

## On the symmetric algebras associated to graphs with loops

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**Abstract.** We study the symmetric algebra of monomial ideals that arise from graphs with loops. The notion of  $s$ -sequence is investigated for such ideals in order to compute standard algebraic invariants of their symmetric algebra in terms of the corresponding invariants of special quotients of the polynomial ring related to the graphs.

*Key words:* Graphs with loops, graph ideals, symmetric algebra,  $s$ -sequences.

### Introduction

In this paper we are interested in examining the symmetric algebra of monomial ideals (see [3], [11]), in particular of some non-squarefree monomial ideals arising from graphs. In order to compute standard invariants of such symmetric algebra, we investigate cases for which the monomial ideals are generated by  $s$ -sequences ([9], [10], [12]). The notion of  $s$ -sequence is employed to compute the invariants of the symmetric algebra of finitely generated modules ([4]). Our proposal is to compute standard invariants of the symmetric algebra of monomial ideals of graphs in terms of the corresponding invariants of special quotients of the polynomial ring related to such graphs. This computation can be obtained for graph ideals generated by an  $s$ -sequence.

Let  $\mathcal{G}$  be a graph on vertex set  $[n] = \{v_1, \dots, v_n\}$ . An algebraic object attached to  $\mathcal{G}$  is the edge ideal  $I(\mathcal{G})$ , a monomial ideal of the polynomial ring in  $n$  variables  $R = K[X_1, \dots, X_n]$ ,  $K$  a field. When  $\mathcal{G}$  is a simple (or loopless) graph,  $I(\mathcal{G})$  is generated by squarefree monomials of degree 2 in  $R$ ,  $I(\mathcal{G}) = (\{X_i X_j \mid \{v_i, v_j\} \text{ is an edge of } \mathcal{G}\})$ , but when  $\mathcal{G}$  is a graph having loops  $\{v_i, v_i\}$ , among the generators of  $I(\mathcal{G})$  there are also non-squarefree monomials  $X_i^2, i = 1, \dots, n$ . In [13] there are results about monomial ideals of  $R$  that can arise from the edges of a simple graph. Recently some properties of monomial ideals associated to graphs with loops were studied in [6], [7], [8].

The aim of this paper is to investigate classes of graphs with loops and prove that the notion of  $s$ -sequence can be considered in this family of monomial ideals in order to compute algebraic invariants of their symmetric algebra. More precisely, we extend some results of [5] about monomial  $s$ -sequences that generate edge ideals of simple graphs to non-squarefree monomial ideals associated to such graphs on which an appropriate loop is put.

The work is organized as follows. In Section 1, preliminary notations about the theory of  $s$ -sequences are given. In Section 2, the notion of  $s$ -sequence is investigated for edge ideals associated to some classes of graphs with loops, namely star graphs with one loop in any vertex,  $n$ -line graphs with one loop in a selected vertex. In Section 3, we give the structure of the annihilator ideals of these edge ideals generated by an  $s$ -sequence and compute the invariants of their symmetric algebra: the Krull dimension, the multiplicity and the Castelnuovo-Mumford regularity. In particular, we achieve formulas for dimension and multiplicity, and when the edge ideal is generated by a strong  $s$ -sequence we give bounds for regularity in terms of the annihilator ideals.

## 1. Preliminaries and notations

Let's recall the theory of  $s$ -sequences in order to apply it to the examined classes of monomial ideals.

Let  $M$  be a finitely generated module on a Noetherian ring  $R$ , and  $f_1, \dots, f_t$  be the generators of  $M$ . Let  $(a_{ij})$ , for  $i = 1, \dots, t$ ,  $j = 1, \dots, p$ , be the relation matrix of  $M$ . Let  $Sym_R(M)$  be the symmetric algebra of  $M$ , then  $Sym_R(M) = R[T_1, \dots, T_t]/J$ , where  $R[T_1, \dots, T_t]$  is a polynomial ring in the variables  $T_1, \dots, T_t$  over  $R$  and  $J$  is the relation ideal of such algebra, generated by  $g_j = \sum_i a_{ij}T_i$ , for  $i = 1, \dots, t$ ,  $j = 1, \dots, p$ .

If we assign degree 1 to each variable  $T_i$  and degree 0 to the elements of  $R$ , then  $J$  is a graded ideal and  $Sym_R(M)$  is a graded algebra on  $R$ .

Set  $S = R[T_1, \dots, T_t]$  and let  $\prec$  be a monomial order on the monomials of  $S$  in the variables  $T_i$ . With respect to this term order, if  $f = \sum a_\alpha \underline{T}^\alpha$ , where  $\underline{T}^\alpha = T_1^{\alpha_1} \cdots T_t^{\alpha_t}$  and  $\alpha = (\alpha_1, \dots, \alpha_t) \in \mathbb{N}^t$ , we put  $\text{in}_\prec(f) = a_\alpha \underline{T}^\alpha$ , where  $\underline{T}^\alpha$  is the largest monomial in  $f$  such that  $a_\alpha \neq 0$ .

So we can define the monomial ideal  $\text{in}_\prec(J) = (\{\text{in}_\prec(f) \mid f \in J\})$ .

For every  $i = 1, \dots, t$ , we set  $M_{i-1} = Rf_1 + \cdots + Rf_{i-1}$  and let  $\mathcal{I}_i =$

$M_{i-1} :_R f_i$  be the colon ideal. Since  $M_i/M_{i-1} \simeq R/\mathcal{I}_i$ ,  $\mathcal{I}_i$  is the annihilator of the cyclic module  $R/\mathcal{I}_i$ .  $\mathcal{I}_i$  is called an *annihilator ideal* of the sequence  $f_1, \dots, f_t$ .

It results  $(\mathcal{I}_1 T_1, \mathcal{I}_2 T_2, \dots, \mathcal{I}_t T_t) \subseteq \text{in}_{\prec}(J)$ , and the two ideals coincide in degree 1.

**Definition 1.1** The sequence  $f_1, \dots, f_t$  is said to be an *s-sequence* for  $M$  if

$$(\mathcal{I}_1 T_1, \mathcal{I}_2 T_2, \dots, \mathcal{I}_t T_t) = \text{in}_{\prec}(J).$$

When  $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \dots \subseteq \mathcal{I}_t$ ,  $f_1, \dots, f_t$  is said to be a *strong s-sequence*.

If  $R = K[X_1, \dots, X_n]$  is the polynomial ring over a field  $K$ , we can use the Gröbner bases theory to compute  $\text{in}_{\prec}(J)$ .

Let  $\prec$  be any term order on  $K[X_1, \dots, X_n; T_1, \dots, T_t]$  with  $X_1 \prec X_2 \prec \dots \prec X_n \prec T_1 \prec T_2 \prec \dots \prec T_t$ . Then, for any Gröbner basis  $B$  for  $J \subset K[X_1, \dots, X_n, T_1, \dots, T_t]$  with respect to  $\prec$ , we have  $\text{in}_{\prec}(J) = (\{\text{in}_{\prec}(f) \mid f \in B\})$ . If the elements of  $B$  are linear in the  $T_i$ , it follows that  $f_1, \dots, f_t$  is an *s-sequence* for  $M$ .

Let  $M = I = (f_1, \dots, f_t)$  be a monomial ideal of  $R = K[X_1, \dots, X_n]$ . Set  $f_{ij} = f_i/[f_i, f_j]$  for  $i \neq j$ , where  $[f_i, f_j]$  is the greatest common divisor of the monomials  $f_i$  and  $f_j$ .  $J$  is generated by  $g_{ij} = f_{ij}T_j - f_{ji}T_i$  for  $1 \leq i < j \leq t$ . The monomial sequence  $f_1, \dots, f_t$  is an *s-sequence* if and only if  $g_{ij}$  for  $1 \leq i < j \leq t$  is a Gröbner basis for  $J$  for any term order in  $K[X_1, \dots, X_n; T_1, \dots, T_t]$  with  $X_i \prec T_j$ , for all  $i, j$ .

Note that the annihilator ideals of the monomial sequence  $f_1, \dots, f_t$  are the ideals  $I_i = (f_{1i}, f_{2i}, \dots, f_{(i-1)i})$ , for  $i = 1, \dots, t$  (see [4]).

**Theorem 1.1** ([4]) *Let  $f_1, \dots, f_t$  be monomials of  $R$ . If, for all  $i, j, k, l \in \{1, \dots, t\}$  with  $i < j, k < l$  and  $i \neq k, j \neq l$ , it is  $[f_{ij}, f_{kl}] = 1$ , then  $f_1, \dots, f_t$  is an s-sequence.*

**Remark 1.1** ([4]) From the theory of Gröbner bases, if  $f_1, \dots, f_t$  is a monomial *s-sequence* with respect to some admissible term order  $\prec$ , then  $f_1, \dots, f_t$  is an *s-sequence* for any other admissible term order.

We now study the symmetric algebra of a class of monomial modules over the polynomial ring  $R = K[X_1, \dots, X_n]$  that are monomial ideals arising from graphs with loops. Let's introduce some preliminary notions.

Let  $\mathcal{G}$  be a graph and  $V(\mathcal{G}) = \{v_1, \dots, v_n\}$  be the set of its vertices. We put  $E(\mathcal{G}) = \{\{v_i, v_j\} \mid v_i \neq v_j, v_i, v_j \in V(\mathcal{G})\}$  the set of edges of  $\mathcal{G}$  and  $L(\mathcal{G}) = \{\{v_i, v_i\} \mid v_i \in V(\mathcal{G})\}$  the set of loops of  $\mathcal{G}$ . Hence  $\{v_i, v_j\}$  is an edge joining  $v_i$  to  $v_j$  and  $\{v_i, v_i\}$  is a loop of the vertex  $v_i$ . Set  $W(\mathcal{G}) = E(\mathcal{G}) \cup L(\mathcal{G})$ .

If  $L(\mathcal{G}) = \emptyset$ , the graph  $\mathcal{G}$  is said simple or loopless, otherwise  $\mathcal{G}$  is said a graph with loops.

If  $V(\mathcal{G}) = \{v_1, \dots, v_n\}$  and  $R = K[X_1, \dots, X_n]$  is the polynomial ring such that each variable  $X_i$  corresponds to the vertex  $v_i$ , the *edge ideal*  $I(\mathcal{G})$  associated to  $\mathcal{G}$  is the ideal  $(\{X_i X_j \mid \{v_i, v_j\} \in W(\mathcal{G})\}) \subset R$ .

Note that the non-zero edge ideals are those generated by monomials of degree 2. This implies that  $I(\mathcal{G})$  is a graded ideal of  $R$  of initial degree 2, that is  $I(\mathcal{G}) = \bigoplus_{i \geq 2} (I(\mathcal{G})_i)$ . If  $W(\mathcal{G}) = \emptyset$ , then  $I(\mathcal{G}) = (0)$ .

## 2. Graphs with loops and $s$ -sequences

In this section we deal with some classes of graphs introduced in [5]. Our purpose is to extend some results about monomial  $s$ -sequences that generate the edge ideals of such graphs to non-squarefree monomial ideals associated to that graphs when an appropriate loop is added to them.

First, let's consider monomial ideals associated to the edge set of a star graph with a loop having vertex set  $[n] = \{\{v_n\}, \{v_1, \dots, v_{n-1}\}\}$ .

**Theorem 2.1** *Let  $R = K[X_1, \dots, X_n]$ . Let  $\mathcal{G}$  be a connected acyclic graph on  $[n]$  vertices with edge ideal  $I(\mathcal{G}) = (X_1 X_n, X_2 X_n, \dots, X_{n-1} X_n, X_r^2)$ , for  $r = 1, \dots, n$ . Then  $I(\mathcal{G})$  is generated by an  $s$ -sequence.*

*Proof.* Let  $f_1 = X_1 X_n, f_2 = X_2 X_n, \dots, f_{n-1} = X_{n-1} X_n, f_n = X_r^2$  be the generators of  $I(\mathcal{G})$ . We have to show that  $[f_{ij}, f_{kl}] = 1$ , for  $i < j, k < l, i \neq k, j \neq l$  with  $i, j, k, l \in \{1, \dots, n\}$ . We compute  $f_{ij} = f_i / [f_i, f_j]$ , for all  $i < j$ .

If  $i \neq r, n$ , because  $[f_i, f_j] = X_n$ , for  $1 \leq i < j < n$ , and  $[f_i, f_n] = 1$ , it results:  $f_{ij} = X_i$ , for  $1 \leq i < j < n$ , and  $f_{ij} = X_i X_n$ , for  $j = n$ .

If  $i = r, i \neq n$ , one has:  $f_{rj} = X_r$ , for  $r + 1 < j < n$ , and  $f_{rj} = X_n$ , for  $j = n$ .

When  $r = n$ , being  $[f_i, f_j] = X_n$ , it is  $f_{ij} = X_i$  for all  $1 \leq i < j \leq n$ .

Hence, for all  $i, j, k, l, [f_{ij}, f_{kl}] = 1$  and, by Theorem 1.1, we conclude that  $f_1, \dots, f_n$  is an  $s$ -sequence.  $\square$

**Remark 2.1** If we change the order of the generators of  $I(\mathcal{G})$  in Theorem 2.1, they may not form an  $s$ -sequence. See [1, Esempio 4.4.1].

Now, let's consider monomial ideals associated to the edge set of a connected graph with a loop, union of a star graph with a line, having vertex set  $[n + 1] = \{v_n\}, \{v_1, \dots, v_{n-1}\}, \{v_{n+1}\}$ .

**Theorem 2.2** Let  $R = K[X_1, \dots, X_{n+1}]$ . Let  $\mathcal{G}$  be a connected acyclic graph on  $[n + 1]$  vertices with edge ideal  $I(\mathcal{G}) = (X_1X_n, X_2X_n, \dots, X_{n-1}X_n, X_rX_{n+1}, X_r^2)$ , for  $r = 1, \dots, n - 1$ . Then  $I(\mathcal{G})$  is generated by an  $s$ -sequence.

*Proof.* Set  $f_i = X_iX_n$ , for  $i = 1, \dots, n - 1$ ;  $f_n = X_rX_{n+1}$ ;  $f_{n+1} = X_r^2$ .

Compute  $f_{ij} = f_i/[f_i, f_j]$  and  $f_{ji} = f_j/[f_i, f_j]$ , for  $1 \leq i < j \leq n + 1$ .

$$f_{ij} = \begin{cases} X_i, & i, j \leq n - 1; \\ X_iX_n, & i \leq n - 1, i \neq r, j \geq n; \\ X_n, & i = r, j \geq n; \\ X_{n+1}, & i = n, j = n + 1. \end{cases}$$

$$f_{ji} = \begin{cases} X_j, & i, j \leq n - 1; \\ X_rX_{n+1}, & i \leq n - 1, i \neq r, j = n; \\ X_{n+1}, & i = r, j = n; \\ X_r, & i = r, n, j = n + 1; \\ X_r^2, & i \neq r, n, j = n + 1. \end{cases}$$

Then the generators  $g_{ij} = f_{ij}T_j - f_{ji}T_i$  of  $J$ , for  $i < j$ , are the linear forms:

$$g_{ij} = \begin{cases} X_iT_j - X_jT_i, & i, j \leq n - 1; \\ X_{n+1}T_{n+1} - X_rT_n, & i = n, j = n + 1; \\ X_iX_nT_n - X_rX_{n+1}T_i, & i \leq n - 1, i \neq r, j = n; \\ X_iX_nT_{n+1} - X_r^2T_i, & i \leq n - 1, i \neq r, j = n + 1; \\ X_nT_n - X_{n+1}T_r, & i = r, j = n; \\ X_nT_{n+1} - X_rT_r, & i = r, j = n + 1. \end{cases}$$

For a suitable term order  $\prec$ , in particular for the reverse lexicographic order, we want to prove that the  $S$ -pairs  $S(g_{ij}, g_{kl}) = (f_{ij}f_{kl}/[f_{ij}, f_{kl}])T_jT_k - (f_{ji}f_{kl}/[f_{ij}, f_{kl}])T_iT_l$ , with  $i, j, k, l \in \{1, \dots, n + 1\}$ ,  $i < j$ ,  $i < k < l$ , have

a standard expression with respect to the set  $B$  of the  $g_{ij}$  with remainder 0. Note that, to get a standard expression of  $S(g_{ij}, g_{hl})$  is equivalent to find some  $g_{st} \in B$  whose initial term divides the initial term of  $S(g_{ij}, g_{hl})$  and substitute a multiple of  $g_{st}$  such that the remaindered polynomial has a smaller initial term and so on up to the remainder is 0. It is:

$$\text{in}_{\prec}(g_{ij}) = \begin{cases} X_i T_j, & i, j \leq n - 1; \\ X_{n+1} T_{n+1}, & i = n, j = n + 1; \\ X_i X_n T_n, & i \leq n - 1, i \neq r, j = n; \\ X_i X_n T_{n+1}, & i \leq n - 1, i \neq r, j = n + 1; \\ X_n T_n, & i = r, j = n; \\ X_n T_{n+1}, & i = r, j = n + 1. \end{cases}$$

The  $S$ -pairs for which  $[\text{in}_{\prec}(g_{ij}), \text{in}_{\prec}(g_{kl})] = 1$  reduce to 0 (see [2, Proposition 2.4.2]).

So we examine only the  $S$ -pairs such that  $[\text{in}_{\prec}(g_{ij}), \text{in}_{\prec}(g_{kl})] \neq 1$ .

Two cases arise: 1.  $j = l$ , and 2.  $j \neq l$ .

1. For  $j = l$ , it is  $i < k < j \leq n + 1$ , and we have:

$$\begin{aligned} S(g_{ij}, g_{kl}) &= S(g_{ij}, g_{kj}) \\ &= T_j \left( \frac{f_{ij} f_{jk}}{[f_{ij}, f_{kj}]} T_k - \frac{f_{ji} f_{kj}}{[f_{ij}, f_{kj}]} T_i \right) \\ &= \frac{T_j}{[f_{ij}, f_{kj}]} (f_{ij} f_{jk} T_k - f_{ji} f_{kj} T_i). \end{aligned}$$

Note that

$$[f_{ij}, f_{kj}] = \begin{cases} X_n, & \text{for at least one of } i, k \text{ different from } r, \text{ and } j \geq n; \\ 1, & \text{otherwise.} \end{cases}$$

In particular, for  $j \geq n$ ,

$$\text{if } i, k \neq r, \text{ then } S(g_{ij}, g_{kj}) = f_{ji} T_j (X_i T_k - X_k T_i) \xrightarrow{G} 0.$$

if  $i = r, k \neq r$ , then

$$S(g_{ij}, g_{kj}) = \begin{cases} X_{n+1}T_j(X_rT_k - X_kT_i) \xrightarrow{G} 0, & j = n; \\ X_rT_j(X_rT_k - X_kT_i) \xrightarrow{G} 0, & j = n + 1. \end{cases}$$

if  $i \neq r, k = r$ , then

$$S(g_{ij}, g_{kj}) = \begin{cases} X_{n+1}T_j(X_iT_k - X_rT_i) \xrightarrow{G} 0, & j = n; \\ X_rT_j(X_iT_k - X_rT_i) \xrightarrow{G} 0, & j = n + 1. \end{cases}$$

otherwise,

$$\text{if } j \leq n - 1, \text{ then } S(g_{ij}, g_{kj}) = X_jT_j(X_iT_k - X_kT_i) \xrightarrow{G} 0.$$

if  $j = n + 1, k = n, i = r$ , then

$$S(g_{ij}, g_{kj}) = X_rT_j(X_nT_k - X_rT_i) \xrightarrow{G} 0.$$

if  $j = n + 1, k = n, i \neq r$ , then

$$S(g_{ij}, g_{kj}) = X_rT_j(X_iX_nT_k - X_{n+1}X_rT_i) \xrightarrow{G} 0.$$

2. For  $j \neq l$ , we have  $[\text{in}_{\prec}(g_{ij}), \text{in}_{\prec}(g_{kl})] \neq 1$  when:

- a)  $j = n, l = n + 1, i, k \leq n - 1, i, k \neq r$ ,
- b)  $j = n, l = n + 1, i \leq n - 1, i \neq r, k = r$ ,
- c)  $j = n, l = n + 1, i = r, k \leq n - 1, k \neq r$ ,
- d)  $j = n + 1, l = n, i, k \leq n - 1, i, k \neq r$ ,
- e)  $j = n + 1, l = n, i = r, k \leq n - 1, k \neq r$ ,
- f)  $j = n + 1, l = n, i = r, k \leq n - 1, k \neq r$ .

If condition a) holds, a computation of the  $S$ -pair  $S(g_{in}, g_{kn+1})$  gives:

$$\begin{aligned} & \frac{f_{in}f_{n+1k}}{[f_{in}, f_{kn+1}]}T_kT_n - \frac{f_{ni}f_{kn+1}}{[f_{in}, f_{kn+1}]}T_iT_{n+1} \\ &= \frac{X_iX_nX_r^2}{x_n}T_kT_n - \frac{X_kX_nX_rX_{n+1}}{x_n}T_iT_{n+1} \\ &= X_iT_n(X_r^2T_k - X_kX_nT_{n+1}) + X_kT_{n+1}(X_iX_nT_n - X_rX_{n+1}T_{n+1}) \\ &= X_kT_{n+1}g_{in} - X_iT_n g_{kn+1} \xrightarrow{G} 0. \end{aligned}$$

In a similar way, we can show that the  $S$ -pairs  $S(g_{ij}, g_{kl})$  corresponding

to b)–f) reduce to 0 with respect to  $B$ .

So  $f_1, \dots, f_{n+1}$  form an  $s$ -sequence.  $\square$

**Corollary 2.1** *Let  $R = K[X_1, \dots, X_{n+1}]$ . Let  $\mathcal{G}$  be a connected acyclic graph on  $[n+1]$  vertices with edge ideal  $I(\mathcal{G}) = (X_1X_n, X_2X_n, \dots, X_{n-1}X_n, X_rX_{n+1}, X_n^2)$ , for  $r = 1, \dots, n-1$ . Then  $I(\mathcal{G})$  is generated by an  $s$ -sequence.*

*Proof.* By setting  $f_i = X_iX_n$ , for  $i = 1, \dots, n-1$ ;  $f_n = X_rX_{n+1}$ ;  $f_{n+1} = X_n^2$ , it results for  $f_{ij}$  and  $f_{ji}$ ,  $1 \leq i < j \leq n+1$ :

$$f_{ij} = \begin{cases} X_i, & i \leq n-1, j \neq n; \\ X_iX_n, & i \leq n-1, i \neq r, j = n; \\ X_n, & i = r, j = n; \\ X_rX_{n+1}, & i = n, j = n+1. \end{cases}$$

$$f_{ji} = \begin{cases} X_j, & i, j \leq n-1; \\ X_rX_{n+1}, & i \leq n-1, i \neq r, j = n; \\ X_{n+1}, & i = r, j = n; \\ X_n, & i \leq n-1, j = n+1; \\ X_n^2, & i = n, j = n+1. \end{cases}$$

Then the generators  $g_{ij} = f_{ij}T_j - f_{ji}T_i$  of  $J$ , for  $i < j$ , are the linear forms:

$$g_{ij} = \begin{cases} X_iT_j - X_jT_i, & i, j \leq n-1; \\ X_iX_nT_n - X_rX_{n+1}T_i, & i \leq n-1, i \neq r, j = n; \\ X_nT_n - X_{n+1}T_i, & i = r, j = n; \\ X_iT_j - X_nT_i, & i \leq n-1, j = n+1; \\ X_rX_{n+1}T_{n+1} - X_n^2T_n, & i = n, j = n+1. \end{cases}$$

For a suitable term order  $\prec$ , in particular for the reverse lexicographic order, it is possible to show, using the same arguments as in the proof of Theorem 2.2, that the  $S$ -pairs  $S(g_{ij}, g_{kl})$ , with  $i, j, k, l \in \{1, \dots, n+1\}$ ,  $i < j$ ,  $i < k < l$ , have a standard expression with respect to the set of the  $g_{ij}$  with remainder 0.

So  $f_1, \dots, f_{n+1}$  form an  $s$ -sequence.  $\square$

**Corollary 2.2** *Let  $R = K[X_1, \dots, X_{n+1}]$ . Let  $\mathcal{G}$  be a connected acyclic*



graph on  $[n+1]$  vertices with edge ideal  $I(\mathcal{G}) = (X_1X_n, X_2X_n, \dots, X_{n-1}X_n, X_rX_{n+1}, X_{n+1}^2)$ , for  $r = 1, \dots, n-1$ . Then  $I(\mathcal{G})$  is generated by an  $s$ -sequence.

*Proof.* By setting  $f_i = X_iX_n$ , for  $i = 1, \dots, n-1$ ;  $f_n = X_rX_{n+1}$ ;  $f_{n+1} = X_{n+1}^2$ , it results for  $f_{ij}$  and  $f_{ji}$ ,  $1 \leq i < j \leq n+1$ :

$$f_{ij} = \begin{cases} X_i, & i, j \leq n-1; \\ X_iX_n, & i \leq n-1, j \geq n, \text{ but } i \neq r, j \neq n; \\ X_n, & i = r, j = n; \\ X_r, & i = n, j = n+1. \end{cases}$$

$$f_{ji} = \begin{cases} X_j, & i, j \leq n-1; \\ X_rX_{n+1}, & i \leq n-1, i \neq r, j = n; \\ X_r, & i = n, j = n+1; \\ X_{n+1}, & i = r, j = n; \\ X_{n+1}^2, & i \leq n-1, j = n+1. \end{cases}$$

The generators  $g_{ij}$  of  $J$ , for  $i < j$ , are the linear forms  $f_{ij}T_j - f_{ji}T_i$ , and for a suitable term order  $\prec$  it is possible to show, likewise to Theorem 2.2 or the above corollary, that the  $S$ -pairs  $S(g_{ij}, g_{kl})$ , with  $i, j, k, l \in \{1, \dots, n+1\}$ ,  $i < j, i < k < l$ , have a standard expression with respect to the set of the  $g_{ij}$  with remainder 0. Consequently,  $f_1, \dots, f_{n+1}$  form an  $s$ -sequence.  $\square$

Finally, let's consider monomial ideals associated to the edge set of a  $n$ -line graph with a loop having vertex set  $[n] = \{v_1, \dots, v_n\}$ .

**Theorem 2.3** *Let  $R = K[X_1, \dots, X_n]$ . Let  $\mathcal{G}$  be a connected acyclic graph on  $[n]$  vertices with edge ideal  $I(\mathcal{G}) = (X_1X_2, X_2X_3, X_3X_4, \dots, X_{n-1}X_n, X_n^2)$ . Then  $I(\mathcal{G})$  is generated by an  $s$ -sequence.*

*Proof.* Let

$$f_i = \begin{cases} X_iX_{i+1}, & i = 1, \dots, n-1; \\ X_i^2, & i = n. \end{cases}$$

be the monomial generators of  $I(\mathcal{G})$ . They form an  $s$ -sequence if the set  $B = \{f_{ij}T_j - f_{ji}T_i \mid 1 \leq i < j \leq n-1\}$  is a Gröbner basis for  $J$ . One has

for  $i < j$

$$[f_i, f_j] = \begin{cases} 1, & j \neq i + 1; \\ X_{i+1}, & j = i + 1. \end{cases}$$

Hence:

$$f_{ij} = \frac{f_i}{[f_i, f_j]} = \begin{cases} X_i X_{i+1}, & j \neq i + 1; \\ X_i, & j = i + 1. \end{cases}$$

Then the generators of  $J$  are the linear forms:

$$g_{ij} = \begin{cases} X_i X_{i+1} T_j - X_j X_{j+1} T_i, & j \neq i + 1; \\ X_i T_j - X_j T_i, & j = i + 1. \end{cases}$$

For a suitable term order  $\prec$ , in particular for the reverse lexicographic order, we want to prove that the  $S$ -pairs  $S(g_{ij}, g_{kl})$ , with  $i, j, k, l \in \{1, \dots, n\}$ ,  $i < j$ ,  $i < k < l$ , have a standard expression with respect to  $B$  with remainder 0. Note that, to get a standard expression of  $S(g_{ij}, g_{hl})$  is equivalent to find some  $g_{st} \in B$  whose initial term divides the initial term of  $S(g_{ij}, g_{hl})$  and substitute a multiple of  $g_{st}$  such that the remaindered polynomial has a smaller initial term and so on up to the remainder is 0. We have:

$$S(g_{ij}, g_{kl}) = \frac{f_{ij} f_{lk}}{[f_{ij}, f_{kl}]} T_j T_k - \frac{f_{ji} f_{kl}}{[f_{ij}, f_{kl}]} T_i T_l.$$

Let

$$\text{in}_{\prec}(g_{ij}) = \begin{cases} X_i X_{i+1} T_j, & j \neq i + 1; \\ X_i T_j, & j = i + 1. \end{cases}$$

It is known that the  $S$ -pairs for which  $[\text{in}_{\prec}(g_{ij}), \text{in}_{\prec}(g_{kl})] = 1$  reduce to 0.

So we examine only the  $S$ -pairs such that  $[\text{in}_{\prec}(g_{ij}), \text{in}_{\prec}(g_{kl})] \neq 1$ .

There are two cases: 1.  $j = l$ , and 2.  $j \neq l$ .

1. For  $j = l$  we have:

$$S(g_{ij}, g_{kl}) = S(g_{ij}, g_{kj})$$

$$\begin{aligned}
 &= T_j \left( \frac{f_{ij}f_{jk}}{[f_{ij}, f_{kj}]} T_k - \frac{f_{ji}f_{kj}}{[f_{ij}, f_{kj}]} T_i \right) \\
 &= \frac{T_j}{[f_{ij}, f_{kj}]} (f_{ij}f_{jk}T_k - f_{ji}f_{kj}T_i) \\
 &= \frac{T_j}{[f_{ij}, f_{kj}]} (f_{ij}f_{jk}T_k + f_{ij}f_{kj}T_j - f_{ij}f_{kj}T_j - f_{ji}f_{kj}T_i) \\
 &= \frac{T_j}{[f_{ij}, f_{kj}]} (f_{kj}(g_{ij}) - f_{ij}(g_{kj})).
 \end{aligned}$$

Note that

$$[f_{ij}, f_{kj}] = \begin{cases} X_k, & k = i + 1, j \neq i + 1; \\ 1, & \text{otherwise.} \end{cases}$$

In fact:

if  $j \neq i + 1, j \neq k + 1$ , we have

$$[f_{ij}, f_{kj}] = [X_i X_{i+1}, X_k X_{k+1}] = \begin{cases} X_k, & k = i + 1; \\ 1, & k \neq i + 1. \end{cases}$$

If  $j \neq i + 1, j = k + 1$ , then

$$[f_{ij}, f_{kj}] = [X_i X_{i+1}, X_k] = \begin{cases} X_k, & k = i + 1; \\ 1, & k \neq i + 1. \end{cases}$$

If  $j = i + 1, j \neq k + 1$ , then  $[f_{ij}, f_{kj}] = [X_i, X_k X_{k+1}] = 1$  being  $i < k$ .

So, if  $[f_{ij}, f_{kj}] = X_k$ , then

$$\begin{aligned}
 S(g_{ij}, g_{kj}) &= \frac{T_j}{[f_{ij}, f_{kj}]} (f_{kj}(g_{ij}) - f_{ij}(g_{kj})) \\
 &= \frac{T_j}{X_k} (X_k X_{k+1}(g_{ij}) - X_i X_{i+1}(g_{kj})) \\
 &= \frac{T_j}{X_k} [X_{k+1}(g_{ij}) - X_i(g_{kj})] \xrightarrow{G} 0.
 \end{aligned}$$

If  $[f_{ij}, f_{kj}] = 1$ , it is obvious.

2. For  $j \neq l$ , we have:

a) if  $j = i + 1$ , then  $[f_{ij}, f_{kl}] = 1$  being  $i < k < l$ . Hence:

$$\begin{aligned} S(g_{ij}, g_{kl}) &= \frac{f_{ij}f_{lk}}{[f_{ij}, f_{kl}]}T_jT_k - \frac{f_{ji}f_{kl}}{[f_{ij}, f_{kl}]}T_iT_l = f_{ij}f_{lk}T_jT_k - f_{ji}f_{kl}T_iT_l \\ &= f_{ij}f_{lk}T_jT_k - f_{ji}f_{kl}T_iT_l - f_{ji}f_{lk}T_iT_k + f_{ji}f_{lk}T_iT_k \\ &= f_{lk}T_k(g_{ij}) - f_{ji}T_i(g_{kl}) \xrightarrow{G} 0. \end{aligned}$$

b) if  $j \neq i + 1$ ,  $[X_iX_{i+1}T_j, X_kX_{k+1}T_l] \neq 1$ , but, as  $j \neq l$ ,  $[X_iX_{i+1}, X_kX_{k+1}] \neq 1$ . Hence  $X_k = X_{i+1}$ , i.e  $k = i + 1$  being  $i < k$ . It follows:

$$\text{for } l \neq k + 1, [f_{ij}, f_{kl}] = [X_iX_{i+1}, X_kX_{k+1}] = X_k,$$

$$\text{for } l = k + 1, [f_{ij}, f_{kl}] = [X_iX_{i+1}, X_k] = X_k.$$

In any case  $[f_{ij}, f_{kl}] = X_k$ , then, if  $l \neq k + 1$  with  $k = i + 1$ , one has:

$$\begin{aligned} S(g_{ij}, g_{kl}) &= \frac{f_{ij}f_{lk}}{[f_{ij}, f_{kl}]}T_jT_k - \frac{f_{ji}f_{kl}}{[f_{ij}, f_{kl}]}T_iT_l \\ &= \frac{X_iX_{i+1}X_lX_{l+1}}{X_k}T_jT_k - \frac{X_kX_{k+1}X_jX_{j+1}}{X_k}T_iT_l \\ &\stackrel{k=i+1}{=} X_iX_lX_{l+1}T_jT_k - X_{k+1}X_jX_{j+1}T_iT_l \\ &= X_iX_lX_{l+1}T_jT_k - X_{k+1}X_jX_{j+1}T_iT_l - X_{k+1}X_iX_{i+1}T_jT_l \\ &\quad + X_{k+1}X_iX_{i+1}T_jT_l \\ &= X_{k+1}T_l(g_{ij}) - X_iT_j(g_{kl}) \xrightarrow{G} 0. \end{aligned}$$

If  $l = k + 1$

$$\begin{aligned} S(g_{ij}, g_{kl}) &= \frac{f_{ij}f_{lk}}{[f_{ij}, f_{kl}]}T_jT_k - \frac{f_{ji}f_{kl}}{[f_{ij}, f_{kl}]}T_iT_l \\ &= \frac{X_iX_{i+1}X_l}{X_k}T_jT_k - \frac{X_kX_jX_{j+1}}{X_k}T_iT_l \\ &\stackrel{k=i+1}{=} X_iX_lT_jT_k - X_jX_{j+1}T_iT_l \end{aligned}$$

$$\begin{aligned}
 &= X_i X_l T_j T_k - X_j X_{j+1} T_i T_l - X_i X_{i+1} T_j T_l + X_i X_{i+1} T_j T_l \\
 &= T_l(g_{ij}) - X_i T_j(g_{kl}) \xrightarrow{\mathcal{G}} 0.
 \end{aligned}$$

Hence all the  $S$ -pairs  $S(g_{ij}, g_{kl})$  reduce to 0 with respect to  $B$ . □

**Remark 2.2** With a procedure similar to that used in Theorem 2.3, it can be shown that:

for  $n$ -line graphs  $\mathcal{G}$  with a loop on vertices  $v_{n-2}$  or  $v_{n-1}$ , the generators of their edge ideal  $I(\mathcal{G})$  form an  $s$ -sequence with respect to the reverse lexicographic order;

for  $n$ -line graphs  $\mathcal{G}$  with a loop on vertices  $v_1$  or  $v_2$ , the generators of their edge ideal  $I(\mathcal{G})$  form an  $s$ -sequence with respect to the degree reverse lexicographic order.

### 3. Invariants of the symmetric algebra

In this section we use the theory of  $s$ -sequences in order to compute standard algebraic invariants of the symmetric algebra of the above-considered edge ideals in terms of their annihilator ideals.

We analyze the following classes of edge ideals associated to a graph with a loop  $\mathcal{G}$ :

- 1)  $I(\mathcal{G}) = (X_1 X_n, X_2 X_n, \dots, X_{n-1} X_n, X_n^2)$ ,
- 2)  $I(\mathcal{G}) = (X_1 X_{n-1}, X_2 X_{n-1}, \dots, X_{n-2} X_{n-1}, X_{n-2} X_n, X_{n-2}^2)$ ,
- 3)  $I(\mathcal{G}) = (X_1 X_2, X_2 X_3, \dots, X_{n-1} X_n, X_n^2)$ .

**Proposition 3.1** *Let  $G$  be the graph with  $n$  vertices having edge ideal  $I(\mathcal{G}) = (X_1 X_n, X_2 X_n, \dots, X_{n-1} X_n, X_n^2) \subset R = K[X_1, \dots, X_n]$ . The annihilator ideals of the generators of  $I(\mathcal{G})$  are  $\mathcal{I}_1 = (0)$ ,  $\mathcal{I}_i = (X_1, \dots, X_{i-1})$ , for  $i = 2, \dots, n$ .*

*Proof.* Let  $I(\mathcal{G}) = (f_1, \dots, f_n)$ , where  $f_1 = X_1 X_n$ ,  $f_2 = X_2 X_n, \dots, f_{n-1} = X_{n-1} X_n$ ,  $f_n = X_n^2$ . Then the annihilator ideals of the monomial sequence are:  $\mathcal{I}_1 = (0)$ ,  $\mathcal{I}_1 = (X_1)$ ,  $\mathcal{I}_2 = (X_1, X_2)$ ,  $\dots$ ,  $\mathcal{I}_i = (X_1, X_2, \dots, X_{i-1})$  for  $i = 1, \dots, n - 1$ .

For  $i = n$  one has  $\mathcal{I}_n = (X_1, X_2, \dots, X_{n-1})$ . □

**Remark 3.1** The  $s$ -sequence  $f_1 = X_1 X_n, f_2 = X_2 X_n, \dots, f_{n-1} =$

$X_{n-1}X_n, f_n = X_n^2$  is strong.

**Theorem 3.1** *Let  $\mathcal{G}, I(\mathcal{G})$  be as in Proposition 3.1. For the symmetric algebra of  $I(\mathcal{G}) \subset R$  it holds:*

1.  $\dim(\text{Sym}_R(I(\mathcal{G}))) = n + 1$
2.  $e(\text{Sym}_R(I(\mathcal{G}))) = n - 1$
3.  $\text{reg}(\text{Sym}_R(I(\mathcal{G}))) \leq 1$ .

*Proof.* 1. By [12]  $\dim(\text{Sym}_R(I(\mathcal{G}))) = n + 1$ .

2. By [4] one has  $e(\text{Sym}_R(I(\mathcal{G}))) = \sum_{i=1}^{n-1} e(R/\mathcal{I}_i)$ . By Proposition 3.1 the annihilator ideals are generated by a regular sequence, then  $e(R/\mathcal{I}_i) = 1$ , for  $i = 2, \dots, n$  and  $e(R/(0)) = 1$ . It follows:  $e(\text{Sym}_R(I(\mathcal{G}))) = \sum_{i=1}^{n-1} e(R/\mathcal{I}_i) = n - 1$ .

3. By [12]:  $\text{reg}(\text{Sym}_R(I(\mathcal{G}))) = \text{reg}(R[T_1, \dots, T_{n-1}]/J) \leq \text{reg}(R[T_1, \dots, T_{n-1}]/\text{in}_<(J)) \leq \max_{2 \leq j \leq n-1} \{ \sum_{i=1}^{j-1} \deg(f_{ij}) - (j-2) \}$ .

Hence  $\text{reg}(\text{Sym}_R(I(\mathcal{G}))) \leq \max_{2 \leq j \leq n-1} \{ \sum_{i=1}^{j-1} \deg(X_i) - (j-2) \} = (j-1) - (j-2) = 1$ . □

**Remark 3.2** Let  $I(\mathcal{G}) = (X_1X_n, X_2X_n, \dots, X_{n-1}X_n, X_n^2) \subset R$ . By [12] it follows that  $\text{depth}(\text{Sym}_R(I(\mathcal{G}))) = \dim(\text{Sym}_R(I(\mathcal{G}))) = n + 1$ , so  $\text{Sym}_R(I(\mathcal{G}))$  is Cohen-Macaulay.

**Proposition 3.2** *Let  $\mathcal{G}$  be the graph with  $n$  vertices having edge ideal  $I(\mathcal{G}) = (X_1X_{n-1}, X_2X_{n-1}, \dots, X_{n-2}X_{n-1}, X_{n-2}X_n, X_{n-2}^2) \subset R = K[X_1, \dots, X_n]$ . The annihilator ideals of the generators of  $I(\mathcal{G})$  are*

$$\mathcal{I}_1 = (0), \quad \mathcal{I}_i = (X_1, \dots, X_{i-1}), \quad \text{for } i = 2, \dots, n-2,$$

$$\mathcal{I}_{n-1} = (X_{n-1}), \quad \mathcal{I}_n = (X_{n-1}, X_n).$$

*Proof.* Let  $I(\mathcal{G}) = (f_1, f_2, \dots, f_n)$ , where  $f_1 = X_1X_n, f_2 = X_2X_n, \dots, f_{n-2} = X_{n-2}X_{n-1}, f_{n-1} = X_{n-2}X_n, f_n = X_{n-2}^2$ . Set  $f_{hk} = f_h/[f_h, f_k]$  for  $h < k, h, k = 1, \dots, n$ . The annihilator ideals of the monomial sequence  $f_1, \dots, f_n$  are  $\mathcal{I}_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$ , for  $i = 1, \dots, n$ . Hence we have  $\mathcal{I}_1 = (0), \mathcal{I}_2 = (f_{12}) = (X_1), \mathcal{I}_3 = (f_{13}, f_{23}) = (X_1, X_2), \dots, \mathcal{I}_{n-2} = (f_{1,n-2}, \dots, f_{n-3,n-2}) = (X_1, X_2, \dots, X_{n-3}), \mathcal{I}_{n-1} = (f_{1,n-1}, \dots, f_{n-2,n-1}) = (X_1X_{n-1}, \dots, X_{n-3}X_{n-1}, X_{n-1}) = (X_{n-1}), \mathcal{I}_n = (f_{1,n}, \dots, f_{n-1,n}) = (X_1X_{n-1}, \dots, X_{n-3}X_{n-1}, X_{n-1}, X_n) = (X_{n-1}, X_n)$ . □

**Theorem 3.2** *Let  $\mathcal{G}, I(\mathcal{G})$  be as in Proposition 3.2. For the symmetric algebra of  $I(\mathcal{G}) \subset R$  it holds:*

- a)  $\dim(\text{Sym}_R(I(\mathcal{G}))) = n + 1$
- b)  $e(\text{Sym}_R(I(\mathcal{G}))) = 3(n - 2)$ .

*Proof.* a) By [4], Proposition 2.4:

$$\dim(\text{Sym}_R(I(\mathcal{G}))) = \max_{\substack{0 \leq r \leq n \\ 1 \leq i_1 \leq \dots \leq i_r \leq n}} \{ \dim(R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})) + r \}$$

By Proposition 3.2 it follows:

$$\begin{aligned} r = 0 : \dim(R) &= n \\ r = 1 : \dim(R/\mathcal{I}_i) &= n - i + 1 \quad \forall i = 1, \dots, n - 2, \\ &\dim(R/\mathcal{I}_{n-1}) = n - 1, \\ &\dim(R/\mathcal{I}_n) = n - 2, \\ r = 2 : \dim(R/(\mathcal{I}_1 + \mathcal{I}_2)) &= \dim(R/(\mathcal{I}_1 + \mathcal{I}_{n-1})) = n - 1, \\ &\dim(R/(\mathcal{I}_i + \mathcal{I}_j)) < n - 1, \text{ for } i \neq j, \text{ otherwise,} \\ r = 3, \dots, n - 1 : \dim(R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})) &\leq n + 1 - r, \\ r = n : \dim(R/(\mathcal{I}_1 + \dots + \mathcal{I}_n)) &= 1. \end{aligned}$$

Hence, in any case, the maximum dimension is  $n + 1 - r$ . Then:

$$\dim(\text{Sym}_R(I(\mathcal{G}))) = \max_{\substack{0 \leq r \leq n \\ 1 \leq i_1 \leq \dots \leq i_r \leq n}} \{ \dim(R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})) + r \} = n + 1.$$

b) By [4, Proposition 2.4],  $e(\text{Sym}_R(I(\mathcal{G}))) = \sum_{1 \leq i_1 < \dots < i_r \leq n} e(R/(\mathcal{I}_{i_1}, \dots, \mathcal{I}_{i_r}))$  with  $\dim(R/(\mathcal{I}_{i_1}, \dots, \mathcal{I}_{i_r})) = d - r$ , where  $d = \dim(\text{Sym}_R(I(\mathcal{G}))) = n + 1$  and  $1 \leq r \leq n$ . Set  $d' = \dim(R/(\mathcal{I}_{i_1}, \dots, \mathcal{I}_{i_r})) = n + 1 - r$ .

The multiplicity  $e(\text{Sym}_R(I(\mathcal{G})))$  is given by the sum of the following terms:

$$\begin{aligned} r = 1 : e(R/\mathcal{I}_1) &= 1, \\ r = 2 : e(R/(\mathcal{I}_1 + \mathcal{I}_2)) &= e(R/(\mathcal{I}_1 + \mathcal{I}_{n-1})) = 1, \end{aligned}$$

$$\begin{aligned}
 r = 3, \dots, n - 2 : e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \dots + \mathcal{I}_r)) \\
 &= e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \dots + \mathcal{I}_{r-1} + \mathcal{I}_{n-1})) \\
 &= e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \dots + \mathcal{I}_{r-2} + \mathcal{I}_{n-1} + \mathcal{I}_n)) = 1, \\
 r = n - 1 : e(R/(\mathcal{I}_1 + \dots + \mathcal{I}_{n-1})) \\
 &= e(R/(\mathcal{I}_1 + \dots + \mathcal{I}_{n-3} + \mathcal{I}_{n-1} + \mathcal{I}_n)) = 1, \\
 r = n : e(R/(\mathcal{I}_1 + \dots + \mathcal{I}_n)) = 1.
 \end{aligned}$$

Hence:  $e(\text{Sym}_R(I(G))) = 3 + 3(n - 4) + 3 = 3(n - 2)$ . □

**Proposition 3.3** *Let  $\mathcal{G}$  be the graph with  $n$  vertices having edge ideal  $I(\mathcal{G}) = (X_1X_2, X_2X_3, \dots, X_{n-1}X_n, X_n^2) \subset R$ . The annihilator ideals of the generators of  $I(\mathcal{G})$  are*

$$\begin{aligned}
 \mathcal{I}_1 = (0), \mathcal{I}_2 = (X_1), \mathcal{I}_3 = (X_2), \dots, \mathcal{I}_i = (X_1X_2, \dots, X_{i-3}X_{i-2}, X_{i-1}), \\
 \text{for } 4 \leq i \leq n.
 \end{aligned}$$

*Proof.* Let  $I(\mathcal{G}) = (f_1, \dots, f_n)$ , where  $f_1 = X_1X_2, f_2 = X_2X_3, \dots, f_{n-1} = X_{n-1}X_n, f_n = X_n^2$ . One computes:  $\mathcal{I}_1 = (0), \mathcal{I}_2 = (X_1), \mathcal{I}_3 = (X_2)$ . For  $i = 4, \dots, n - 1 : \mathcal{I}_i = (X_1X_2, \dots, X_{i-3}X_{i-2}, X_{i-1})$ . For  $i = n : \mathcal{I}_n = (X_1X_2, \dots, X_{n-3}X_{n-2}, X_{n-1})$ . Hence the thesis. □

**Theorem 3.3** *Let  $\mathcal{G}, I(\mathcal{G})$  be as in Proposition 3.3. For the symmetric algebra of  $I(\mathcal{G}) \subset R$  it holds:*

1.  $\dim(\text{Sym}_R(I(\mathcal{G}))) = n + 1$
2.  $e(\text{Sym}_R(I(\mathcal{G}))) = \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \dots$

*Proof.* By [4], Proposition 2.4:

$$\begin{aligned}
 \dim(\text{Sym}_R(I(\mathcal{G}))) &= \max_{\substack{0 \leq r \leq n \\ 1 \leq i_1 \leq \dots \leq i_r \leq n}} \{ \dim(R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})) + r \} \\
 e(\text{Sym}_R(I(\mathcal{G}))) &= \sum_{\substack{0 \leq r \leq n \\ 1 \leq i_1 \leq \dots \leq i_r \leq n \\ \dim(R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})) = d - r}} e(R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})).
 \end{aligned}$$



1. We consider  $R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})$ , for  $r = 0, \dots, n$ ,  $1 \leq i_1 \leq \dots \leq i_r \leq n$ .  
 By Proposition 3.3 one has:  $\mathcal{I}_1 = (0)$ ,  $\mathcal{I}_2 = (X_1)$ ,  $\mathcal{I}_3 = (X_2), \dots, \mathcal{I}_i = (X_1 X_2, \dots, X_{i-3} X_{i-2}, X_{i-1})$ , for  $i = 4, \dots, n$ . Then:

$$\begin{aligned} r = 0 : \dim(R) &= n, \\ r = 1 : \dim(R/\mathcal{I}_1) &= \dim(R) = n, \\ &\dim(R/\mathcal{I}_2) = \dim(R/\mathcal{I}_3) = n - 1, \\ &\dim(R/\mathcal{I}_i) \leq n - 1 \quad \forall i = 4, \dots, n, \\ r = 2 : \dim(R/(\mathcal{I}_1 + \mathcal{I}_2)) &= \dim(R/(\mathcal{I}_1 + \mathcal{I}_3)) = n - 1, \\ &\dim(R/(\mathcal{I}_i + \mathcal{I}_j)) < n - 1, \text{ for } i \neq j, \text{ otherwise,} \\ r = 3, \dots, n - 1 : \dim(R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})) &\leq n + 1 - r, \\ r = n : \dim(R/(\mathcal{I}_1 + \dots + \mathcal{I}_n)) &= 1. \end{aligned}$$

Hence, in any case, the maximum dimension is  $n + 1 - r$ . It follows:

$$\dim(\text{Sym}_R(I(\mathcal{G}))) = \max_{\substack{0 \leq r \leq n \\ 1 \leq i_1 \leq \dots \leq i_r \leq n}} \{ \dim(R/(\mathcal{I}_{i_1} + \dots + \mathcal{I}_{i_r})) + r \} = n + 1.$$

2. Set  $d' = \dim(R/(\mathcal{I}_{i_1}, \dots, \mathcal{I}_{i_r})) = n + 1 - r$ .  
 We consider  $R/(\mathcal{I}_{i_1}, \dots, \mathcal{I}_{i_r})$  of dimension  $d'$ , then the multiplicity  $e(\text{Sym}_R(I(\mathcal{G})))$  is given by the sum of the following terms:

$$\begin{aligned} r = 1 : e(R/\mathcal{I}_1) &= 1, \\ r = 2 : e(R/(\mathcal{I}_1 + \mathcal{I}_2)) &= e(R/(\mathcal{I}_1 + \mathcal{I}_3)) = 1, \\ r = 3 : e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) &= e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4)) = 1 \\ &e(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_4)) = e(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_5)) = 1, \end{aligned}$$

and so on, for  $r = 1, \dots, n - 1$ .

$$\begin{aligned} r = n : \dim(R/(\mathcal{I}_1 + \mathcal{I}_2 + \dots + \mathcal{I}_n)) &= 1, \\ \text{then } e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \dots + \mathcal{I}_n)) &= 1. \end{aligned}$$

Let  $F_1 = 1, F_2 = 1, \dots, F_i = F_{i-2} + F_{i-1}, i \geq 3$ , be the Fibonacci sequence. It results:

- if  $n = 2$

$$e(\text{Sym}_R(I(\mathcal{G}))) = e(R/(\mathcal{I}_1)) + e(R/(\mathcal{I}_1 + \mathcal{I}_2)) = 2 = 3 - 1 = F_4 - 1 = \binom{n}{1},$$

- if  $n = 3$

$$\begin{aligned} e(\text{Sym}_R(I(\mathcal{G}))) &= e(R/(\mathcal{I}_1)) + e(R/(\mathcal{I}_1 + \mathcal{I}_2)) + e(R/(\mathcal{I}_1 + \mathcal{I}_3)) \\ &\quad + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) \\ &= 4 = 5 - 1 = F_5 - 1 = \binom{n}{1}, \end{aligned}$$

- if  $n = 4$

$$\begin{aligned} e(\text{Sym}_R(I(\mathcal{G}))) &= e(R/(\mathcal{I}_1)) + e(R/(\mathcal{I}_1 + \mathcal{I}_2)) + e(R/(\mathcal{I}_1 + \mathcal{I}_3)) \\ &\quad + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4)) \\ &\quad + e(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_4)) + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4)) \\ &= 7 = 8 - 1 = F_6 - 1 = n - \binom{1}{2} + \binom{n}{1}, \end{aligned}$$

- if  $n = 5$

$$\begin{aligned} e(\text{Sym}_R(I(\mathcal{G}))) &= e(R/(\mathcal{I}_1)) + e(R/(\mathcal{I}_1 + \mathcal{I}_2)) + e(R/(\mathcal{I}_1 + \mathcal{I}_3)) \\ &\quad + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4)) \\ &\quad + e(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_4)) + e(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_5)) \\ &\quad + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4)) + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_5)) \\ &\quad + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4 + \mathcal{I}_5)) + e(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5)) \\ &\quad + e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5)) \\ &= 12 = 13 - 1 = F_7 - 1 = n - \binom{2}{3} + n - \binom{1}{2} + \binom{n}{1}, \end{aligned}$$

and so on.

Hence one obtains  $e(\text{Sym}_R(I(\mathcal{G}))) = F_{n+2} - 1$ ; so the assertion follows taking in consideration the Lucas' formula.  $\square$

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