\gamma'(s) &= \sum_{j=1}^{7} K_{\bar{k}j}(s) y_j(s) + \sum_{l=0}^{7} K_{\bar{k}\bar{l}}(s) y_{\bar{l}}(s), \end{aligned} \tag{4.4}$$

for $1 \leq i \leq 7$, $0 \leq k \leq 7$. Then, since $\nabla R = 0$ and the vector fields

$y_i(s)$, $y_{\bar{k}}(s)$ $(1 \le i \le 7,\ 0 \le k \le 7)$ are parallel along γ, we easily see that $K_{ij}(s)\big(=K_{ji}(s)\big)$, $K_{i\bar{k}}(s)\big(=K_{\bar{k}i}(s)\big)$, $K_{\bar{k}\bar{l}}(s)\big(=K_{\bar{l}\bar{k}}(s)\big)$ are all constant along γ. Thus, from (4.4) taking account of (3.10), we have

$$\begin{aligned} K_{ij}(s) &= K_{ji}(s) = -\delta_{ij}, \\ K_{i\bar{k}}(s) &= K_{\bar{k}i}(s) = 0, \\ K_{\bar{k}\bar{l}}(s) &= K_{\bar{l}\bar{k}} = -\frac{1}{4}\delta_{kl}, \end{aligned} \tag{4.5}$$

for $1 \le i,j \le 7$, $0 \le k,l \le 7$. Since $Y_i(s)$, $Y_{\bar{k}}(s)$ $(1 \le i \le 7,\ 0 \le k \le 7)$ are Jacobi vector fields along the geodesic, from (4.3), taking account of (4.4) with (4.5), we have the following system of differential equations along γ:

$$\begin{gathered} a''_{ij} + a_{ij} = 0, \\ a''_{i\bar{k}} = 0, \quad a''_{\bar{l}i} = 0, \\ a''_{\bar{k}\bar{l}} + \frac{1}{4}a_{\bar{k}\bar{l}} = 0. \end{gathered} \tag{4.6}$$

Solving (4.6) under the initial conditions (4.2), we have

$$\begin{aligned} a_{ij}(s) &= \delta_{ij}\sin s, \\ a_{i\bar{k}}(s) &= a_{\bar{k}i}(s) = 0, \\ a_{\bar{k}\bar{l}}(s) &= 2\delta_{kl}\sin\frac{1}{2}s, \end{aligned} \tag{4.7}$$

for $1 \le i,j \le 7$, $0 \le k,l \le 7$.

Now, we define $15 \times 15-$matrix $A(s)$ by

$$A(s) = \begin{pmatrix} a_{ij}(s) & a_{il}(s) \\ a_{\bar{k}j}(s) & a_{\bar{k}\bar{l}}(s) \end{pmatrix} \tag{4.8}$$

for $1 \le i,j \le 7$, $0 \le k,l \le 7$. Then, it is well-known that the following equality

$$\Theta_o(\gamma(s)) = s^{15}\theta_o(\gamma(s)) = \det A(s) \tag{4.9}$$

holds along the geodesic γ. From (4.8) with (4.7), we have

$$\det A(s) = (\sin s)^7 \left(2 \sin \frac{1}{2} s\right)^8 \\ = 16^2 (\sin s)^7 \left(\sin \frac{1}{2} s\right)^8. \tag{4.10}$$

Thus, from (4.9) and (4.10), we have

$$\Theta_o(\gamma(s)) = 16^2 (\sin s)^7 \left(\sin \frac{1}{2} s\right)^8. \tag{4.11}$$

Here, since the Cayley projective plane $\mathfrak{C}P^2 = (F_4/\mathrm{Spin}(9), g)$ is a harmonic manifold, the volume density function θ_o (and hence, the function Θ_o) is a radial function on a normal neighborhood U_o centered at the origin o. Thus, Θ_o is determined by its value along the geodesic γ. Thus, from (4.11), we easily see the function Θ_o is given by

$$\Theta_o(q) = \det A(s) \\ = 16^2 (\sin s)^7 \left(\sin \frac{1}{2} s\right)^8, \tag{4.12}$$

where $q = \gamma(s) \in U_o - \{o\}$ ([3, pp. 269]).

Now, let $\phi(s)$ be a smooth function of s ($0 < s < \epsilon$, $\epsilon > 0$), and consider the function $f(q)$ on U_o defined by $f(q) = \phi(s)$, $s = d(0, q)$, $q \in U_o$. Then, the following equality holds as in [5] with the sign difference:

$$\triangle f = \phi''(s) + \frac{(\Theta_o(\gamma(s)))'}{\Theta_o(\gamma(s))} \phi'(s), \quad q = \gamma(s), \tag{4.13}$$

where $\triangle$ denotes the Laplace-Beltrami operator of $\mathfrak{C}P^2 = (F_4/\mathrm{Spin}(9), g)$. Here, from (4.12), we get

$$\frac{(\Theta_o(\gamma(s)))'}{\Theta_o(\gamma(s))} \\ = \frac{7(\sin s)^6 (\sin((1/2)s))^8 \cos s + 4(\sin s)^7 (\sin((1/2)s))^7 \cos(1/2)s}{(\sin s)^7 (\sin((1/2)s))^8} \\ = 7 \cot s + 4 \cot \frac{1}{2} s$$

$$= \frac{15}{2} \cot \frac{1}{2} s - \frac{7}{2} \tan \frac{1}{2} s. \tag{4.14}$$

We here consider the special case where $\phi(s) = (1/2)s^2$ $(s > 0)$. Then, from (4.13) and (4.14), by direct calculation, we see that

$$\triangle \Omega = 1 + \sqrt{\frac{\Omega}{2}} \left\{ 15 \cot \sqrt{\frac{\Omega}{2}} - 7 \tan \sqrt{\frac{\Omega}{2}} \right\} \tag{4.15}$$

holds on $U_o - \{o\}$. This completes the proof of Theorem 3.

Next, we shall give an outline of the proof of Theorem 4. Let ${}^*\mathfrak{C}\mathrm{P}^2 = ({}^*\mathfrak{C}\mathrm{P}^2, {}^*g)$ be the non-compact dual of the Cayley projective plane $\mathfrak{C}\mathrm{P}^2 = (F_4/\mathrm{Spin}(9), g)$. Then, we see that ${}^*\mathfrak{C}\mathrm{P}^2$ is isometric to the Cayley hyperbolic plane $\mathfrak{C}\mathrm{H}^2 = (F_{4(-20)}/\mathrm{Spin}(9), {}^*g)$ and the corresponding Cartan decomposition of the Lie algebra $\mathbf{F}_{4(-20)}$ of the Lie group $F_{4(-20)}$ is given by

$$\mathbf{F}_{4(-20)} = \mathbf{D}_4 \oplus \sqrt{-1}\mathfrak{m} \tag{4.16}$$

in the complexification $\mathbf{F}_{4(-20)}$ of the Lie algebra $\mathbf{F}_4$. Thus, taking account of (4.16), we easily check that the curvature tensor of $\mathfrak{C}\mathrm{H}^2$ is only sign difference of curvature tensor R of $\mathfrak{C}\mathrm{P}^2$ algebraically. Thus, by suitably modifying the arguments for the case of the Cayley projective plane suitably, we have Theorem 4.

5. Characteristic functions of rank one symmetric spaces

Summing up the results in [6], [7], [9] and ours of the present paper, we have the following list of the characteristic functions for the rank one symmetric spaces.

We here denote by $S^m(1)$, $\mathrm{H}^m(-1)$, $\mathbb{C}\mathrm{P}^n(1)$, $\mathbb{C}\mathrm{H}^n(-1)$, $\mathbb{H}\mathrm{P}^n(1)$, $\mathbb{H}\mathrm{H}^n(-1)$, $\mathfrak{C}\mathrm{P}^2(1)$, $\mathfrak{C}\mathrm{H}^2(-1)$ the m-dimensional sphere of constant sectional curvature 1, m-dimensional hyperbolic space of constant sectional curvature -1, $2n$-dimensional complex projective space of constant holomorphic sectional curvature 1, $2n$-dimensional complex hyperbolic space of constant holomorphic sectional curvature -1, $4n$-dimensional quaternion projective space of constant Q-sectional curvature 1, $4n$-dimensional quaternion hyperbolic space of constant Q-sectional curvature -1, Cayley projective plane and Cayley hyperbolic plane, respectively.

Space	Characteristic function
$S^m(1)$	$f(\Omega) = 1 + (m-1)\sqrt{2\Omega}\cot\left(\sqrt{2\Omega}\right)$
$\mathrm{H}^m(-1)$	$f(\Omega) = 1 + (m-1)\sqrt{2\Omega}\coth\left(\sqrt{2\Omega}\right)$
$\mathbb{C}\mathrm{P}^n(1)$	$f(\Omega) = 1 + \sqrt{\frac{\Omega}{2}}\left\{(2n-1)\cot\left(\sqrt{\frac{\Omega}{2}}\right) - \tan\left(\sqrt{\frac{\Omega}{2}}\right)\right\}$
$\mathbb{C}\mathrm{H}^n(-1)$	$f(\Omega) = 1 + \sqrt{\frac{\Omega}{2}}\left\{(2n-1)\coth\left(\sqrt{\frac{\Omega}{2}}\right) + \tanh\left(\sqrt{\frac{\Omega}{2}}\right)\right\}$
$\mathbb{H}\mathrm{P}^n(1)$	$f(\Omega) = 1 + \sqrt{\frac{\Omega}{2}}\left\{(4n-1)\cot\left(\sqrt{\frac{\Omega}{2}}\right) - 3\tan\left(\sqrt{\frac{\Omega}{2}}\right)\right\}$
$\mathbb{H}\mathrm{H}^n(-1)$	$f(\Omega) = 1 + \sqrt{\frac{\Omega}{2}}\left\{(4n-1)\coth\left(\sqrt{\frac{\Omega}{2}}\right) + 3\tanh\left(\sqrt{\frac{\Omega}{2}}\right)\right\}$
$\mathfrak{C}\mathrm{P}^2(1)$	$f(\Omega) = 1 + \sqrt{\frac{\Omega}{2}}\left\{15\cot\left(\sqrt{\frac{\Omega}{2}}\right) - 7\tan\left(\sqrt{\frac{\Omega}{2}}\right)\right\}$
$\mathfrak{C}\mathrm{H}^2(-1)$	$f(\Omega) = 1 + \sqrt{\frac{\Omega}{2}}\left\{15\coth\left(\sqrt{\frac{\Omega}{2}}\right) + 7\tanh\left(\sqrt{\frac{\Omega}{2}}\right)\right\}$

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