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The Fermat septic and the Klein quartic as moduli spaces of hypergeometric Jacobians

Dedicated to the 70th birthday of Professor Hironori Shiga.

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Abstract. We study the Schwarz triangle function with the monodromy group $\Delta(7,7,7)$, and we construct its inverse by theta constants. As consequences, we give uniformizations of the Klein quartic curve and the Fermat septic curve as Shimura curves parametrizing Abelian 6-folds with endomorphisms $\mathbb{Z}[\zeta_7]$.

Key words: Shimura curves, Hypergeometric functions, Theta functions.

1. Introduction

The Gauss hypergeometric differential equation

$$E(a,b,c) : z(z-1)u'' + \{(a+b+1)z - c\}u' + abu = 0$$

is regular on $\mathbb{C} - \{0,1\}$ for general parameters a, b and c, and the solution space is spanned by Euler type integrals

$$\int_\gamma x^{a-c}(x-1)^{c-b-1}(x-z)^{-a}dx,$$

that are regarded as period integrals for algebraic curves if $a, b, c \in \mathbb{Q}$. Two independent solutions $f_0(z), f_1(z)$ define a multi-valued analytic function $\mathfrak{s}(z) = f_0(z)/f_1(z)$ (Schwarz map), and monodromy transformations for $\mathfrak{s}(z)$ are given by fractional linear transformations.

If parameters satisfy the conditions

$$|1-c| = \frac{1}{p}, \quad |c-a-b| = \frac{1}{q}, \quad |a-b| = \frac{1}{r}, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

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with $p, q, r \in \mathbb{N} \cup \{\infty\}$, the monodromy group is isomorphic to a triangle group

$$\Delta(p,q,r) = \langle M_0,\ M_1,\ M_\infty \mid M_0^p = M_1^q = M_\infty^r = M_0 M_1 M_\infty = 1\rangle$$

(the condition $M_0^p = 1$ is omitted if $p = \infty$, and so on). In this case, the upper half plane is mapped by $\mathfrak{s}$ to a triangle with vertices $\mathfrak{s}(0)$, $\mathfrak{s}(1)$ and $\mathfrak{s}(\infty)$, angles π/p, π/q and π/r, respectively, and so is the lower half plane. Copies of these two triangles give a tessellation of a disk $\mathbb{D}$ by the monodromy action, and we have an isomorphism $\overline{\mathbb{D}/\Delta(p,q,r)} \cong \overline{\mathbb{C} - \{0,1\}} = \mathbb{P}^1$. For example, $E(1/2, 1/2, 1)$ is known as the Picard-Fuchs equation for the Legendre family of elliptic curves $y^2 = x(x-1)(x-z)$ and the monodromy group $\Delta(\infty,\infty,\infty)$ is projectively isomorphic to the congruence subgroup $\Gamma(2)$ in $\mathrm{SL}_2(\mathbb{Z})$ of level 2. Also a triangle group $\Delta(n,n,n)$ with $n \geq 4$ is interesting, since its commutator subgroup N_n gives a uniformization of the Fermat curve $\mathcal{F}_n$ of degree n. More precisely, the natural projection $\mathbb{D}/N_n \to \mathbb{D}/\Delta(n,n,n) = \mathbb{P}^1$ is an Abelian covering branched at $0, 1$ and ∞ with the covering group $\Delta(n,n,n)/N_n \cong (\mathbb{Z}/n\mathbb{Z})^2$ (see [CIW94]).

In [T77], Takeuchi determined all arithmetic triangle groups. According to it, $\Delta(n,n,n)$ is arithmetic (and hence the Fermat curve $\mathcal{F}_n$ is a Shimura curve) for $n \in FT = \{4,5,6,7,8,9,12,15\}$. These groups come from the Picard-Fuchs equation for algebraic curves $X_t : y^m = x(x-1)(x-t)$ with $m = n$ (resp. $m = 2n$) if $n \in FT$ is odd (resp. even). Among them, $n = 5$ and 7 are special in the sense that a Jacobian $J(X_t)$ is simple in general, and Picard-Fuchs equations describe variations of Hodge structure on the whole of $H^1(X_t, \mathbb{Q})$, rather than sub Hodge structures. These two families are treated by Shimura as examples of PEL families in [Sm64]. Also de Jong and Noot studied them as counter examples of Coleman's conjecture (which asserts the finiteness of the number of CM Jacobians for a fixed genus $g \geq 4$) for $g = 4, 6$ in [dJN91] (see also [R09] and [MO13] for this direction).

For $n = 5$, we gave $\mathfrak{s}^{-1}$ by theta constants in [K03] as a byproduct of study of the moduli space of ordered five points on $\mathbb{P}^1$. In present paper, we compute the monodromy group, Riemann's period matrices and the Riemann constant with an explicit symplectic basis for $n = 7$. Using them, we express the Schwarz inverse map $\mathfrak{s}^{-1}$ by Riemann's theta constants (Theorem 4.1). As a consequence, we give explicit modular interpretations of the Klein quartic curve $\mathcal{K}_4$ and the Fermat septic curve $\mathcal{F}_7$ as modular varieties parametrizing Abelian 6-folds with endomorphisms $\mathbb{Z}[\zeta_7]$. (Corollary 4.1 and Corollary 4.2). The Klein quartic is classically known to be isomorphic to the elliptic modular curve of level 7. In [E99], Elkies studied it as a Shimura curve parametrizing a family of QM Abelian 6-folds. Our interpretation of $\mathcal{K}_4$ gives the third face as a modular variety. Our expression of $\mathfrak{s}^{-1}$ is a variant of Thomae's formula. This kind of formula for cyclic coverings was studied in general context by Bershadsky-Radul ([BR87], [BR88]), Nakayashiki ([Na97]) and Enolski-Grava ([EG06]), but our standpoint is more moduli theoretic as a classical work of Picard ([P1883]) which produces modular forms on a 2-dimensional complex ball. In [Sh88], Shiga determined Picard modular forms explicitly, and his results were applied to number theory and cryptography (see [KS07] and [KW04]). We expect that also our concrete results will give a good example to develop a generalization of arithmetic theory of elliptic curves. Here we mention that there are several studies of automorphic forms for triangle groups (e.g. [Mi75], [W81], [H05] and [DGMS13]). However it seems that we have very few explicit constructions of autmorphic forms for co-compact triangle groups in the view point of the Picard's work.

Our Schwarz map is regarded also as a periods map of K3 surfaces. In pioneer work [Sh79,81], Shiga studied families of elliptic K3 surfaces with period maps to complex balls. These K3 surfaces have a non-symplectic automorphism of order 3, which induces a Hermitian structure on the transcendental lattice. Now K3 surfaces with non-symplectic automorphisms of prime order are classified (see [AST11]), and many of them are known to be quotients of product surfaces ([GP]). In the last section, we give elliptic K3 surfaces S_t associated to X_t and compute the Neron-Severi group and the Mordell-Weil lattice of S_t.

2. Uniformization of Fermat Curves

2.1. Hypergeometric integral

We compute monodromy groups and invariant Hermitian forms for hypergeometric integrals

$$u(t) = \int \Omega_\alpha(x), \qquad \Omega_\alpha(x) = \{x(x-1)(x-t)\}^{-\alpha} dx$$

according to [Y97, Chapter IV], for $\alpha = k/(2k+1)$ and $(2k-1)/4k$ with $k \geq 2$. They satisfy differential equations $E(k/(2k+1), (k-1)/(2k+1), 2k/(2k+1))$ and $E((2k-1)/4k, (2k-3)/4k, (2k-1)/2k)$ with monodromy groups $\Delta(n,n,n)$, $n = 2k+1$ and $2k$ respectively. Let us consider decompositions

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{H}_+ \cup \mathbb{P}^1(\mathbb{R}) \cup \mathbb{H}_-, \qquad \mathbb{P}^1(\mathbb{R}) = I_0 \cup I_1 \cup I_2 \cup I_3,$$
$$I_0 = (-\infty, 0), \quad I_1 = (0,t), \quad I_2 = (t,1), \quad I_3 = (1,\infty),$$

where $\mathbb{H}_+$ and $\mathbb{H}_-$ are the upper and lower half planes respectively, and I_k are (oriented) real intervals. (As the initial position of t, we assume that $0 < t < 1$.) Modifying boundaries $\partial\mathbb{H}_+$ and $\partial\mathbb{H}_-$ to avoid $0, t, 1$ and ∞ as

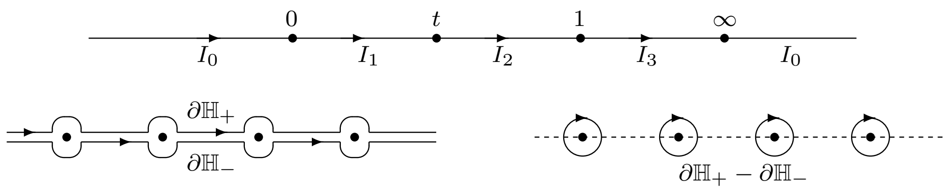

Figure 1. oriented interval I_k.

in Figure 1, we fix a branch of $\Omega_\alpha(x)$ on a simply connected domain $\mathbb{H}_-$ and define integrals $u_k(t) = \int_{I_k} \Omega_\alpha(x)$ by this branch. By the Cauchy integral theorem, they satisfy

$$0 = \int_{\partial\mathbb{H}_-} \Omega_\alpha(x) = u_0(t) + u_1(t) + u_2(t) + u_3(t),$$
$$0 = \int_{\partial\mathbb{H}_+} \Omega_\alpha(x) = u_0(t) + cu_1(t) + c^2u_2(t) + c^3u_3(t), \quad c = \exp(2\pi i\alpha),$$

since $\Omega_\alpha(x)$ is multiplied by $\exp(2\pi i\alpha)$ if x travels around $0, t$ or 1 in clockwise direction. Hence we have

$$u_2(t) = -\frac{1}{1+c}\{u_1(t) + (1+c+c^2)u_3(t)\}.$$

Now let δ_0 and δ_1 be paths to make a half turn around 0 and 1 respectively in counter clockwise direction, starting from the initial point of

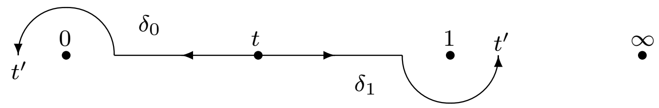

Figure 2. δ_0 and δ_1.

t (Figure 2). Corresponding analytic continuations are represented by connection matrices h_0 and h_1:

$$\delta_0 : \begin{bmatrix} u_1(t) \\ u_3(t) \end{bmatrix} \dashrightarrow \begin{bmatrix} -c^{-1}u_1(t') \\ u_3(t') \end{bmatrix} = h_0 \begin{bmatrix} u_1(t') \\ u_3(t') \end{bmatrix}, \qquad h_0 = \begin{bmatrix} -c^{-1} & 0 \\ 0 & 1 \end{bmatrix},$$

$$\delta_1 : \begin{bmatrix} u_1(t) \\ u_3(t) \end{bmatrix} \dashrightarrow \begin{bmatrix} u_1(t') + u_2(t') \\ c^{-1}u_2(t') + u_3(t') \end{bmatrix} = h_1 \begin{bmatrix} u_1(t') \\ u_3(t') \end{bmatrix},$$

$$h_1 = \begin{bmatrix} \dfrac{c}{c+1} & -\dfrac{c^2+c+1}{c+1} \\ -\dfrac{1}{c^2+c} & -\dfrac{1}{c^2+c} \end{bmatrix},$$

where $u_1(t'), \dots, u_4(t')$ are integrals over oriented intervals $I'_1, \dots, I'_4$ defined for new configurations $-\infty < t' < 0 < 1 < \infty$ and $-\infty < 0 < 1 < t' < \infty$. The monodromy group **Mon** is generated by

$$g_0 = h_0^2 = \begin{bmatrix} c^{-2} & 0 \\ 0 & 1 \end{bmatrix}, \qquad g_1 = h_1^2 = \begin{bmatrix} \dfrac{c^2+1}{c^2+c} & \dfrac{1-c^3}{c^2+c} \\ \dfrac{1-c}{c^3+c^2} & \dfrac{c^2+1}{c^3+c^2} \end{bmatrix}.$$

It is known that there exists a unique monodromy-invariant Hermitian form up to constant (see e.g. [B07] and [Y97]). In fact, we can easily check that h_0 and h_1 belong to a unitary group

$$U_H = \{g \in \mathrm{GL}_2(\mathbb{C}) \mid {}^t\bar{g}Hg = H\}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & 1+c+c^{-1} \end{bmatrix},$$

and hence $\mathbf{Mon} \subset U_H$. The value of $1+c+c^{-1}$ is negative for $c = \exp(2\pi i \alpha)$ with $\alpha = k/(2k+1)$ and $(2k-1)/4k$ $(k \geq 2)$, and H is indefinite. Therefore two domains

$$\mathbb{D}_H^{\pm} = \{u \in \mathbb{C}^2 \mid \pm{}^t\bar{u}Hu < 0\}/\mathbb{C}^\times \subset \mathbb{P}^1(\mathbb{C}).$$

are disks, and U_H acts on each domain. Now the image of the Schwarz map

$$\mathfrak{s} : \mathbb{C} - \{0,1\} \longrightarrow \mathbb{P}^1(\mathbb{C}), \quad t \mapsto [u_1(t) : u_3(t)]$$

is contained in either $\mathbb{D}_H^+$ or $\mathbb{D}_H^-$, which is tessellated by Schwarz triangles. Since we have

$$\mathfrak{s}(0) = \lim_{t \to 0}[u_1(u) : u_3(t)] = [0 : u_3(0)] \in \mathbb{D}_H^+,$$

we see that $\mathbb{D}_H^+/\mathbf{Mon} \cong \mathbb{P}^1(\mathbb{C})$ and $\mathbb{D}_H^+/[\mathbf{Mon}, \mathbf{Mon}] \cong \mathcal{F}_n$ (see [CIW94]), where $\mathcal{F}_n$ is the Fermat curve of degree n with $n = 2k+1$ (resp. $2k$) if $\alpha = k/(2k+1)$ (resp. $(2k-1)/4k$).

Remark 2.1 (1) Putting $\zeta_d = \exp(2\pi i/d)$, we have

$$1 + c + c^{-1} = \begin{cases} 1 + (\zeta_{2k+1})^k + (\zeta_{2k+1})^{k+1} & (n = 2k+1), \\ 1 + (\zeta_{4k})^{2k-1} + (\zeta_{4k})^{2k+1} & (n = 2k). \end{cases}$$

(2) In the case of $n = 2k+1$, we have

$$g_0 = \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = \frac{1}{1+\zeta^k} \begin{bmatrix} \zeta^k + \zeta^{k+1} & \zeta^{k+1} - \zeta^{2k} \\ \zeta - \zeta^{k+1} & 1 + \zeta \end{bmatrix},$$

where $\zeta = \zeta_{2k+1}$. Since $1/(1+\zeta^k) = -(\zeta + \zeta^2 + \cdots + \zeta^k)$ and $\det g_1 = \zeta$, the monodromy group $\mathbf{Mon}$ is a subgroup of $U_H \cap \mathrm{GL}_2(\mathbb{Z}[\zeta])$.

(3) In the case of $n = 2k$, we have

$$g_0 = \begin{bmatrix} \zeta^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = \frac{1}{1+\zeta^{2k-1}} \begin{bmatrix} \zeta^{2k+1} + \zeta^{2k-1} & \zeta^{2k+1} - \zeta^{4k-2} \\ \zeta^2 - \zeta^{2k+1} & 1 + \zeta^2 \end{bmatrix},$$

where $\zeta = \zeta_{4k}$. Note that the cyclotomic polynomial $\Phi_{4k}(x)$ satisfies $\Phi_{4k}(1) = 1$ if $4k \neq 2^m$. In this case, $1 - \zeta$ is a unit in $\mathbb{Z}[\zeta]$, and so is $1/(1+\zeta^{2k-1}) = \zeta/(\zeta - 1)$. Hence $\mathbf{Mon}$ is a subgroup of $U_H \cap \mathrm{GL}_2(\mathbb{Z}[\zeta])$ if $4k \neq 2^m$.

2.2. Fermat curve as a Shimura variety

A triangle group $\Delta(n,n,n)$ is arithmetic for

$$n \in FT = \{4, 5, 6, 7, 8, 9, 12, 15\},$$

and the Fermat curve $\mathcal{F}_n$ is a Shimura curve. Let us see corresponding families of hypergeometric curves

$$X_t : y^m = x(x-1)(x-t)$$

to these case. By the Riemann-Hurwitz formula, the genus of X_t is $g = m-1$ if $3 \nmid m$, and $g = m-2$ if $3 \mid m$. Let ρ be the covering automorphism $(x,y) \to (x, \zeta_m y)$, where $\zeta_m = \exp(2\pi i/m)$. By this action, we can decompose $H^1(X_t, \mathbb{Q})$ into irreducible representations of ρ, and $H^1(X_t, \mathbb{C})$ into eigenspaces of ρ. Let $V(\lambda)$ be the λ-eigenspace of ρ. If m is not prime, the covering $X_t \to \mathbb{P}^1$ has intermediate curves Y_t, and the pullback of $H^1(Y_t, \mathbb{C})$ consists of $V(\zeta_m^k)$ such that $(m,k) \neq 1$. Conversely, such $V(\zeta_m^k)$ descends to a quotient curve. From explicit basis of $H^{1,0}(X_t)$, we see that the Prym part

$$H^1_{Prym}(X_t, \mathbb{Q}) = \left[\bigoplus_{(k,m)=1} V(\zeta_m^k)\right] \cap H^1(X_t, \mathbb{Q})$$

has a Hodge structure of type

$$\begin{aligned} &H^1_{Prym}(X_t, \mathbb{C}) \\ &\quad = \underbrace{V(\lambda_1) \oplus \cdots \oplus V(\lambda_{d-1})}_{\text{contained in } H^{1,0}} \oplus \underbrace{V(\lambda_d)}_{\text{split}} \oplus \underbrace{V(\lambda_{d+1})}_{\text{split}} \oplus \underbrace{V(\lambda_{d+2}) \oplus \cdots \oplus V(\lambda_{2d})}_{\text{contained in } H^{0,1}}, \end{aligned}$$

where $2d = [\mathbb{Q}(\zeta_m) : \mathbb{Q}]$, $\lambda_1, \ldots, \lambda_{2d}$ are primitive roots of unity $\zeta_m, \ldots, \zeta_m^{m-1}$ such that $\bar{\lambda_i} = \lambda_{2d+1-i}$ and $\dim V(\lambda_i) = 2$ for $i = 1, \ldots, 2d$ (see Table 1). Therefore the Hodge structure on $H^1_{Prym}(X_t, \mathbb{Q})$ with the action of ρ is determined by a decomposition $V(\lambda_d) = V(\lambda_d)^{1,0} \oplus V(\lambda_d)^{0,1}$ (the decomposition of $V(\lambda_{d+1})$ is automatically determined as the complex conjugate of $V(\lambda_d)$, and vice versa), that is, determined by periods of $\Omega_\alpha(x) \in V(\lambda_d)^{1,0}$. In the cases $n = 5$ and 7, the monodromy group has a nice representation. Put

$$\Gamma = U_H \cap \mathrm{GL}_2(\mathbb{Z}[\zeta_n]), \qquad \Gamma(\mathfrak{m}) = \{g \in \Gamma \mid g \equiv 1 \mod \mathfrak{m}\} \quad (\mathfrak{m} \in \mathbb{Z}[\zeta_n]).$$

The arithmetic quotient $\mathbb{D}_H^+/\Gamma$ is the moduli space of Jacobians of curves $y^n = x^3 + ax + b$ $(n = 5, 7)$ as a PEL-family (see [Sm64]). Therefore we have the following diagram

Table 1. $H^{1,0}(X_t)_{Prym}$.

$\Delta(n,n,n)$	m	g	$[\mathbb{Q}(\zeta_m):\mathbb{Q}]$	$x^a dx/y^b$ with the following (a,b) give a basis of $H^{1,0}(X_t)_{Prym}$
(4,4,4)	8	7	4	(0,3), (0,5), (0,7), (1,7)
(5,5,5)	5	4	4	(0,2), (0,3), (0,4), (1,4)
(6,6,6)	12	10	4	(0,5), (0,7), (0,11), (1,11)
(7,7,7)	7	6	6	(0,3), (0,4), (0,5), (1,5), (0,6),(1,6)
(8,8,8)	16	15	8	(0,7), (0,9), (0,11), (1,11), (0,13),(1,13), (0,15),(1,15)
(9,9,9)	9	7	6	(0,4), (0,5), (0,7), (1,7),(0,8),(1,8)
(12,12,12)	24	22	8	(0,11), (0,13), (0,17), (1,17), (0,19),(1,19), (0,23),(1,23)
(15,15,15)	15	13	8	(0,7), (0,8), (0,11), (1,11), (0,13),(1,13), (0,14),(1,14)

$$\begin{array}{ccc} \mathbb{D}_H^+/\mathbf{Mon} & \longrightarrow & \mathbb{P}^1 = \overline{\{\text{ordered distinct } (3+1) \text{ points } (0,1,t,\infty)\}} \\ \downarrow & & \downarrow \\ \mathbb{D}_H^+/\Gamma & \longrightarrow & \mathbb{P}^1/S_3 = \overline{\{\text{unordered distinct 3 points in } \mathbb{C}\}/\sim} \end{array}$$

where horizontal arrow are isomorphisms, and $\sim$ is the equivalence relation by affine transformations. From this fact, we see that $\Gamma/\mathbf{Mon}$ is isomorphic to S_3 up to the center.

Remark 2.2 For $n=5$, the Hermitian form H is same with one given in [Sm64]:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1+\zeta_5^2+\zeta_5^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (1-\sqrt{5})/2 \end{bmatrix}.$$

For $n=7$, the Hermitian form given in [Sm64] is

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -\dfrac{\sin(3\pi/7)}{\sin(2\pi/7)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(\zeta_7+\zeta_7^6) \end{bmatrix} = {}^t\bar{A}HA,$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \zeta_7 + \zeta_7^6 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}[\zeta_7]).$$

Proposition 2.1 ([YY84] for $n = 5$) *Let us denote the image of $G \subset \mathrm{GL}_2(\mathbb{Z}[\zeta_n])$ in $P\mathrm{GL}_2(\mathbb{Z}[\zeta_n])$ by $\overline{G}$. For $n = 5$ and 7,*

(1) *the projective modular group $\overline{\Gamma}$ is projectively generated by h_0 and h_1,*

(2) *we have*

$$\overline{\mathbf{Mon}} = \overline{\Gamma(1-\zeta_n)}, \qquad \overline{[\mathbf{Mon}, \mathbf{Mon}]} = \overline{\Gamma((1-\zeta_n)^2)}$$

as automorphisms of $\mathbb{D}_H^+$.

Proof. We show these facts only for $n = 7$, but the case $n = 5$ can be shown by the same way (also see [YY84] and [K03] for $n = 5$). The quotient group $\Gamma/\Gamma(1-\zeta_7)$ is isomorphic to a subgroup of the finite orthogonal group

$$\mathrm{O}(Q, \mathbb{F}_7) = \{g \in \mathrm{GL}_2(\mathbb{F}_7) \mid {}^t g Q g = Q\}, \quad \mathbb{F}_7 = \mathbb{Z}[\zeta_7]/(1-\zeta_7), \;\; Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

The group $\mathrm{O}(Q, \mathbb{F}_7)$ is isomorphic to $S_3 \times \{\pm 1\}$, since elements of $\mathrm{O}(Q, \mathbb{F}_7)/\{\pm 1\}$ are

$$\text{order } 2: \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}, \qquad \text{order } 3: \quad \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 5 \\ 3 & 3 \end{bmatrix}.$$

Since we have

$$h_0 \equiv \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad h_1 \equiv \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \qquad \bmod 1 - \zeta_7,$$

the group $\Gamma/\Gamma(1-\zeta_7)$ is generated by h_0, h_1 and ± 1, and isomorphic to $S_3 \times \{\pm 1\}$. Moreover we see that

$$g_0 = h_0^2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = h_1^2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \bmod 1 - \zeta_7,$$

and $\mathbf{Mon} = \langle g_0, \; g_1 \rangle \subset \Gamma(1-\zeta)$. Therefore $\overline{\mathbf{Mon}}$ coincides with $\overline{\Gamma(1-\zeta_7)}$ since we have $\overline{\Gamma}/\overline{\mathbf{Mon}} = S_3$ as mentioned earlier. Hence $\overline{\Gamma}$ is generated by h_0 and h_1. A homomorphism

$$\nu : \Gamma(1-\zeta_7) \longrightarrow \mathrm{M}_2(\mathbb{F}_7), \qquad \nu(g) = \frac{1}{1-\zeta_7}(g-1) \mod 1-\zeta_7$$

has the kernel $\Gamma((1-\zeta_7)^2)$, and the image is generated by

$$\nu(g_0) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \nu(g_1) = \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}.$$

Therefore we have $\Gamma(1-\zeta_7)/\Gamma((1-\zeta_7)^2) \cong (\mathbb{Z}/7\mathbb{Z})^2$. Since we have

$$[\Gamma(1-\zeta_7), \Gamma(1-\zeta_7)] \subset \Gamma((1-\zeta_7)^2), \qquad \mathbf{Mon}/[\mathbf{Mon}, \mathbf{Mon}] \cong (\mathbb{Z}/7\mathbb{Z})^2,$$

we conclude that $\overline{[\mathbf{Mon}, \mathbf{Mon}]} = \overline{\Gamma((1-\zeta_n)^2)}$. □

3. Heptagonal Curves

3.1. Hodge structure and Periods

From now, we concentrate in the case $n = 7$, that is, a 1-dimensional family of algebraic curves

$$X_t : y^7 = x(x-1)(x-t).$$

We denote $\zeta_7 = \exp(2\pi i/7)$ simply by ζ. As a Riemann surface, X_t is obtained by glueing seven sheets $\Sigma_1, \ldots, \Sigma_7$, each of which is a copy of $\mathbb{P}^1$ with cuts (see Figure 1) and satisfying $\rho(\Sigma_i) = \Sigma_{i+1}$ where indices are considered modulo 7. Let $\mathfrak{i}_i(x_1, x_2)$ be an oriented real interval from x_1 to x_2 on Σ_i. We define 1-cycles

$$\begin{aligned} \gamma_1 &= \mathfrak{i}_1(0,t) + \mathfrak{i}_2(t,0) = (1-\rho)\mathfrak{i}_1(0,t), \\ \gamma_2 &= \mathfrak{i}_1(t,1) + \mathfrak{i}_2(1,t) = (1-\rho)\mathfrak{i}_1(t,1), \\ \gamma_3 &= \mathfrak{i}_1(1,\infty) + \mathfrak{i}_2(\infty,1) = (1-\rho)\mathfrak{i}_1(1,\infty). \end{aligned}$$

For computation of intersection numbers, we use deformations of γ_1 and γ_3 as in Figure 3. Let $\mathbf{Int}_k$ be the intersection matrix $[\rho^i(\gamma_k) \cdot \rho^j(\gamma_k)]_{0 \le i,j \le 5}$. We have

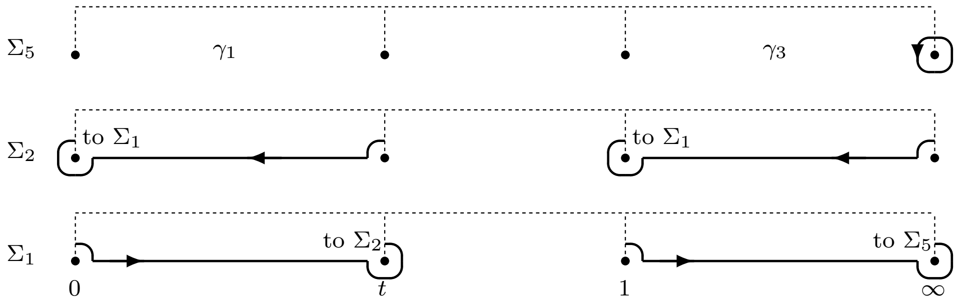

Figure 3. γ_1 and γ_3.

$$\mathbf{Int}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad \mathbf{Int}_3 = \begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

and $\det(\mathbf{Int}_1) = \det(\mathbf{Int}_3) = 1$. Since $\rho^i(\gamma_1) \cdot \rho^j(\gamma_3) = 0$, the intersection matrix of twelve 1-cycles $\gamma_1, \rho(\gamma_1), \dots, \rho^5(\gamma_1)$ and $\gamma_3, \rho(\gamma_3), \dots, \rho^5(\gamma_3)$ is unimodular, and they form a basis of $H_1(X_t, \mathbb{Z})$. Hence $\{\gamma_1,\ \gamma_3\}$ gives a basis of $H_1(X_t, \mathbb{Z}) \cong \mathbb{Z}[\rho]^2$ as a $\mathbb{Z}[\rho]$-module.

Similarly we have $H^1(X_t, \mathbb{Z}) \cong \mathbb{Z}[\rho]^2$ and the decomposition of $H^1(X_t, \mathbb{C}) \cong \mathbb{Z}[\rho]^2 \otimes \mathbb{C}$ into eigenspaces of ρ:

$$H^1(X_t, \mathbb{C}) = V(\zeta) \oplus V(\zeta^2) \oplus \cdots \oplus V(\zeta^6), \qquad \dim V(\zeta^k) = 2.$$

Let $P_0,\ P_1, P_t$ and P_∞ be four ramification points of X_t over $0, 1, t$ and ∞. We denote the divisor of a rational function (or a rational 1-form) f by $\mathrm{div}(f)$. Then we see that

$$\mathrm{div}(x) = 7P_0 - 7P_\infty, \qquad \mathrm{div}(y) = P_0 + P_1 + P_t - 3P_\infty,$$
$$\mathrm{div}(dx) = 6(P_0 + P_1 + P_t) - 8P_\infty,$$

and holomorphic 1-forms

$$\omega_1 = \frac{dx}{y^3}, \quad \omega_2 = \frac{dx}{y^4}, \quad \omega_3 = \frac{dx}{y^5}, \quad \omega_4 = \frac{xdx}{y^5}, \quad \omega_5 = \frac{dx}{y^6}, \quad \omega_6 = \frac{xdx}{y^6}$$

on X_t give a basis of $H^{1,0}(X_t)$.

Remark 3.1 As stated in the previous section, we have

$$V(\zeta)\oplus V(\zeta^2)\subset H^{1,0}(X_t),\qquad V(\zeta^5)\oplus V(\zeta^6)\subset H^{0,1}(X_t)$$

and the Hodge structure on $H^1(X_t,\mathbb{Z})$ is determined by a decomposition of $V(\zeta^4)$.

The following 1-cycles

$$\begin{array}{lll}
B_1=\gamma_1, & B_2=(1+\rho^2)(\gamma_1), & B_3=(1+\rho^2+\rho^4)(\gamma_1),\\
A_1=\rho(\gamma_1), & A_2=\rho^3(\gamma_1), & A_3=\rho^5(\gamma_1),\\
B_4=\rho^5(\gamma_3), & B_5=\rho^3(\gamma_3), & B_6=(1+\rho-\rho^4-\rho^5)(\gamma_3),\\
A_4=(1+\rho^2)(\gamma_3), & A_5=(-\rho+\rho^4+\rho^5)(\gamma_3), & A_6=(1+\rho+\rho^2)(\gamma_3),
\end{array}$$

give a symplectic basis of $H_1(X_t,\mathbb{Z})$ such that

$$A_i\cdot A_j=0,\quad B_i\cdot B_j=0,\quad B_i\cdot A_j=\delta_{ij}.$$

The associated period matrix is

$$\Pi_A=\left[\int_{A_i}\omega_j\right]=\begin{bmatrix}\int_{\gamma_1}\vec{\omega}R\\ \int_{\gamma_1}\vec{\omega}R^3\\ \int_{\gamma_1}\vec{\omega}R^5\\ \int_{\gamma_3}\vec{\omega}(I+R^2)\\ \int_{\gamma_3}\vec{\omega}(-R+R^4+R^5)\\ \int_{\gamma_3}\vec{\omega}(I+R+R^2)\end{bmatrix},$$

$$\Pi_B=\left[\int_{B_i}\omega_j\right]=\begin{bmatrix}\int_{\gamma_1}\vec{\omega}\\ \int_{\gamma_1}\vec{\omega}(I+R^2)\\ \int_{\gamma_1}\vec{\omega}(I+R^2+R^4)\\ \int_{\gamma_3}\vec{\omega}R^5\\ \int_{\gamma_3}\vec{\omega}R^3\\ \int_{\gamma_3}\vec{\omega}(I+R-R^4-R^5)\end{bmatrix}$$

where $\vec{\omega} = (\omega_1, \ldots, \omega_6)$ and R is a diagonal matrix $\mathrm{diag}(\zeta^4, \zeta^3, \zeta^2, \zeta^2, \zeta, \zeta)$. The normalized period matrix $\tau = \Pi_A \Pi_B^{-1}$ belongs to the Siegel upper half space $\mathbb{H}_6$, consisting of symmetric matrices of degree 6 whose imaginary part is positive definite. The symplectic group

$$Sp_{12}(\mathbb{Z}) = \{g \in \mathrm{GL}_{12}(\mathbb{Z}) \mid {}^t g J g = J\}, \quad J = \begin{bmatrix} 0 & \mathrm{I}_6 \\ -\mathrm{I}_6 & 0 \end{bmatrix},$$

acts on $\mathbb{H}_6$ by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$, and $\mathcal{A}_6 = \mathbb{H}_6 / Sp_{12}(\mathbb{Z})$ is the moduli space of principally polarized abelian varieties (p.p.a.v.) of dimension 6.

Remark 3.2 For a suitable choice of a branch of $\Omega_\alpha(x)$ in the previous section, we have

$$\int_{\gamma_k} \omega_1 = (1 - \zeta^4) u_k(t) \qquad (k = 1, 2, 3).$$

Since we use u_k for projective coordinates mainly, hereafter we denote $\int_{\gamma_k} \omega_1$ by u_k for simplicity.

3.2. Modular embedding

Let $M \in Sp_{12}(\mathbb{Z})$ be the symplectic representation of ρ with respect to the above basis:

$$(\rho(A_1), \ldots, \rho(A_6), \rho(B_1), \ldots, \rho(B_6)) = (A_1, \ldots, A_6, B_1, \ldots, B_6)^t M.$$

Explicit form of M is given in Appendix. By definition, we have $M \left[\begin{smallmatrix} \Pi_A \\ \Pi_B \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Pi_A \\ \Pi_B \end{smallmatrix}\right] R$. Therefore $\Pi_A \Pi_B^{-1}$ belongs to a domain $\mathbb{H}_6^M = \{\tau \in \mathbb{H}_6 \mid M \cdot \tau = \tau\}$, which parametrizes p.p.a.v of dimension 6 with an automorphism M (see section 5 in [vG92]). We know that this domain is 1-dimensional, and hence isomorphic to $\mathbb{D}_H^+$ ([BL92, Chapter 9] and [Sm64]). The centralizer of M in $Sp_{12}(\mathbb{Z})$ is

$$Sp_{12}^M(\mathbb{Z}) = \{g \in Sp_{12}(\mathbb{Z}) \mid gM = Mg\},$$

which acts on the domain $\mathbb{H}_6^M$.

Proposition 3.1 *There exist a group isomorphisms $\phi : \Gamma \to Sp_{12}^M(\mathbb{Z})$ and an analytic isomorphism $\Phi : \mathbb{D}_H^+ \to \mathbb{H}_6^M$ such that $\Phi(gu) = \phi(g)\Phi(u)$. We* *have the following commutative diagram.*

$$\begin{tikzcd}
\mathbb{D}_H^+ \arrow[r, "\Phi"] \arrow[d] & \mathbb{H}_6^M \arrow[d] \\
\mathbb{D}_H^+/\Gamma \arrow[r] & \mathbb{H}_6^M/Sp_{12}^M(\mathbb{Z})
\end{tikzcd}$$

Proof. Now we have

$$\begin{aligned}
\Pi_{A,1} &= {}^t\left[\int_{A_1}\omega_1,\dots,\int_{A_6}\omega_1\right] \\
&= {}^t[\zeta^4 u_1,\ \zeta^5 u_1,\ \zeta^6 u_1,\ (1+\zeta)u_3,\ (\zeta^2-\zeta^4+\zeta^6)u_3,\ (1+\zeta+\zeta^4)u_3], \\
\Pi_{B,1} &= {}^t\left[\int_{B_1}\omega_1,\dots,\int_{B_6}\omega_1\right] \\
&= {}^t[u_1,(1+\zeta)u_1,\ (1+\zeta+\zeta^2)u_1,\ \zeta^6 u_3,\ \zeta^5 u_3,\ (1+\zeta^4-\zeta^2-\zeta^6)u_3].
\end{aligned}$$

This correspondence $\left[\begin{smallmatrix} u_1 \\ u_3 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \Pi_{A,1} \\ \Pi_{B,1} \end{smallmatrix}\right]$ define a linear map $\Phi_1 : \mathbb{C}^2 \to \mathbb{C}^{12}$. Since coefficients of u_1 (or u_3) in $\Pi_{A,1}$ and $\Pi_{B,1}$ give a $\mathbb{Z}$-basis of $\mathbb{Z}[\zeta]$, there exists a homomorphism $\phi : \mathrm{GL}_2(\mathbb{Z}[\zeta]) \to \mathrm{GL}_{12}(\mathbb{Z})$ such that $\Phi_1(gu) = \phi(g)\Phi_1(u)$. Especially, we have $\phi(\zeta^4 I_2) = M$ and the image of ϕ is the centralizer of M. We can easily check that the condition

$$|u_1|^2 + (1+\zeta^3+\zeta^4)|u_3|^2 < 0$$

for $\mathbb{D}_H^+$ is equivalent to Riemann's relation ([M83])

$$\mathrm{Im}\left(\sum_{i=1}^{6}\overline{\int_{B_i}\omega_1}\int_{A_i}\omega_1\right) > 0,$$

and hence $\phi(\Gamma) = Sp_{12}^M(\mathbb{Z})$. We give the map Φ, which is compatible with Φ_1, explicitly in Appendix. $\square$

Remark 3.3 Let us define a homomorphism

$$\lambda : H_1(X_t,\mathbb{Z}) = \langle\gamma_1,\gamma_3\rangle_{\mathbb{Z}[\rho]} \longrightarrow \mathbb{Z}[\zeta]^2,$$

$$F_1(\rho)\gamma_1 + F_3(\rho)\gamma_3 \mapsto (F_1(\zeta^4), F_3(\zeta^4)).$$

By explicit computation, we see that the intersection form (which gives the polarization) on $H_1(X_t, \mathbb{Z})$ is given by

$$E(x, y) = \frac{1}{7} Tr_{\mathbb{Q}(\zeta)/\mathbb{Q}}((\zeta^3 - \zeta^4)^t \overline{\lambda(x)} H^{-1} \lambda(y)).$$

4. Schwarz inverse and theta function

4.1. Abel-Jacobi map

For the normalized holomorphic 1-forms

$$\vec{\xi} = (\xi_1, \ldots, \xi_6) = (\omega_1, \ldots, \omega_6)\Pi_B^{-1}$$

with respect to A_i and B_i in the previous section, period integrals satisfy

$$\tau = \left[\int_{A_i} \vec{\xi}\right]_{1 \le i \le 6} \in \mathbb{H}_6^M, \qquad \left[\int_{B_i} \vec{\xi}\right]_{1 \le i \le 6} = I_6.$$

Let $Div(X_t)$ be the group of divisors on X_t, and $J(X_t)$ be the Jacobian variety $\mathbb{C}^6/(\mathbb{Z}^6\tau + \mathbb{Z}^6)$. The Abel-Jacobi map with the base point P_∞ is

$$Div(X_t) \longrightarrow J(X_t), \qquad \sum m_i Q_i \mapsto \sum m_i \int_{P_\infty}^{Q_i} \vec{\xi} \mod \mathbb{Z}^6\tau + \mathbb{Z}^6.$$

We denote this homomorphism by $\overline{\mathfrak{A}}$, and a lift of $\overline{\mathfrak{A}}(D)$ by $\mathfrak{A}(D)$ (Hence $\mathfrak{A} : Div(X_t) \to \mathbb{C}^6$ is a multi-valued map). As is well known, $\overline{\mathfrak{A}}$ factors through

$$Div(X_t) \longrightarrow Pic(X_t) = Div(X_t)/\{\text{principal divisors}\}.$$

Since the base point is fixed by ρ, the map $\overline{\mathfrak{A}}$ is ρ-equivariant. Therefore the image of a ρ-invariant divisor belongs to the set of fixed points of ρ, that is, the $(1-\rho)$-torsion subgroup

$$J(X_t)_{1-\rho} = \{z \in J(X_t) \mid (1-\rho)z = 0\}.$$

Lemma 4.1 *The $(1-\rho)$-torsion subgroup is*

$$J(X_t)_{1-\rho} = \{\overline{\mathfrak{A}}(mP_0 + nP_1) \mid m, n \in \mathbb{Z}\} \cong (\mathbb{Z}/7\mathbb{Z})^2.$$

More explicitly, we have

$$\mathfrak{A}(mP_0 + nP_1) \equiv a_{m,n}\tau + b_{m,n} \mod \mathbb{Z}^6\tau + \mathbb{Z}^6$$

with

$$\begin{aligned} a_{m,n} &= \frac{1}{7}(m,\, 2m,\, 3m,\, 2m+3n,\, 2m+3n,\, 0) \in \frac{1}{7}\mathbb{Z}^6, \\ b_{m,n} &= \frac{1}{7}(-m,\, -m,\, -m,\, 3m+n,\, 5m+4n,\, m+5n) \in \frac{1}{7}\mathbb{Z}^6. \end{aligned}$$

Proof. It is obvious that $Ker(1-\rho) \cong (\mathbb{Z}[\zeta]/(1-\zeta))^2 \cong (\mathbb{Z}/7\mathbb{Z})^2$. Recall that

$$\gamma_1 = (1-\rho)\mathfrak{i}_1(0,1), \quad \gamma_2 = (1-\rho)\mathfrak{i}_1(1,\infty), \quad \gamma_3 = (1-\rho)\mathfrak{i}_1(t,1).$$

Computing intersection numbers, we see that

$$\gamma_2 = A_1 + A_2 + A_3 + B_4 + B_5 = \rho(\gamma_1) + \rho^3(\gamma_1) + \rho^5(\gamma_1) + \rho^5(\gamma_3) + \rho^3(\gamma_3).$$

Therefore we have

$$\begin{aligned} \mathfrak{i}_1(0,t) &= \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_1 \\ &= \frac{1}{7}(5A_1 + 3A_2 + A_3 + 2B_1 + 2B_2 + 2B_3), \\ \mathfrak{i}_1(1,\infty) &= \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_3 \\ &= \frac{1}{7}(-3A_4 + 4A_5 + 7A_6 - B_4 + 3B_5 + 2B_6), \\ \mathfrak{i}_1(t,1) &= \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_2 \\ &= \frac{1}{7}(A_1 + 2A_2 + 3A_3 - B_1 - B_2 - B_3) \\ &\quad + \frac{1}{7}(A_4 + A_5 - 7A_6 + 5B_4 - B_5 + 4B_6), \end{aligned}$$

namely,

$$\int_0^t \vec{\xi} \equiv \frac{1}{7}(5,3,1,0,0,0)\tau + \frac{1}{7}(2,2,2,0,0,0),$$
$$\int_1^\infty \vec{\xi} \equiv \frac{1}{7}(0,0,0,4,4,0)\tau + \frac{1}{7}(0,0,0,6,3,2),$$
$$\int_t^1 \vec{\xi} \equiv \frac{1}{7}(1,2,3,1,1,0)\tau + \frac{1}{7}(6,6,6,5,6,4) \mod \mathbb{Z}^6 + \tau\mathbb{Z}^6.$$

As combinations of these integrals, we obtain explicit values of $\overline{\mathfrak{A}}(P_0)$ and $\overline{\mathfrak{A}}(P_1)$. □

4.2. Theta function and Riemann constant

Let us consider Riemann's theta function

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}^6} \exp[\pi i n\tau {}^t n + 2\pi i n {}^t z], \quad (z,\tau) \in \mathbb{C}^6 \times \mathbb{H}_6.$$

The Abel-Jacobi map $\overline{\mathfrak{A}}$ induces a birational morphism from $\mathrm{Sym}^6 X_t$ to $J(X_t)$, and $W_{\mathfrak{A}}^5 = \overline{\mathfrak{A}}(\mathrm{Sym}^5 X_t)$ is a translation of the theta divisor

$$\Theta = \{z \in J(X_t) \mid \vartheta(z) = 0\}.$$

More precisely, there exist a constant vector $\kappa \in \mathbb{C}^6$ such that $\vartheta(e,\tau) = 0$ if and only if

$$e \equiv \kappa - \mathfrak{A}(Q_1 + \cdots + Q_5) \mod \mathbb{Z}^6\tau + \mathbb{Z}^6$$

for some $Q_1, \ldots, Q_5 \in X_t$. The constant κ (or its image $\overline{\kappa}$ in $J(X_t)$) is called the Riemann constant. It is the image of a half canonical class by $\mathfrak{A}$ ([M83, Chapter II], Appendix to Section 3), and depends only on a symplectic basis A_i, B_i and the base point of $\mathfrak{A}$. Since $\mathrm{div}(\omega_5) = 10P_\infty$, the image of the canonical class by $\overline{\mathfrak{A}}$ is 0 and κ must be a half period. Hence we have $\kappa = a\tau + b$ for some $a, b \in (1/2)\mathbb{Z}^6$. By the same argument as the proof of Lemma 5.4 in [K03], the corresponding theta characteristic (a,b) is invariant under the action of M on $\mathbb{Q}^{12}/\mathbb{Z}^{12}$:

$$M \cdot (a,b) = (a,b)M^{-1} + \frac{1}{2}((C^tD)_0, (A^tB)_0), \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $(M)_0$ is the diagonal vector of M. By explicit computations, we have

Lemma 4.2 *The M-invariant theta characteristics are $(a_{m,n} + a_0, b_{m,n} + b_0)$ with*

$$a_0 = \frac{1}{2}(1,0,1,0,0,1), \quad b_0 = \frac{1}{2}(1,1,1,0,1,0).$$

Especially, we have $\kappa \equiv a_0\tau + b_0$. Since $\vartheta(-e) = \vartheta(e)$ and κ is a half period, we have

$$\overline{\kappa} - W^5_{\mathfrak{A}} = \Theta = -\Theta = \overline{\kappa} + W^5_{\mathfrak{A}},$$

that is $W^5_{\mathfrak{A}} = -W^5_{\mathfrak{A}}$.

Let us consider $J(X)_{1-\rho} \cap W^5_{\mathfrak{A}}$. By definition, we have

$$\overline{\mathfrak{A}}(mP_0 + nP_1) \in W^5_{\mathfrak{A}} = -W^5_{\mathfrak{A}}$$

for $0 \leq m, n \leq 6$ such that $m + n \leq 5$ or $(7-m) + (7-n) \leq 5$. The rest of $J(X)_{1-\rho}$ are $\overline{\mathfrak{A}}(mP_0 + nP_1)$ with the following (m,n):

$$\begin{gathered}(1,5),\ (1,6),\ (2,4)\ (2,5),\ (2,6),\ (3,3),\ (3,4),\ (3,5),\\ (4,2),\ (4,3),\ (4,4),\ (5,1),\ (5,2),\ (5,3),\ (6,1),\ (6,2).\end{gathered}$$

Moreover we have the following reductions:

$$(6P_0 + P_1) = (2P_1 + P_t + 4P_\infty) + \mathrm{div}\left(\frac{x}{y}\right),$$

$$(3P_0 + 3P_1) = (4P_t + 2P_\infty) + \mathrm{div}\left(\frac{x(x-1)}{y^4}\right),$$

$$(5P_0 + P_1) = (3P_1 + 2P_t + P_\infty) + \mathrm{div}\left(\frac{x}{y^2}\right),$$

$$(4P_0 + 3P_1) = (P_0 + 4P_t + 2P_\infty) + \mathrm{div}\left(\frac{x(x-1)}{y^4}\right),$$

that is,

$$\overline{\mathfrak{A}}(6P_0+P_1),\quad \overline{\mathfrak{A}}(3P_0+3P_1),\quad \overline{\mathfrak{A}}(5P_0+P_1),\quad \overline{\mathfrak{A}}(4P_0+3P_1)\in W^5_{\mathfrak{A}}.$$

By the equality $W^5_{\mathfrak{A}} = -W^5_{\mathfrak{A}}$ and the symmetry for P_0, P_1, we see that $\overline{\mathfrak{A}}(mP_0+nP_1)\in W^5_{\mathfrak{A}}$ if

$$(m,n)\neq(2,4),\ (2,5),\ (3,5),\ (4,2),\ (5,2),\ (5,3).$$

The converse is also true:

Lemma 4.3 *We have* $\overline{\mathfrak{A}}(mP_0+nP_1)\notin W^5_{\mathfrak{A}}$ *for*

$$(m,n)=(2,4),\ (2,5),\ (3,5),\ (4,2),\ (5,2),\ (5,3).$$

Proof. To prove this, note that

$$(5P_0+2P_1)=(4P_1+2P_t+P_\infty)+\operatorname{div}\left(\frac{x}{y^2}\right),$$

and hence $\overline{\mathfrak{A}}(5P_0+2P_1)=\overline{\mathfrak{A}}(4P_1+2P_t)$. Moreover we have

$$\overline{\mathfrak{A}}(3P_i+5P_j)=-\overline{\mathfrak{A}}(4P_i+2P_j),\qquad i,j\in\{0,1\}.$$

By symmetry for P_0, P_1 and P_t, it suffices to prove that $\overline{\mathfrak{A}}(4P_0+2P_1)\notin W^5_{\mathfrak{A}}$.

Applying the Riemann-Roch formula for $4P_0+2P_1$, we have

$$\ell(4P_0+2P_1)=\ell(K-4P_0-2P_1)+1,$$

where $\ell(D)=\dim H^0(X_t,\mathcal{O}(D))$ and K is the canonical class. From the vanishing order of ω_i:

	ω_5	ω_3	ω_2	ω_1	ω_6	ω_4
at P_0	0	1	2	3	7	8

,

	ω_5	ω_6	ω_3	ω_4	ω_2	ω_1
at P_1	0	0	1	1	2	3

,

we see that there does not exist a holomorphic 1-form ω such that $\operatorname{div}(\omega)-4P_0-2P_1$ is positive. Therefore we have $\ell(4P_0+2P_1)=1$ and $H^0(X_t,\mathcal{O}(4P_0+2P_1))$ contains only constant functions. This implies $\overline{\mathfrak{A}}(4P_0+2P_1)\notin W^5_{\mathfrak{A}}$. □

4.3. Jacobi inversion

We apply Theorem 4 in [Si71, Chapter 4, Section 11], for rational functions

$$f : X_t \longrightarrow \mathbb{P}^1,\ (x,y) \mapsto x, \qquad g : X_t \longrightarrow \mathbb{P}^1,\ (x,y) \mapsto 1-x$$

on X_t. Then we have

$$f(Q_1) \times \cdots \times f(Q_6) = \frac{1}{E} \prod_{k=1}^{7} \frac{\vartheta(\kappa - \mathfrak{A}(Q_1 + \cdots + Q_6) + \int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(Q_1 + \cdots + Q_6), \tau)}, \quad (1)$$

$$g(Q_1) \times \cdots \times g(Q_6) = \frac{1}{E'} \prod_{k=1}^{7} \frac{\vartheta(\kappa - \mathfrak{A}(Q_1 + \cdots + Q_6) + \int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(Q_1 + \cdots + Q_6), \tau)}, \quad (2)$$

where constants E and E' are independent of $Q_1, \ldots, Q_6$, integrals $\int_{\mathrm{i}_k(\infty,*)} \vec{\xi} \in \mathbb{C}^6$ are chosen such that

$$\int_{\mathrm{i}_1(\infty,*)} \vec{\xi} + \cdots + \int_{\mathrm{i}_7(\infty,*)} \vec{\xi} = 0,$$

and $\mathfrak{A}(Q_1 + \cdots + Q_6) \in \mathbb{C}^6$ takes the same value in the numerator and the denominator.

Substituting $4P_1 + 2P_t$ and $2P_1 + 4P_t$ for $Q_1 + \cdots + Q_6$ in (1), and taking their ratio, we have an expression of t^2 by theta values:

$$\begin{aligned} t^2 &= (f(P_1)^2 f(P_t)^4) / (f(P_1)^4 f(P_t)^2) \\ &= \prod_{k=1}^{7} \frac{\vartheta(\kappa - \mathfrak{A}(2P_1 + 4P_t) + \int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(2P_1 + 4P_t), \tau)} \Bigg/ \\ &\qquad \prod_{k=1}^{7} \frac{\vartheta(\kappa - \mathfrak{A}(4P_1 + 2P_t) + \int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(4P_1 + 2P_t), \tau)} \\ &= \prod_{k=1}^{7} \frac{\vartheta(\kappa + a_{4,2}\tau + b_{4,2} + \int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta(\kappa + a_{4,2}\tau + b_{4,2}, \tau)} \Bigg/ \\ &\qquad \prod_{k=1}^{7} \frac{\vartheta(\kappa + a_{5,2}\tau + b_{5,2} + \int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta(\kappa + a_{5,2}\tau + b_{5,2}, \tau)}. \end{aligned} \quad (3)$$

Similarly, substituting $4P_1 + 2P_t$ and $2P_1 + 4P_t$ for $Q_1 + \cdots + Q_6$ in (2), we have

$$\begin{aligned}
(1-t)^2 &= (g(P_0)^2 g(P_t)^4)\big/(g(P_1)^4 g(P_t)^2) \\
&= \prod_{k=1}^{7} \frac{\vartheta(\kappa - \mathfrak{A}(2P_0 + 4P_t) + \int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(2P_0 + 4P_t), \tau)} \Bigg/ \\
&\qquad\qquad \prod_{k=1}^{7} \frac{\vartheta(\kappa - \mathfrak{A}(4P_0 + 2P_t) + \int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(4P_0 + 2P_t), \tau)} \\
&= \prod_{k=1}^{7} \frac{\vartheta(\kappa + a_{2,4}\tau + b_{2,4} + \int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta(\kappa + a_{2,4}\tau + b_{2,4}, \tau)} \Bigg/ \\
&\qquad\qquad \prod_{k=1}^{7} \frac{\vartheta(\kappa + a_{5,2}\tau + b_{5,2} + \int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta(\kappa + a_{5,2}\tau + b_{5,2}, \tau)}. \qquad (4)
\end{aligned}$$

The above expressions are simplified by introducing theta functions with characteristcs $a, b \in \mathbb{Q}^6$:

$$\begin{aligned}
\vartheta_{a,b}(z,\tau) &= \exp[\pi i a \tau {}^t a + 2\pi i a^t (z+b)] \vartheta(z + a\tau + b, \tau) \\
&= \sum_{n \in \mathbb{Z}^6} \exp[\pi i (n+a) \tau^t (n+a) + 2\pi i (n+a)^t (z+b)].
\end{aligned}$$

We denote a theta constant $\vartheta_{a,b}(0,\tau)$ by $\vartheta_{a,b}(\tau)$. Let $\vartheta_{[m,n]}(z,\tau)$ be $\vartheta_{a,b}(z,\tau)$ with characteristics $a = a_{m,n} + a_0$, $b = b_{m,n} + b_0$ in Lemma 4.2. With this notation, theta expressions (3) and (4) are

$$t^2 = \prod_{k=1}^{7} \frac{\vartheta_{[2,5]}(\tau)\, \vartheta_{[4,2]}(\int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta_{[4,2]}(\tau)\, \vartheta_{[2,5]}(\int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)},$$

$$(1-t)^2 = \prod_{k=1}^{7} \frac{\vartheta_{[5,2]}(\tau)\, \vartheta_{[2,4]}(\int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta_{[2,4]}(\tau)\, \vartheta_{[5,2]}(\int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)}.$$

Putting

$$\int_{\mathrm{i}_k(\infty,x)} \vec{\xi} = \begin{cases} a_{1,0}\tau + b_{1,0} \ (x=0) \\ a_{0,1}\tau + b_{0,1} \ (x=1) \end{cases} \qquad (1 \le k \le 6),$$

$$\int_{\mathrm{i}_7(\infty,x)} \vec{\xi} = \begin{cases} -6(a_{1,0}\tau + b_{1,0}) \ (x=0) \\ -6(a_{1,0}\tau + b_{1,0}) \ (x=1) \end{cases}$$

and using formulas

$$\vartheta_{a,b}(a'\tau + b', \tau) = \exp[-\pi i a'\tau^t a' - 2\pi i a'^t(b+b')]\vartheta_{a+a',b+b'}(0,\tau), \qquad a', b' \in \mathbb{Q}^6,$$

$$\theta_{(a+a',b+b')}(z,\tau) = \exp(2\pi\sqrt{-1}a^t b')\theta_{(a,b)}(z,\tau), \qquad a', b' \in \mathbb{Z}^6,$$

we see that

$$\prod_{k=1}^{7} \frac{\vartheta_{[4,2]}(\int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta_{[2,5]}(\int_{\mathrm{i}_k(\infty,0)} \vec{\xi}, \tau)} = \zeta^3 \frac{\vartheta_{[5,2]}(\tau)^7}{\vartheta_{[3,5]}(\tau)^7},$$

$$\prod_{k=1}^{7} \frac{\vartheta_{[2,4]}(\int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta_{[5,2]}(\int_{\mathrm{i}_k(\infty,1)} \vec{\xi}, \tau)} = \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[5,3]}(\tau)^7}.$$

Since $\vartheta_{-a,-b}(-z,\tau) = \vartheta_{a,b}(z,\tau)$, we can easily show the following equalities

$$\vartheta_{[2,5]}(\tau) = \vartheta_{[5,2]}(\tau), \quad \vartheta_{[2,4]}(\tau) = \vartheta_{[5,3]}(\tau), \quad \vartheta_{[3,4]}(\tau) = \vartheta_{[4,3]}(\tau).$$

Therefore the above expressions of t^2 and $(1-t)^2$ are simply

$$t^2 = \zeta^3 \frac{\vartheta_{[5,2]}(\tau)^{14}}{\vartheta_{[4,2]}(\tau)^{14}}, \qquad (1-t)^2 = \frac{\vartheta_{[2,5]}(\tau)^{14}}{\vartheta_{[2,4]}(\tau)^{14}}.$$

Namely, there exist constants $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$ such that

$$t = \zeta^3 \varepsilon_1 \frac{\vartheta_{[5,2]}(\tau)^7}{\vartheta_{[4,2]}(\tau)^7}, \qquad 1 - t = \varepsilon_2 \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[2,4]}(\tau)^7}. \tag{5}$$

For $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}_{2g}(\mathbb{Z})$, theta constants $\vartheta_{a,b}(\tau)$ satisfy the transformation formula

$$\vartheta_{g\cdot(a,b)}(g\tau) = \mu(g)\exp[2\pi i\lambda_{a,b}(g)]\det(C\tau+D)^{1/2}\vartheta_{(a,b)}(\tau)$$

where

$$g\cdot(a,b) = (a,b)g^{-1} + \frac{1}{2}((C^tD)_0,(A^tB)_0),$$

$$\lambda_{a,b}(g) = -\frac{1}{2}({}^ta^tDBa - 2^ta^tBCb + {}^tb^tCAb) + \frac{1}{2}({}^ta^tD -^t b^tC)(A^tB)_0,$$

and $\mu(g)$ is a certain 8-th root of 1 depending only on g. Therefore, as coordinates of $\mathbb{P}^2(\mathbb{C})$, we have

$$\begin{aligned}&[\vartheta_{g[2,4]} : \vartheta_{g[2,5]} : \vartheta_{g[3,5]}](g\cdot\tau)\\ &\quad = [\mathbf{e}[\lambda_{[2,4]}(g)]\vartheta_{[2,4]} : \mathbf{e}[\lambda_{[2,5]}(g)]\vartheta_{[2,5]} : \mathbf{e}[\lambda_{[3,5]}(g)]\vartheta_{[3,5]}](\tau)\end{aligned} \tag{6}$$

where $\mathbf{e}[-] = \exp[2\pi i-]$.

By explicit forms of $\sigma_0 = \phi(h_0)$ and $\sigma_1 = \phi(h_1)$ in Appendix, we see that

$$\begin{aligned}&\lambda_{[2,4]}(\sigma_0) = 53/56, \quad \lambda_{[2,5]}(\sigma_0) = 53/56, \quad \lambda_{[3,5]}(\sigma_0) = 7/8,\\ &\lambda_{[2,4]}(\sigma_1) = 25/56, \quad \lambda_{[2,5]}(\sigma_1) = 19/392, \quad \lambda_{[3,5]}(\sigma_1) = 79/392,\end{aligned}$$

and

$$\begin{aligned}&\vartheta_{\sigma_0[2,4]} = \mathbf{e}[5/14]\vartheta_{[5,2]}, \quad \vartheta_{\sigma_0[2,5]} = -\vartheta_{[5,3]}, \quad \vartheta_{\sigma_0[3,5]} = \mathbf{e}[13/14]\vartheta_{[4,2]},\\ &\vartheta_{\sigma_1[2,4]} = -\vartheta_{[5,3]}, \quad \vartheta_{\sigma_1[2,5]} = \mathbf{e}[9/14]\vartheta_{[4,2]}, \quad \vartheta_{\sigma_1[3,5]} = \mathbf{e}[4/7]\vartheta_{[5,2]}.\end{aligned}$$

Applying these for (6), we obtain

$$\begin{aligned}&[\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}](\sigma_0\cdot\tau) = [-\vartheta_{[2,5]} : \mathbf{e}[9/14]\vartheta_{[2,4]} : \vartheta_{[3,5]}](\tau),\\ &[\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}](\sigma_1\cdot\tau) = [\vartheta_{[2,4]} : \mathbf{e}[67/98]\vartheta_{[3,5]} : \mathbf{e}[45/98]\vartheta_{[2,5]}](\tau).\end{aligned} \tag{7}$$

Theorem 4.1 (1) *The inverse of the Schwarz map*

$$\mathfrak{s} : \mathbb{C} - \{0,1\} \longrightarrow \mathbb{D}_H^+, \qquad t \mapsto u = [u_1(t) : u_3(t)]$$

is given by $\Gamma(1-\zeta)$*-invariant function* $\mathfrak{t}(u) = \zeta^5(\vartheta_{[2,5]}(\Phi(u))^7/\vartheta_{[3,5]}(\Phi(u))^7)$, *where* $\Phi : \mathbb{D}_H^+ \to \mathbb{H}_6^M$ *is the modular embedding given in Appendix. In other* *words,* $\Phi(u) \in \mathbb{H}_6^M$ *is the period matrix of an algebraic curve*

$$y^7 = x(x-1)(x-\mathfrak{t}(u)).$$

(2) *The analytic map*

$$\begin{aligned} &Th : \mathbb{D}_H^+ \longrightarrow \mathbb{P}^2(\mathbb{C}), \\ &u \mapsto [\mathbf{e}[5/49]\vartheta_{[2,4]}\vartheta_{[2,5]} : \vartheta_{[2,5]}\vartheta_{[3,5]} : -\vartheta_{[2,4]}\vartheta_{[3,5]}](\Phi(u)) \end{aligned}$$

induces an isomorphism $\mathbb{D}_H^+/\Gamma((1-\zeta)^2)$ *and the Fermat septic curve*

$$\mathcal{F}_7 : X^7 + Y^7 + Z^7 = 0, \qquad [X : Y : Z] \in \mathbb{P}^2(\mathbb{C}).$$

Proof. From (5), we have

$$1 = \varepsilon_1 \zeta^5 \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[3,5]}(\tau)^7} + \varepsilon_2 \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[2,4]}(\tau)^7}.$$

Since this equation must be invariant under actions of $\sigma_0 = \phi(h_0)$ and $\sigma_1 = \phi(h_1)$ in (7) (otherwise, the image of Th is not irreducible), we see that $\varepsilon_1 = \varepsilon_2 = 1$ and

$$t = \zeta^5 \frac{\vartheta_{[2,5]}(\Phi(u))^7}{\vartheta_{[3,5]}(\Phi(u))^7}.$$

Let us recall that $\Gamma(1-\zeta)$ is projectively generated by h_0^2 and h_1^2, and $\Gamma((1-\zeta)^2)$ is projectively isomorphic to the commutator subgroup of $\Gamma(1-\zeta)$. From (7), we see that

$$\begin{aligned} &[\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}](\sigma_0^2 \cdot \tau) = [\zeta\vartheta_{[2,4]} : \zeta\vartheta_{[2,5]} : \vartheta_{[3,5]}](\tau), \\ &[\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}](\sigma_1^2 \cdot \tau) = [\vartheta_{[2,4]} : \zeta\vartheta_{[2,5]} : \zeta\vartheta_{[3,5]}](\tau). \end{aligned}$$

Therefore the commutator subgroup of $\Gamma(1-\zeta)$ acts trivially on

$$[\vartheta_{[2,4]}(\Phi(u)) : \vartheta_{[2,5]}(\Phi(u)) : \vartheta_{[3,5]}(\Phi(u))] \in \mathbb{P}^2,$$

and the map Th gives a $(\mathbb{Z}/7\mathbb{Z})^2$-equivariant isomorphism of $\mathbb{D}_H^+/\Gamma((1-\zeta)^2)$ and the Fermat septic curve. $\square$

4.4. Klein quartic

It is known that the Klein quartic curve

$$\mathcal{K}_4 : X^3Y + Y^3Z + Z^3X = 0, \qquad [X : Y : Z] \in \mathbb{P}^2(\mathbb{C}),$$

is the quotient of $\mathcal{F}_7$ by an automorphism

$$\alpha : \mathcal{F}_7 \longrightarrow \mathcal{F}_7, \qquad [X : Y : Z] \mapsto [\zeta X : \zeta^3 Y : Z]$$

which is induced by $g_0 g_1^3 \in \Gamma(1-\zeta)$ via the map Th. The quotient map is given by

$$\mathcal{F}_7 \longrightarrow \mathcal{K}_4, \qquad [X : Y : Z] \mapsto [XY^3 : YZ^3 : ZX^3].$$

The Klein quartic $\mathcal{K}_4$ is isomorphic to the elliptic modular curve $\mathcal{X}(7)$ of level 7, and also to a Shimura curve parametrizing a family of QM Abelian 6-folds (see [E99]). The following Corollary gives a new moduli interpretation of $\mathcal{K}_4$.

Corollary 4.1 *The Klein quartic curve $\mathcal{K}_4$ is isomorphic to $\mathbb{D}_H^+/\Gamma_{Klein}$, where*

$$\Gamma_{Klein} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1-\zeta) \,\middle|\, a \equiv 1 \mod (1-\zeta)^2 \right\}.$$

Proof. Let us recall the homomorphism

$$\nu : \Gamma(1-\zeta) \longrightarrow \mathrm{M}_2(\mathbb{F}_7), \qquad \nu(g) = \frac{1}{1-\zeta}(g-1) \mod 1-\zeta$$

in the proof of Proposition 2.1. The kernel of ν is $\Gamma((1-\zeta)^2)$ and the image is generated by

$$\nu(g_0) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \nu(g_1) = \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}.$$

Since we have $\nu(g_0^a g_1^b) = \left[\begin{smallmatrix} -a+5b & b \\ 5b & b \end{smallmatrix}\right]$, the group Γ_{Klein} is generated by $\Gamma((1-\zeta)^2)$ and $g_0 g_1^3$. Namely we have $\mathbb{D}_H^+/\Gamma_{Klein} = \mathcal{F}_7/\langle\alpha\rangle$. $\square$

Let (A, E, ρ, λ) be a 4-tuple

(1) A is a 6-dimensional complex Abelian variety V/Λ, where V is isomorphic to the tangent space T_0A and Λ is isomorphic to $H_1(A,\mathbb{Z})$.
(2) $E:\Lambda\times\Lambda\to\mathbb{Z}$ is a principal polarization.
(3) ρ is an automorphism of order 7 preserving E, and the induced action on T_0A has eigenvalues $\zeta,\zeta,\zeta^2,\zeta^2,\zeta^3,\zeta^4$.
(4) $\lambda:\Lambda\to\mathbb{Z}[\zeta]^2$ is an isomorphism such that

$$\lambda(\rho(x))=\zeta^4\lambda(x),\qquad E(x,y)=\frac{1}{7}Tr_{\mathbb{Q}(\zeta)/\mathbb{Q}}\big((\zeta^3-\zeta^4)^t\overline{\lambda(x)}H^{-1}\lambda(y)\big)$$

(see Remark 3.3). Note that λ induces an isomorphism of the torsion subgroup A_{tor} and $(\mathbb{Q}(\zeta)/\mathbb{Z}[\zeta])^2$.

An isomorphism $f:(A,E,\rho,\lambda)\to(A',E',\rho',\lambda')$ is defined as an isomorphism of Abelian varieties $f:A\to A'$ such that $f^*E'=E$, $f\circ\rho=\rho'\circ f$ and $\lambda=\lambda'\circ f$. Then we see the following.

Corollary 4.2 *We have isomorphisms*

$$\mathbb{D}_H^+/\Gamma(\mathfrak{m})\cong\left\{\begin{array}{c}\textit{Set of } (A,E,\rho,\lambda) \textit{ modulo isomorphisms } f \textit{ such that}\\ \lambda^{-1}\equiv(\lambda'\circ f)^{-1}\quad \textit{on}\quad (\mathfrak{m}^{-1}\mathbb{Z}[\zeta]/\mathbb{Z}[\zeta])^2\end{array}\right\},$$

$$\mathbb{D}_H^+/\Gamma_{Klein}\cong\left\{\begin{array}{l}\textit{Set of } (A,E,\rho,\lambda) \textit{ modulo isomorphisms } f \textit{ such that}\\ \quad \text{(i) } \lambda^{-1}\left(\dfrac{1}{(1-\zeta)^2},0\right)=(\lambda'\circ f)^{-1}\left(\dfrac{1}{(1-\zeta)^2},\dfrac{b}{1-\zeta}\right)\\ \qquad \textit{for } {}^{\exists}b\in\mathbb{Z}[\zeta]\\ \quad \text{(ii) } \lambda^{-1}\left(0,\dfrac{1}{1-\zeta}\right)=(\lambda'\circ f)^{-1}\left(0,\dfrac{1}{1-\zeta}\right)\end{array}\right\}.$$

5. K3 surface

In this final section, we construct K3 surfaces with a non-symplectic automorphism of order 7 attached to X_t, according to Garbagnati and Penegini ([GP]). For generalities on K3 surfaces and elliptic surfaces, see [SS10] and references therein. Let us consider two curves

$$X_t: y_1^7=x_1(x_1-1)(x_1-t),\qquad X_\infty: y_2^7=x_2^2-1$$

and an affine algebraic surface

$$S_t : y^2 = x(x-z)(x-tz) + z^{10}.$$

Note that X_t is a hyperelliptic curve of genus 3. The surface S_t is birational to the quotient of $X_t \times X_\infty$ by an automorphism

$$\rho \times \rho : X_t \times X_\infty \longrightarrow X_t \times X_\infty, \quad (x_1, y_1) \times (x_2, y_2) \mapsto (x_1, \zeta y_1) \times (x_2, \zeta y_2),$$

and the rational quotient map $X_t \times X_\infty \dashrightarrow S_t$ is given by

$$z = y_1/y_2, \qquad y = z^5 x_2, \qquad x = z x_1.$$

The minimal smooth compact model of S_t (denoted by the same symbol S_t) is a K3 surface with an elliptic fibration

$$\pi : S_t \longrightarrow \mathbb{P}^1, \quad (x, y, z) \mapsto z.$$

To see this, let us consider a minimal Weierstrass form

$$\begin{aligned} S'_t : y^2 &= x^3 + G_2(z)x + G_3(z) \\ G_2(z) &= -\frac{1}{3}(t^2 - t + 1)z^2, \\ G_3(z) &= z^{10} - \frac{1}{27}(2t-1)(t+1)(t-2)z^3 \end{aligned}$$

and the discriminant

$$\begin{aligned} \Delta(z) &= 4G_2(z)^3 + 27G_3(z)^2 \\ &= z^6\{27z^{14} - 2(2t-1)(t+1)(t-2)z^7 - t^2(t-1)^2\}. \end{aligned}$$

From this, we see that S_t is a K3 surface, and it has a singular fiber of type I_0^* at $z = 0$, of type IV at $z = \infty$ and fourteen fibers of type I_1 on $\mathbb{P}^1 - \{0, \infty\}$. Note that

$$\frac{dx_1}{y_1^3} \otimes \frac{y_2^2 dy_2}{x_2} \in H^0(X_t, \Omega^1) \otimes H^0(X_\infty, \Omega^1)$$

is the unique $(\rho \times \rho)$ - invariant element up to constants, and descents to a holomorphic 2-form on S_t (see [GP, Section 3]). Therefore the period map for a family of K3 surface S_t is given by the Schwarz map $\mathfrak{s}$. Note also that an automorphism $\rho \times \mathrm{id}$ of $X_t \times X_\infty$ descends to S_t:

$$\rho \times \mathrm{id} : S_t \longrightarrow S_t, \quad (x, y, z) \mapsto (\zeta x, \zeta^5 y, \zeta z).$$

Since $S_t / \langle \rho \times \mathrm{id} \rangle$ is birational to a rational surface $X_t / \langle \rho \rangle \times X_\infty / \langle \rho \rangle$, the automorphism $\rho \times \mathrm{id}$ is non-symplectic. Hence the transcendental lattice T_{S_t} is a free $\mathbb{Z}[\rho \times \mathrm{id}]$-module ([Ni79]). Since our family has positive dimensional moduli, we have $\operatorname{rank} \mathrm{T}_{S_t} \geq 12$ and $\operatorname{rank} \mathrm{NS}(S_t) \leq 10$ for a general $t \in \mathbb{C} - \{0, 1\}$, where $\mathrm{NS}(S_t)$ is the Néron-Severi lattice.

Let us compute the Néron-Severi lattice and the Mordell-Weil group $\mathrm{MW}(S_t)$. Let o be the zero section of $\pi : S_t \to \mathbb{P}^1$. We have three sections

$$s_a : \mathbb{P}^1 \longrightarrow S_t, \quad z \mapsto (x, y, z) = (az, z^5, z), \qquad a = 0,\ 1,\ t$$

such that $s_0 + s_1 + s_t = o$ in $\mathrm{MW}(S_t)$. Let $2\ell_0 + \ell_1 + \ell_2 + \ell_3 + \ell_4$ be the irreducible decomposition of $\pi^{-1}(0)$, and $\ell'_1 + \ell'_2 + \ell'_3$ be that of $\pi^{-1}(\infty)$. For a suitable choice of indeces, intersection numbers of these curves are given by the following graph; the self intersection number of each curve is -2, two curves are connected by an edge if they intersect and intersection numbers are 1 except $s_a \cdot s_b = 2$ (Figure 4). Let $N \subset \mathrm{NS}(S_t)$ be the lattice generated by $o,\ s_0,\ s_1,\ s_t,\ \ell_0,\ \ell_1,\ \ell_2,\ \ell_3,\ \ell_4,\ \ell'_1$. The rank of N is 10 and the discriminant is -49. Hence the Picard number of S_t is generically 10 and the rank of $\mathrm{MW}(S_t)$ is 2 by the Shioda-Tate formula ([SS10, Corollary 6.13]). Since the fixed locus $S_t^{\rho \times \mathrm{id}}$ is contained in $\pi^{-1}(0) \cup \pi^{-1}(\infty)$ and no elliptic curve contained in $S_t^{\rho \times \mathrm{id}}$, we see that $\mathrm{NS}(S_t) = \mathrm{U}(7) \oplus \mathrm{E}_8$ by the classification theorem of Artebani, Sarti and Taki ([AST11, Section 6]). Therefore we have $\mathrm{NS}(S_t) = N$. Let L be the lattice generated by the zero section and vertical divisors. It is known that $\mathrm{MW}(S_t) \cong \mathrm{NS}(S_t)/L$

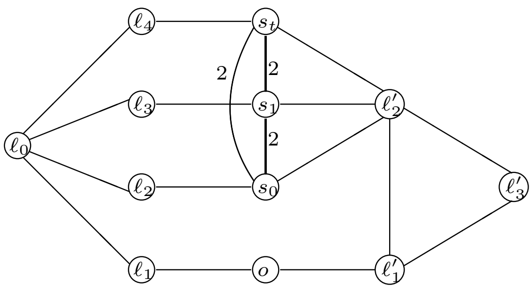

Figure 4. intersection graph.

([SS10, Theorem 6.3]). Now it is obvious that $\mathrm{MW}(S_t) = \mathbb{Z}s_0 \oplus \mathbb{Z}s_1 \cong \mathbb{Z}^2$. Summarizing the above, we have the following proposition.

Proposition 5.1 *For a general $t \in \mathbb{C} - \{0, 1\}$, an elliptic K3 surface S_t has transcendental lattice $\mathrm{T}_{S_t} = \mathrm{U} \oplus \mathrm{U}(7) \oplus \mathrm{E}_8$ and the Mordell-Weil group $\mathrm{MW}(S_t) \cong \mathbb{Z}^2$. By the automorphism $(x, y, z) \mapsto (\zeta x, \zeta^5 y, \zeta z)$, we have $\mathrm{T}_{S_t} \cong \mathbb{Z}[\zeta]^2$ and the period map for this 1-parameter family is given by the Scwarz triangle mapping $\mathfrak{s}(t)$ with the monodromy group $\Delta(7, 7, 7)$. Therefore the Schwarz inverse $\mathfrak{t}(u)$ is an example of "K3 modular function" ([Sh79,81]).*

A. Appendix

A.1 Symplectic representation

Here we give the representation matrix $M \in Sp_{12}(\mathbb{Z})$ of ρ with respect to symplectic basis B_i and A_i, and images of h_0 and h_1 by the homomorphism ϕ in Proposition 3.1:

$$M = \left[\begin{array}{ccc|ccc|ccc|ccc}
0 & 0 & 0 & & & & -1 & 1 & 0 & & & \\
0 & 0 & 0 & & \mathbf{O} & & 0 & -1 & 1 & & \mathbf{O} & \\
-1 & -1 & -1 & & & & 0 & 0 & -1 & & & \\
\hline
& & & -1 & 0 & 1 & & & & 0 & 1 & 0 \\
& \mathbf{O} & & 0 & 0 & -2 & & \mathbf{O} & & 1 & -1 & 1 \\
& & & 0 & -1 & 1 & & & & 0 & 1 & -1 \\
\hline
1 & 0 & 0 & & & & & & & & & \\
1 & 1 & 0 & & \mathbf{O} & & & \mathbf{O} & & & \mathbf{O} & \\
1 & 1 & 1 & & & & & & & & & \\
\hline
& & & 1 & -1 & -2 & & & & 0 & -1 & 0 \\
& \mathbf{O} & & -1 & 1 & 1 & & \mathbf{O} & & -1 & 0 & 0 \\
& & & -1 & 0 & 3 & & & & -1 & 1 & -1
\end{array}\right],$$

$$\phi(h_0) = \left[\begin{array}{ccc|ccc|ccc|ccc}
0 & 0 & 0 & & & & 1 & -1 & 0 & & & \\
0 & 0 & 0 & & \mathbf{O} & & 0 & 1 & -1 & & \mathbf{O} & \\
1 & 1 & 1 & & & & 0 & 0 & 1 & & & \\
\hline
& & & 1 & 0 & 0 & & & & & & \\
& \mathbf{O} & & 0 & 1 & 0 & & \mathbf{O} & & & \mathbf{O} & \\
& & & 0 & 0 & 1 & & & & & & \\
\hline
-1 & 0 & 0 & & & & & & & & & \\
-1 & -1 & 0 & & \mathbf{O} & & & \mathbf{O} & & & \mathbf{O} & \\
-1 & -1 & -1 & & & & & & & & & \\
\hline
& & & & & & & & & 1 & 0 & 0 \\
& \mathbf{O} & & & \mathbf{O} & & & \mathbf{O} & & 0 & 1 & 0 \\
& & & & & & & & & 0 & 0 & 1
\end{array}\right],$$

$$\phi(h_1) = \left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 \\
1 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
-1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\
-1 & -2 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & -1 & -1 \\
-1 & -2 & -3 & 0 & -1 & -1 & -1 & -1 & 0 & -2 & -2 & -1 \\
\hline
1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & -2 & 1 & -1 & -1 & 0 & -1 & 1
\end{array}\right].$$

A.2. Period matrix

Let ζ be $\exp[2\pi i/7]$ and put

$$\alpha = 1 + \zeta + \zeta^2 + \zeta^4 = \frac{1+\sqrt{-7}}{2},$$

$$\beta_1 = \zeta - 2\zeta^2 - 2\zeta^4, \qquad \beta_2 = -(2\zeta^3 + 1 - \zeta^6 + 2\zeta^5).$$

The modular embedding $\Phi : \mathbb{D}_H^+ \to \mathbb{H}_6^M$ in Proposition 3.1 is given by

$$\Phi(u) = \frac{1}{\Delta}\left(\begin{bmatrix} A_{11} & O \\ O & D_{11} \end{bmatrix} u_1^2 + \begin{bmatrix} O & B_{12} \\ {}^tB_{12} & O \end{bmatrix} u_1 u_2 + \begin{bmatrix} A_{22} & O \\ O & D_{22} \end{bmatrix} u_2^2 \right),$$

where

$$\Delta = (\zeta^2 + \zeta + 1)(2\zeta^2 - \zeta + 2)u_1^2 + 3(\zeta + 1)u_2^2$$

and

$$A_{11} = (\zeta^2 + \zeta + 1)(2\zeta^2 - \zeta + 2)\begin{bmatrix} \alpha & 0 & -1 \\ 0 & \alpha - 1 & -\alpha \\ -1 & -\alpha & 1 \end{bmatrix},$$

$$D_{11} = (\zeta^2 + 1)\begin{bmatrix} 2\zeta^6 + \zeta^5 - \zeta^3 - 1 & 2\zeta^6 - \zeta^3 & -\zeta^3 \\ 2\zeta^6 - \zeta^3 & \zeta^2 - \zeta^3 & \zeta^6 - \alpha \\ -\zeta^3 & \zeta^6 - \alpha & \alpha \end{bmatrix},$$

$$B_{12} = (\zeta^3 - \zeta^5)\begin{bmatrix} -\beta_1 & -\beta_2 & -1 \\ \zeta^5\beta_1 & \zeta^5\beta_2 & \zeta^5 \\ (1+\zeta^6)\beta_1 & (1+\zeta^6)\beta_2 & (1+\zeta^6) \end{bmatrix},$$

$$A_{22} = -3\begin{bmatrix} \zeta(\zeta^5-\zeta^2-1) & \zeta-1 & \zeta^3+1 \\ \zeta-1 & \zeta^5-\zeta^2+1 & \zeta^2 \\ \zeta^3+1 & \zeta^2 & -\zeta^2(\zeta+1) \end{bmatrix},$$

$$D_{22} = (\zeta+1)\begin{bmatrix} 3\alpha-2 & \alpha-1 & \alpha \\ \alpha-1 & 2\alpha-1 & -2 \\ \alpha & -2 & \alpha+1 \end{bmatrix}.$$

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