Hokkaido Mathematical Journal Vol. 46 (2017) p. 15–27

A generalization of starlike functions of order alpha

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(Received July 29, 2014; Revised December 15, 2015)

Abstract. For every $q \in (0,1)$ and $0 \leq \alpha < 1$ we define a class of analytic functions, the so-called q-starlike functions of order α, on the open unit disk. We study this class of functions and explore some inclusion properties with the well-known class of starlike functions of order α. The paper is also devoted to the discussion on the Herglotz representation formula for analytic functions $zf'(z)/f(z)$ when $f(z)$ is q-starlike of order α. As an application we also discuss the Bieberbach conjecture problem for the q-starlike functions of order α.

Key words: Starlike functions, q-starlike functions, order of starlikeness, order of q-starlikeness, q-difference operator, Bieberbach's conjecture, infinite product, uniform convergence, Herglotz representation, probability measure.

1. Introduction and Main Results

In view of the well-known Riemann mapping theorem in classical complex analysis, the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is usually considered as a standard domain. The analytic functions such as convex, starlike, and close-to-convex functions defined in the unit disk have been extensively studied and found numerous applications to various problems in complex analysis and related topics. Part of this development is the study of subclasses of the class of univalent functions, more general than the classes of convex, starlike, and close-to-convex functions. Analytic and geometric characterizations of such functions are of quite interesting to all function theorists in general. Background knowledge in this theory can be found from standard books (see for instance [8], [11]).

In 1916, Bieberbach first posed a conjecture on the coefficient estimate of univalent functions. This conjecture was a long standing open problem in univalent function theory and was a challenge to all mathematicians. In this regard a lot of methods and concepts were developed. One of the important concepts is the *Herglotz representation theorem* for univalent functions with positive real part. Initially, the Bieberbach conjecture was proved for first

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2010 Mathematics Subject Classification : 28A25, 30B10, 30C45, 30C50, 30C55, 33B10, 39A13, 39A70, 40A20, 46G05, 47B38, 47B39.

few coefficients of univalent functions. Then the conjecture was considered in many special cases. In one direction, it was considered for certain subclasses of univalent functions like starlike, convex, close-to-convex, typically real functions etc. The concept of order for the starlike and convex was also introduced, which are the subclasses of the class of starlike and convex functions respectively, and the conjecture was proved in these subclasses. In other direction, discussion on many conjectures, namely, the Zalcman conjecture, the Robertson conjecture, the Littlewood-Paley conjecture, etc. were investigated to prove the Bieberbach conjecture. Finally, the full conjecture for univalent functions was settled down by L. de Branges in 1985 [6].

In 1990, Ismail et al. [12] introduced a link between starlike functions and the q-theory by introducing a q-analog of the starlike functions. We call these functions as q-starlike functions. They proved the Bieberbach conjecture for the q-starlike functions through the Herglotz representation theorem for these functions. In this connection, we aim to introduce the concept of order of q-starlikeness and prove the Bieberbach conjecture for q-starlike functions in terms of their order. In particular, we also discuss several other basic properties on the order of q-starlike functions.

We now collect some standard notations and basic definitions used in the sequel. We denote by $\mathcal{H}(\mathbb{D})$, the set of all analytic (or holomorphic) functions in $\mathbb{D}$. We use the symbol $\mathcal{A}$ for the class of functions $f \in \mathcal{H}(\mathbb{D})$ with the standard normalization $f(0) = 0 = f'(0) - 1$. This means that the functions $f \in \mathcal{A}$ have the power series representation of the form $z + \sum_{n=2}^{\infty} a_n z^n$. The principal value of the logarithmic function $\log z$ for $z \neq 0$ is denoted by $\operatorname{Log} z := \ln|z| + i \operatorname{Arg}(z)$, where $-\pi \leq \operatorname{Arg}(z) < \pi$.

For $0 < q < 1$, *the q-difference operator* denoted as $D_q f$ is defined by the equation

$$(D_q f)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \neq 0, \quad (D_q f)(0) = f'(0).$$

The operator $D_q f$ plays an important role in the theory of basic hypergeometric series (see [2], [3], [10], [20]); see also Section 4 in this paper. It is evident that, when $q \to 1^-$, the difference operator $D_q f$ converges to the ordinary differential operator $Df = df/dz = f'$.

A function $f \in \mathcal{A}$ is called starlike of order α, $0 \leq \alpha < 1$, if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

We use the notation $\mathcal{S}^*(\alpha)$ for the class of starlike functions of order α. Set $\mathcal{S}^* := \mathcal{S}^*(0)$, the class of all starlike functions.

One way to generalize the starlike functions of order α is to replace the derivative function f' by the q-difference operator $D_q f$ and replace the right-half plane $\{w : \mathrm{Re}\, w > \alpha\}$ by a suitable domain in the above definition of the starlike functions of order α. The appropriate definition turned out to be the following:

Definition 1.1 A function $f \in \mathcal{A}$ is said to *belong to the class* $\mathcal{S}_q^*(\alpha)$, $0 \le \alpha < 1$, if

$$\left| \frac{\dfrac{z(D_q f)(z)}{f(z)} - \alpha}{1-\alpha} - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

The following is the equivalent form of Definition 1.1.

$$f \in \mathcal{S}_q^*(\alpha) \iff \left| \frac{z(D_q f)(z)}{f(z)} - \frac{1-\alpha q}{1-q} \right| \le \frac{1-\alpha}{1-q}.$$

Observe that as $q \to 1^-$ the closed disk $|w - (1-\alpha q)(1-q)^{-1}| \le (1-\alpha)(1-q)^{-1}$ becomes the right-half plane $\mathrm{Re}\, w \ge \alpha$ and the class $\mathcal{S}_q^*(\alpha)$ reduces to $\mathcal{S}^*(\alpha)$, $0 \le \alpha < 1$. In particular, when $\alpha = 0$, the class $\mathcal{S}_q^*(\alpha)$ coincides with the class $\mathcal{S}_q^* := \mathcal{S}_q^*(0)$, which was first introduced by Ismail et al. [12] in 1990 and later (also recently) it has been considered in [2], [17], [18], [19]. In words we call $\mathcal{S}_q^*(\alpha)$, the class of *q-starlike functions of order α.*

The main objective in this paper is to prove the following theorems. The first main theorem describes the *Herglotz Representation* for functions belonging to the class $\mathcal{S}_q^*(\alpha)$ in the form of a Poisson-Stieltjes integral (see Herglotz Representation Theorem for analytic functions with positive real part in [8, pp. 22]).

Theorem 1.1 *Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}_q^*(\alpha)$ if and only if there exists a probability measure μ supported on the unit circle such that*

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F'_{q,\alpha}(\sigma z) \mathrm{d}\mu(\sigma)$$

where

$$F_{q,\alpha}(z) = \sum_{n=1}^{\infty} \frac{(-2)\left(\ln \frac{q}{1-\alpha(1-q)}\right)}{1-q^n} z^n, \quad z \in \mathbb{D}. \tag{1}$$

Remark 1.2 When q approaches 1, Theorem 1.1 leads to the Herglotz Representation Theorem for starlike functions of order α (see for instance [11, Problem 3, pp. 172]). Note that the coefficients of the function $F_{q,\alpha}$ are all positive.

Our second main theorem concerns about the Bieberbach conjecture problem for functions in $\mathcal{S}_q^*(\alpha)$. The extremal function is also explicitly obtained in terms of exponential of the function $F_{q,\alpha}(z)$. This exponential form generalizes the Koebe function $k_\alpha(z) = z/(1-z)^{2(1-\alpha)}$, $z \in \mathbb{D}$. That is, when $q \to 1^-$, the exponential form $G_{q,\alpha}(z) := z \exp[F_{q,\alpha}(z)]$ representing the extremal function for the class $\mathcal{S}_q^*(\alpha)$ turns into the Koebe function $k_\alpha(z)$.

Theorem 1.3 *Let*

$$G_{q,\alpha}(z) := z \exp[F_{q,\alpha}(z)] = z + \sum_{n=2}^{\infty} c_n z^n. \tag{2}$$

Then $G_{q,\alpha} \in \mathcal{S}_q^*(\alpha)$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_q^*(\alpha)$, then $|a_n| \le c_n$ with equality holding for all n if and only if f is a rotation of $G_{q,\alpha}$.

Remark 1.4 When q approaches 1, Theorem 1.3 leads to the Bieberbach conjecture for starlike functions of order α (see for instance [11, Theorem 2, pp. 140]).

Motivation behind this comes from the work of Ismail et al., where the q-analog of starlike functions was introduced in 1990 (see [12]). The q-theory has important role in special functions and quantum physics (see for instance [3], [9], [10], [14], [15], [20]). For updated research work in function theory related to q-analysis, readers can refer [2], [12], [17], [18], [19]. In [12], the authors have obtained the Herglotz representation for functions of the class

$\mathcal{S}_q^*$ in the following form:

Theorem A ([12, Theorem 1.15]) *Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}_q^*$ if and only if there exists a probability measure μ supported on the unit circle such that*

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F_q'(\sigma z) \mathrm{d}\mu(\sigma)$$

where

$$F_q(z) = \sum_{n=1}^{\infty} \frac{-2\ln q}{1-q^n} z^n, \quad z \in \mathbb{D}.$$

Also they have proved the Bieberbach conjecture problem for q-starlike functions in the following form:

Theorem B ([12, Theorem 1.18]) *Let*

$$G_q(z) := z \exp[F_q(z)] = z + \sum_{n=2}^{\infty} c_n z^n.$$

Then $G_q \in \mathcal{S}_q^*$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_q^*$, then $|a_n| \le c_n$ with equality holding for all n if and only if f is a rotation of G_q.

Remark 1.5 Note that Theorem 1.1 and Theorem 1.3 are respectively generalizations of Theorem A and Theorem B.

Structure of rest of the paper is as follows. Section 2 is devoted for basic interesting properties of the class $\mathcal{S}_q^*(\alpha)$, which are used in the proof of main theorems. In Section 3, we prove our main results. Finally, we focus on concluding remarks with few questions in Section 4 for future research in this direction.

2. Properties of the class $\mathcal{S}_q^*(\alpha)$

As a matter of fact the following proposition says that a function f in $\mathcal{S}_q^*(\alpha)$ can be obtained in terms of a function g in $\mathcal{S}_q^*$. The proof is obvious and it follows from the definition of $\mathcal{S}_q^*(\alpha)$, $0 \le \alpha < 1$.

Proposition 2.1 *Let $f \in \mathcal{S}_q^*(\alpha)$. Then there exists a unique function*

$g \in \mathcal{S}_q^*$ such that

$$\frac{\dfrac{z(D_qf)(z)}{f(z)}-\alpha}{1-\alpha}=\frac{z(D_qg)(z)}{g(z)} \quad \textit{or} \quad \frac{f(qz)-\alpha q f(z)}{(1-\alpha)f(z)}=\frac{g(qz)}{g(z)} \tag{3}$$

holds. Similarly, for a given function $g \in \mathcal{S}_q^*$ there exists $f \in \mathcal{S}_q^*(\alpha)$ satisfying the above relation. Uniqueness follows trivially.

Next we present an easy characterization of functions in the class $\mathcal{S}_q^*(\alpha)$. This shows that if $f \in \mathcal{S}_q^*(\alpha)$ then $f(z)=0$ implies $z=0$, otherwise $f(qz)/f(z)$ would have a pole at a zero of $f(z)$ with least nonzero modulus.

Theorem 2.2 *Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}_q^*(\alpha)$ if and only if*

$$\left|\frac{f(qz)}{f(z)}-\alpha q\right| \le 1-\alpha, \quad z \in \mathbb{D}.$$

Proof. The proof can be easily obtained from the fact

$$\frac{z(D_qf)(z)}{f(z)}=\left(\frac{1}{1-q}\right)\left(1-\frac{f(qz)}{f(z)}\right)$$

and the definition of $\mathcal{S}_q^*(\alpha)$. □

The next result is an immediate consequence of Theorem 2.2.

Corollary 2.3 *The class $\mathcal{S}_q^*(\alpha)$ satisfies the inclusion relation*

$$\bigcap_{q<p<1} \mathcal{S}_p^*(\alpha) \subset \mathcal{S}_q^*(\alpha) \quad \textit{and} \quad \bigcap_{0<q<1} \mathcal{S}_q^*(\alpha) = \mathcal{S}^*(\alpha).$$

Proof. The inclusions

$$\bigcap_{q<p<1} \mathcal{S}_p^*(\alpha) \subset \mathcal{S}_q^*(\alpha) \quad \text{and} \quad \bigcap_{0<q<1} \mathcal{S}_q^*(\alpha) \subset \mathcal{S}^*(\alpha)$$

clearly hold. It remains to show that

$$\mathcal{S}^*(\alpha) \subset \bigcap_{0<q<1} \mathcal{S}_q^*(\alpha)$$

holds. For this, we let $f \in \mathcal{S}^*(\alpha)$. Then it is enough to show that $f \in \mathcal{S}_q^*(\alpha)$ for all $q \in (0,1)$. Since $f \in \mathcal{S}^*(\alpha)$ there exists a unique $g \in \mathcal{S}^*$ satisfying

$$\frac{\dfrac{zf'(z)}{f(z)} - \alpha}{1-\alpha} = \frac{zg'(z)}{g(z)}, \quad |z| < 1.$$

Since $\mathcal{S}^* = \bigcap_{0<q<1} \mathcal{S}_q^*$, it follows that $g \in \mathcal{S}_q^*$ for all $q \in (0,1)$. Thus, by Proposition 2.1 there exists a unique $h \in \mathcal{S}_q^*(\alpha)$ satisfying the identity (3) with $h(z) = f(z)$. The proof now follows immediately. $\square$

We now define two sets and proceed to prepare some basic results which are being used to prove our main results in this section. They are

$$B_q = \{g : g \in \mathcal{H}(\mathbb{D}),\ g(0) = q \text{ and } g : \mathbb{D} \to \mathbb{D}\} \quad \text{and}$$
$$B_q^0 = \{g : g \in B_q \text{ and } 0 \notin g(\mathbb{D})\}.$$

Lemma 2.4 *If $h \in B_q$ then the infinite product $\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n) + \alpha q)/q\}$ converges uniformly on compact subsets of $\mathbb{D}$.*

Proof. We set $(1-\alpha)h(z) + \alpha q = g(z)$. Since $h \in B_q$, it easily follows that $g \in B_q$. By [12, Lemma 2.1], the conclusion of our lemma follows. $\square$

Lemma 2.5 *If $h \in B_q^0$ then the infinite product $\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n) + \alpha q)/q\}$ converges uniformly on compact subsets of $\mathbb{D}$ to a nonzero function in $\mathcal{H}(\mathbb{D})$ with no zeros. Furthermore, the function*

$$f(z) = \frac{z}{\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n) + \alpha q)/q\}} \tag{4}$$

belongs to $\mathcal{S}_q^*(\alpha)$ and $h(z) = ((f(qz)/f(z)) - \alpha q)/(1-\alpha)$.

Proof. The convergence of the infinite product is proved in Lemma 2.4. Since $h \in B_q^0$, we have $h(z) \neq 0$ in $\mathbb{D}$ and the infinite product does not vanish in $\mathbb{D}$. Thus, the function $f \in \mathcal{A}$ and we find the relation

$$\frac{f(qz)}{f(z)} = (1-\alpha)h(z) + \alpha q, \quad \text{equivalently} \quad \frac{\dfrac{f(qz)}{f(z)} - \alpha q}{1-\alpha} = h(z).$$

Since $h \in B_q^0$, we get $f \in \mathcal{S}_q^*(\alpha)$ and the proof of our lemma is complete. $\square$

We define two classes $B_{q,\alpha}$ and $B_{q,\alpha}^0$ by

$$B_{q,\alpha} = \left\{ g : g \in \mathcal{H}(\mathbb{D}),\ g(0) = \frac{q}{1-\alpha(1-q)} \text{ and } g : \mathbb{D} \to \mathbb{D} \right\}$$

and

$$B_{q,\alpha}^0 = \{g : g \in B_{q,\alpha} \text{ and } 0 \notin g(\mathbb{D})\}.$$

Lemma 2.6 *A function $g \in B_{q,\alpha}^0$ if and only if it has the representation*

$$g(z) = \exp\left\{ \left(\ln \frac{q}{1-\alpha(1-q)} \right) p(z) \right\}, \tag{5}$$

where $p(z)$ belongs to the class

$$\mathcal{P} = \{p : p \in \mathcal{H}(\mathbb{D}),\ p(0) = 1 \textit{ and } \mathrm{Re}\{p(z)\} > 0 \textit{ for } z \in \mathbb{D}\}.$$

Proof. For $g \in B_{q,\alpha}^0$, define the function $L(z) = \mathrm{Log}\, g(z)$. Then it is easy to show that the function $p(z) = L(z)/(\ln(q/(1-\alpha(1-q)))) \in \mathcal{P}$ and satisfies (5). Conversely, if g is given by (5), then it is obvious that $g \in B_{q,\alpha}^0$. $\square$

Theorem 2.7 *The mapping $\rho : \mathcal{S}_q^*(\alpha) \to B_q^0$ defined by*

$$\rho(f)(z) = \frac{\dfrac{f(qz)}{f(z)} - \alpha q}{1-\alpha}$$

is a bijection.

Proof. For $h \in B_q^0$, define a mapping $\sigma :\ B_q^0 \to \mathcal{A}$ by

$$\sigma(h)(z) = \frac{z}{\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n) + \alpha q)/q\}}.$$

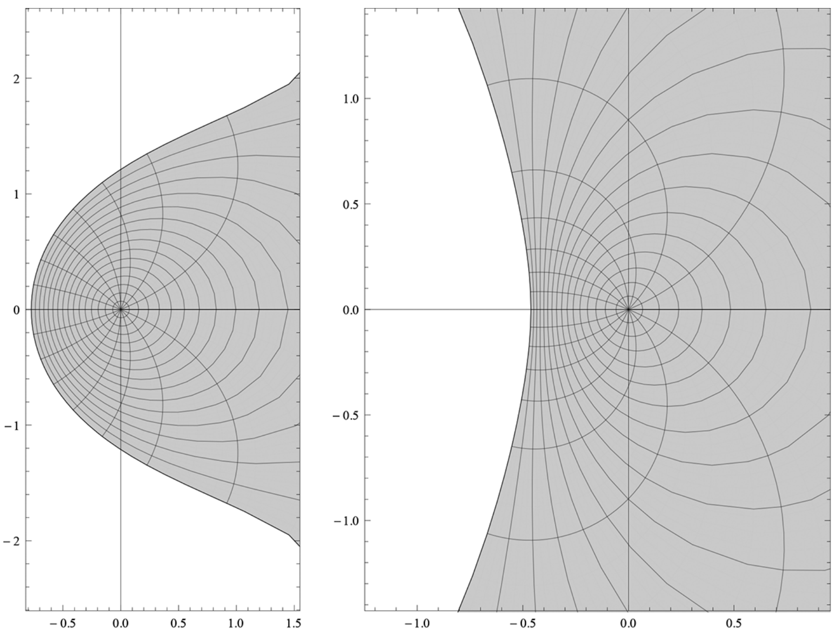
It is clear from Lemma 2.5 that $\sigma(h) \in \mathcal{S}_q^*(\alpha)$ and $(\rho \circ \sigma)(h) = h$. Considering the composition mapping $\sigma \circ \rho$ we compute that

$$(\sigma \circ \rho)(f)(z) = \frac{z}{\prod_{n=0}^{\infty}\{(f(zq^{n+1})/qf(zq^n)\}} = \frac{z}{z/f(z)} = f(z).$$

Hence $\sigma \circ \rho$ and $\rho \circ \sigma$ are identity mappings and σ is the inverse of ρ, i.e. the map $\rho(f)$ is invertible. Hence $\rho(f)$ is a bijection. This completes the proof of our theorem. $\square$

3. Proof of the main theorems

This section is devoted to the proofs of main theorems using the supplementary results proved in Section 2.



Graph of $F_{5/6,1/2}(z)$, $|z| < 1$ Graph of $G_{5/6,1/2}(z)$, $|z| < 1$

Figure 1. Graphs of the functions $F_{5/6,1/2}(z)$ and $G_{5/6,1/2}(z)$ for $|z| < 1$.

Proof of Theorem 1.1. For $0 < q < 1$ and $0 \le \alpha < 1$, let $F_{q,\alpha}$ be defined by (1). Geometry of $F_{q,\alpha}$ is described in Figure 1 for different ranges over the parameters q and α. Suppose that $f \in \mathcal{S}_q^*(\alpha)$. Then by Theorem 2.7 and Lemma 2.5, it is clear that $f(z)$ has the representation (4) with $h \in B_q^0$. The logarithmic derivative of f gives

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{n=0}^{\infty} \frac{(1-\alpha)zq^n h'(zq^n)}{(1-\alpha)h(zq^n)+\alpha q}. \tag{6}$$

Now, let us assume that

$$g(z) = \frac{(1-\alpha)h(z)+\alpha q}{1-\alpha(1-q)}.$$

Clearly, $g \in B^0_{q,\alpha}$ and hence Lemma 2.6 guarantees that $g(z)$ has the representation (5). Taking the logarithmic derivative of g we have

$$\frac{zg'(z)}{g(z)} = \left(\ln \frac{q}{1-\alpha(1-q)}\right) zp'(z), \tag{7}$$

where $\mathrm{Re}\{p(z)\} \geq 0$. By Herglotz representation of $p(z)$, there exists a probability measure μ supported on the unit circle $|\sigma| = 1$ such that

$$zp'(z) = \int_{|\sigma|=1} 2\sigma z(1-\sigma z)^{-2} d\mu(\sigma). \tag{8}$$

Using (7) and (8) in (6), we have

$$\begin{aligned}
\frac{zf'(z)}{f(z)} &= 1 - 2\left(\ln \frac{q}{1-\alpha(1-q)}\right) \sum_{n=0}^{\infty} \int_{|\sigma|=1} \sigma z q^n (1-\sigma z q^n)^{-2} d\mu(\sigma) \\
&= 1 - 2\left(\ln \frac{q}{1-\alpha(1-q)}\right) \int_{|\sigma|=1} \left\{ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m\sigma^m z^m q^{mn} \right\} d\mu(\sigma) \\
&= 1 - 2\left(\ln \frac{q}{1-\alpha(1-q)}\right) \int_{|\sigma|=1} \left\{ \sum_{m=1}^{\infty} m\sigma^m z^m \frac{1}{1-q^m} \right\} d\mu(\sigma) \\
&= 1 + \int_{|\sigma|=1} \sigma z F'_{q,\alpha}(\sigma z) \mathrm{d}\mu(\sigma).
\end{aligned}$$

This completes the proof of our theorem. $\square$

Proof of Theorem 1.3. For $0 < q < 1$ and $0 \leq \alpha < 1$, let $G_{q,\alpha}$ be defined by (2). Geometry of the mapping $G_{q,\alpha}$ is described in Figure 1 for different ranges over the parameters q and α. As a special case to Theorem 1.1, when the measure has a unit mass, it is clear that $G_{q,\alpha} \in \mathcal{S}_q^*(\alpha)$. Let $f \in \mathcal{S}_q^*(\alpha)$. Then by Theorem 2.7, there exists a function $h \in B_q^0$ such that $h(z) = ((f(qz)/f(z)) - \alpha q)/(1-\alpha)$. Since $h \in B_q^0$, $g(z) = ((1-\alpha)h(z) + \alpha q)/(1-\alpha(1-q)) \in B_{q,\alpha}^0$. By Lemma 2.6, $g(z)$ has the representation (5) and on solving we get,

$$\frac{f(qz)}{f(z)} = (1-\alpha(1-q)) \exp\left\{\left(\ln \frac{q}{1-\alpha(1-q)}\right)p(z)\right\}.$$

Define the function $\phi(z) = \mathrm{Log}\{f(z)/z\}$ and set

$$\phi(z) = \mathrm{Log}\,\frac{f(z)}{z} = \sum_{n=1}^{\infty} \phi_n z^n. \tag{9}$$

On solving, we get

$$\ln \frac{q}{1-\alpha(1-q)} + \phi(qz) = \phi(z) + \left(\ln \frac{q}{1-\alpha(1-q)}\right)p(z).$$

This implies

$$\phi_n = p_n\left(\ln \frac{q}{1-\alpha(1-q)}\right)\Big/(q^n - 1).$$

Since $|p_n| \le 2$, we have

$$|\phi_n| \le \frac{(-2)\left(\ln \frac{q}{1-\alpha(1-q)}\right)}{1-q^n}.$$

From this inequality, together with the expression of $G_{q,\alpha}(z)$ and (9), the conclusion follows. □

4. Concluding remarks

At the beginning of the last century, studies on q-difference equations appeared in intensive works especially by Jackson [13], Carmichael [7], Mason [16], Adams [1], Trjitzinsky [21], and later by others such as Poincaré, Picard, Ramanujan. Unfortunately, from the thirties up to the beginning of the eighties only non-significant interest in this area was investigated.

Recently some research in this topic is carried out by Bangerezako [4]; see also references therein for other related work. Research works in connection with function theory and q-theory together were first introduced by Ismail et al. [12]. Later it is also studied in [2], [17], [18], [19]. Since only few work have been carried out in this direction, as indicated in [2], there are a lot can be done. For instance, q-analog of convexity of analytic functions in the unit disk and even more general in arbitrary simply connected domains may be interesting for researchers in this field. Recently, the concept of q-convexity for basic hypergeometric functions is considered in [5]. Bieberbach conjecture problem for q-close-to-convex functions is estimated optimally in a recent paper [19]. In fact sharpness of this result is still an open problem and concerning this, a conjecture is stated there.

Acknowledgement. The work of the first author is supported by University Grants Commission, New Delhi (grant no. F.2-39/2011 (SA-I)). The authors would like to thank the referee for his/her careful reading of the manuscript and valuable suggestions.

References

[1] Adams C. R., *On the linear ordinary q-difference equation.* Am. Math. Ser. II **30** (1929), 195–205.

[2] Agrawal S. and Sahoo S. K., *Geometric properties of basic hypergeometric functions.* J. Difference Equ. Appl. **20** (2014), 1502–1522.

[3] Andrews G. E., *Applications of basic hypergeometric functions.* SIAM Rev. **16** (1974), 441–484.

[4] Bangerezako G., *An Introduction to q-Difference Equations*, University of Burundi, Bujumbura, 2007, Preprint.

[5] Baricz A. and Swaminathan A., *Mapping properties of basic hypergeometric functions.* J. Class. Anal. **5** (2014), 115–128.

[6] de Branges L., *A proof of the Bieberbach conjecture.* Acta Math. **154** (1985), 137–152.

[7] Carmichael R. D., *The general theory of linear q-difference equations.* Amer. J. Math. **34** (1912), 147–168.

[8] Duren P. L., *Univalent Functions*, Springer-Verlag, New York, 1983.

[9] Ernst T., *The History of q-calculus and a New Method*, Licentiate Dissertation, Uppsala, 2001.

[10] Fine N. J., *Basic Hypergeometric Series and Applications*, Mathematical Surveys No. 27, Amer. Math. Soc. Providence, 1988.

[11] Goodman A. W., *Univalent Functions*, Volume 1, Mariner Publishing Company, Inc., Florida, 1983.

[12] Ismail M. E. H., Merkes E. and Styer D., *A generalization of starlike functions.* Complex Variables **14** (1990), 77–84.

[13] Jackson F. H., *On q-definite integrals.* Quart. J. Pure and Appl. Math. **41** (1910), 193–203.

[14] Kac V. and Cheung P., *Quantum calculus*, Universitext, Springer-Verlag, New York, 2002.

[15] Kirillov A. N., *Dilogarithm identities.* Progr. Theoret. Phys. Suppl. **118** (1995), 61–142.

[16] Mason T. E., *On properties of the solution of linear q-difference equations with entire fucntion coefficients.* Amer. J. Math. **37** (1915), 439–444.

[17] Raghavendar K. and Swaminathan A., *Close-to-convexity of basic hypergeometric functions using their Taylor coefficients.* J. Math. Appl. **35** (2012), 111–125.

[18] Rønning F., A Szegő quadrature formula arising from q-starlike functions: In *continued fractions and orthogonal functions* (Loen, 1992), 345–352, Dekker, New York.

[19] Sahoo S. K. and Sharma N. L., *On a generalization of close-to-convex functions.* Ann. Polon. Math. **113** (2015), 93–108.

[20] Slater L. J., *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[21] Trjitzinsky W. J., *Analytic theory of linear q-difference equations.* Acta Math. **61** (1933), 1–38.

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