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A Note on Vertex-transitive Kähler graphs

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Abstract. In this paper we construct finite vertex-transitive Kähler graphs, which may be considered as discrete models of Hermitian symmetric spaces admitting Kähler magnetic fields. We give a condition on cardinality of the set of vertices and the principal and the auxiliary degrees for a vertex-transitive Kähler graphs. Also we give some examples of Kähler graphs corresponding typical vertex-transitive ordinary graphs.

Key words: Kähler graphs, regular graphs, vertex-transitive graphs, complete graphs, partition functions.

1. Introduction

A graph consists of a set of vertices and a set of edges, and forms a 1-dimensional CW-complex. From geometrical point of view, a graph is a discrete model of a Riemannian manifold. Paths, which are chains of edges, on a graph are considered as geodesics (see [10] and its references for analysis on graphs). In his paper [2], the second author introduced the notion of Kähler graphs as discrete models of Riemannian manifolds with magnetic fields. A closed 2-form on a Riemannian manifold is said to be a magnetic field because it can be regarded as a generalization of static magnetic fields on a Euclidean 3-space (see [9], for example). Under the influence of a magnetic field, a charged particle makes its motion of constant speed. For the trivial magnetic field, the case that there are no influence of magnetic fields, the motion of a charged particle is a geodesic. We are hence interested in motions of charged particles under some uniform magnetic fields. Here, a uniform magnetic field means a magnetic field whose strength of Lorentz force does not depend on places and on directions. Typical examples of uniform magnetic fields are constant multiples of the Kähler form on a Kähler manifold. Trajectories for Kähler magnetic fields are circles, which are simplest curves next to geodesics (see [1]).

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Since graphs are 1-dimensional objects, in order to show trajectories for some magnetic fields we need something to indicate "geodesic curvature" of paths. Our idea is to use two kinds of edges. A Kähler graph is a compound graph which consists of a principal graph and an auxiliary graph. We consider paths on the principal graph to be geodesics. Given a pair (p, q) of relatively prime positive integers, we take a q-step path on the auxiliary graph which follows to a p-step path on the principal graph. We consider such distinguishing $(p+q)$-step paths as trajectories for a magnetic field of uniform strength q/p.

In geometry it is important to consider nice examples. Many results on geodesics are based on the study of geodesics on real space forms. Similarly, studies on trajectories for Kähler magnetic fields are based on the study of them on complex space forms, which are model spaces of Kähler manifolds (see [1], [3]). The authors hence think that the first step of the study of classical treatment of magnetic fields on graphs should be recorded by giving models of Kähler graphs. In this paper we consider the condition that automorphism groups of Kähler graphs acts on the set of vertices transitively. We give a condition on degrees of the principal and the auxiliary graphs of a vertex-transitive Kähler graph and give some examples of such graphs.

2. Kähler graphs

A graph (V, E) is pair of a set V of vertices and a set E of edges. In this paper we do not consider orientations of edges. Also, we suppose that graphs do not have loops and multiple edges. That is, endpoints of each edge do not coincide, and for distinct two vertices there is at most one edge joining them. We call two vertices $v, w \in V$ adjacent to each other if there is an edge joining them. In this case we denote as $v \sim w$. A graph (V, E) is said to be *Kähler* if the set E is divided into two disjoint subsets $E^{(p)}, E^{(a)}$ and satisfies the following condition: At each vertex $v \in V$ there exist at least four edges emanating from v; two of them are in $E^{(p)}$ and two of them are in $E^{(a)}$. In order to distinguish Kähler graphs clearly from other graphs we shall say a graph to be an ordinary graph. For a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ we call the graph $(V, E^{(p)})$ *principal*, and call the graph $(V, E^{(a)})$ *auxiliary*. We say that two vertices v, w are principally adjacent to each other if they are adjacent in the principal graph, and denote as $v \sim_p w$. We say that two vertices v, w are auxiliary adjacent to each other if they are adjacent in the auxiliary graph, and denote as $v \sim_a w$. We here give some examples.

Example 1 We take a lattice $\mathbb{Z}^{2m}$ ($\subset \mathbb{R}^{2m}$). We define that two points $z = (z_1, \dots, z_{2m})$, $z' = (z'_1, \dots, z'_{2m}) \in \mathbb{Z}^{2m}$ are principally adjacent to each other if and only if there is j $(1 \le j \le m)$ satisfying $z'_{2j-1} = z_{2j-1} \pm 1$ and $z'_i = z_i$ for i $(1 \le i \le 2m,\ i \ne 2j-1)$, and that they are auxiliary adjacent to each other if and only if there is k $(1 \le k \le m)$ satisfying $z'_{2k} = z_{2k} \pm 1$ and $z'_i = z_i$ for i $(1 \le i \le 2n,\ i \ne 2k)$. Then $(\mathbb{Z}^{2m}, E^{(p)} \cup E^{(a)})$ is a Kähler graph. We shall call this a complex Euclidean lattice graph.

Example 2 Given a finite ordinary graph $G = (V, E)$, we take its complement graph $G^c = (V, E^c)$. Here, G^c is defined so that two distinct vertices are adjacent in G^c if and only if they are not adjacent in G. We denote by n_G the cardinality of the set V. If the degree $d_G(v)$ at $v \in V$, which is the cardinality of the set of vertices adjacent to v, satisfies $2 \le d_G(v) \le n_G - 3$ for each vertex $v \in V$, then $G^K = (V, E \cup E^c)$ is a Kähler graph. We shall say such a graph to be a complement-filled Kähler graph.

Example 3 Let $G = (V, E)$, $H = (W, F)$ be two ordinary graphs which do not have vertices of degree 1. Given two points (v, w), $(v', w') \in V \times W$ we define $(v, w) \sim_p (v', w')$ if and only if $v \sim v'$ in G and $w = w'$, and define $(v, w) \sim_a (v', w')$ if and only if $v = v'$ and $w \sim w'$ in H. Then we obtain a Kähler graph $G \widehat{\square} H = (V \times W, E^{(p)} \cup E^{(a)})$. We shall call this graph a Kähler graph of Cartesian product type.

We here explain briefly why we regard Kähler graphs as discrete models of Kähler manifolds admitting Kähler magnetic fields. Let $G = (V, E^{(p)} \cup E^{(a)})$ be a Kähler graph. For a pair (p, q) of relatively prime positive integers, we say a $(p+q)$-step path $\gamma = (v_0, v_1, \dots, v_{p+q}) \in V \times \cdots \times V$ to be a (p, q)-primitive bicolored path if it satisfies the following conditions:

i) $v_{i-1} \ne v_{i+1}$ for $1 \le i \le p+q-1$,
ii) $v_{i-1} \sim_p v_i$ for $1 \le i \le p$,
iii) $v_{i-1} \sim_a v_i$ for $p+1 \le i \le p+q$.

When we treat graphs we sometimes give coloring to their vertices or to their edges. If we distinguish principal and auxiliary edges by giving coloring to them with two colors, such $(p+q)$-step paths are bicolored. We therefore use the terminology of "bicolored" paths. For a vertex $v \in V$ we denote by $d_G^{(p)}(v)$, $d_G^{(a)}(v)$ the cardinalities of the sets $\{w \in V \mid w \sim_p v\}$ and $\{w \in V \mid w \sim_a v\}$, respectively. We call them the *principal* and the *auxiliary degrees* at v. For a (p,q)-primitive bicolored path $\gamma = (v_0, \dots, v_{p+q})$, we set $\omega(\gamma) = \bigl\{d_G^{(a)}(v_p) \prod_{i=p+1}^{p+q-1}\{d_G^{(a)}(v_i) - 1\}\bigr\}^{-1}$ and call it its probabilistic weight. We consider motions of charged particles on this graph in the following manner. Without influence of magnetic fields, motions of charged particles are expressed as paths on the principal graph. Under the influence of a magnetic field paths are bended. When the strength of a magnetic field is q/p with relatively prime positive integers p, q, a p-step path on the principal graph is bended and its terminus turns to the terminus of a (p,q)-primitive bicolored path whose first p-step coincides with the given one. As graphs do not have 2-dimensional objects, we can not show the direction of the action of the Lorentz force. Therefore we treat the terminus probabilistically. This is our way of treating magnetic fields on a graph.

3. Regular Kähler graphs

A Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ is said to be *regular* if both the principal and the auxiliary graphs are regular. That is, the degree functions $d_G^{(p)}$, $d_G^{(a)} : V \to \mathbb{N}$ are constant functions. If a finite ordinary graph G is regular and has degree $2 \le d_G \le n_G - 3$, it is clear that the Kähler graph G^K formed by G and its complement is regular.

By handshaking lemma (see Proposition 1.2.1 in [5]), we have the following.

Proposition 1 *Let $G = (V, E^{(p)} \cup E^{(a)})$ be a finite regular Kähler graph. If the cardinality $n_G = \sharp(V)$ of the set V is odd, then the principal and the auxiliary degrees $d_G^{(p)}$, $d_G^{(a)}$ are even.*

We now consider the converse of this proposition. Let $G = (V, E)$, $H = (W, F)$ be two ordinary graphs. A map $f : V \to W$ is said to be a homomorphism of G to H if it satisfies $f(v) \sim f(v')$ for arbitrary v, $v' \in V$ with $v \sim v'$. A bijection $f : V \to W$ is called an isomorphism of G to H if it and its inverse $f^{-1} : W \to V$ are homomorphisms. We call a graph G is vertex-transitive if for arbitrary distinct two vertices v, $v' \in V$ there is an isomorphism (automorphism) $f : V \to V$ of G satisfying $f(v) = v'$. A typical example of vertex-transitive graphs is a Cayley graph. Let $\mathcal{G}$ be a group and S be a subset of $\mathcal{G}$ which does not contain the identity and which is invariant under the action of the inverse operation. That is, $S = S^{-1} = \{s^{-1} \mid s \in S\}$. If we put $V = \mathcal{G}$ and define $E = E(\mathcal{G}; S)$ as the set of pairs $g, h \in \mathcal{G}$ satisfying $gh^{-1} \in S$, then the Cayley graph $G(\mathcal{G}; S) = (V, E)$ is vertex-transitive. See [4], [8] and also [6] for more detail on vertex-transitive graphs.

We extend the notion of vertex-transitivity to Kähler graphs. Let $G = (V, E^{(p)} \cup E^{(a)})$, $H = (W, F^{(p)} \cup F^{(a)})$ be two Kähler graphs. A map $f : V \to W$ is said to be a homomorphism of G to H if it induces homomorphisms between principal graphs and between auxiliary graphs. That is, if two vertices v, $v' \in V$ satisfy $v \sim_p v'$ in G then $f(v) \sim_p f(v')$ in H and if they satisfy $v \sim_a v'$ in G then $f(v) \sim_a f(v')$ in H. We call a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ *vertex-transitive* if for arbitrary distinct vertices v, $v' \in V$ there is an isomorphism (automorphism) $f : V \to V$ of G satisfying $f(v) = v'$. For example, a complex Euclidean lattice graph $(\mathbb{Z}^{2m}, E^{(p)} \cup E^{(a)})$ is vertex-transitive.

Example 4 If a finite graph G is vertex-transitive and has degree $2 \leq d_G \leq n_G - 3$, then the complement-filled Kähler graph G^K of G is a vertex-transitive Kähler graph.

Example 5 Let $\mathcal{G}$ be a group and S_1, S_2 be disjoint subsets of $\mathcal{G}$ which do not contain the identity and which are invariant under the action of the inverse operation. If we put $V = \mathcal{G}$ and $E^{(p)} = E(\mathcal{G}; S_1)$, $E^{(a)} = E(\mathcal{G}; S_2)$, then we find that $G(\mathcal{G}; S_1, S_2) = (V, E^{(p)} \cup E^{(a)})$ is a vertex-transitive Kähler graph. We shall call such a graph as a Cayley Kähler graph.

We take two groups $\mathcal{G}_1$, $\mathcal{G}_2$ and their subsets $S_i \subset \mathcal{G}_i$ $(i = 1, 2)$ which do not contain the identities and which are invariant under the action of the inverse operation. If we set $\widehat{S}_1 = S_1 \times \{1\}$, $\widehat{S}_2 = \{1\} \times S_2$, then $G = (\mathcal{G}_1 \times \mathcal{G}_2; \widehat{S}_1, \widehat{S}_2)$ is a vertex-transitive Kähler graph, and is the Kähler graph of Cartesian product type of $G(\mathcal{G}_1; S_1)$ and $G(\mathcal{G}_2; S_2)$.

Example 6 Let G, H be vertex-transitive ordinary graphs. Then their Kähler graph $G \widehat{\Box} H$ of Cartesian product type is vertex-transitive.

It is clear that vertex-transitive Kähler graphs are regular Kähler graphs. In this section, as the converse of Proposition 1, we shall show the following.

Theorem 1 *Let n, $d^{(p)}$, $d^{(a)}$ be positive integers satisfying $n \geq 5$, $d^{(p)} \geq 2$, $d^{(a)} \geq 2$ and $d^{(p)} + d^{(a)} \leq n - 1$. There exists a vertex-transitive finite*

Kähler graph whose cardinality of the set of vertices is n *and whose principal and auxiliary degrees are* $d^{(p)}$ *and* $d^{(a)}$ *if and only if one of the following conditions holds*;

1) n *is odd and both* $d^{(p)}$, $d^{(a)}$ *are even,*
2) n *is even.*

Proof. We shall show this step by step. We take $V = \{v_0, v_1, \dots, v_{n-1}\}$. We give principal and auxiliary edges by considering the indices of vertices modulo n.

(1) The case that both $d^{(p)}$, $d^{(a)}$ (≥ 2) are even (and n (≥ 5) is arbitrary).

We denote $d^{(p)}, d^{(a)}$ as $d^{(p)} = 2d_1$ and $d^{(a)} = 2d_2$ with positive integers d_1, d_2. We define principal edges so that each vertex v_i is principally adjacent to v_{i+j} for $j = \pm 1, \pm 2, \dots, \pm d_1$, and define auxiliary edges so that each vertex v_i is auxiliary adjacent to v_{i+j} for $j = \pm(d_1+1)$, $\pm(d_1+2), \dots, \pm(d_1+d_2)$ (see Figure 1). Since $d^{(p)} + d^{(a)} \leq n-1$, this graph does not have multiple edges. By our way of construction, if we consider rotations $f_k : V \to V$ $(k = 1, 2, \dots, n-1)$ given by $f(v_i) = v_{i+k}$, they are automorphisms of our Kähler graph $(V, E^{(p)} \cup E^{(a)})$. Therefore our Kähler graph is vertex-transitive.

(2) The case that n (≥ 5) and $d^{(p)}$ (≥ 2) are even and $d^{(a)}$ (≥ 2) is odd.

We denote $n, d^{(p)}, d^{(a)}$ as $n = 2m$, $d^{(p)} = 2d_1$ and $d^{(a)} = 2d_2 + 1$ with positive integers m, d_1, d_2. We define principal edges so that each vertex v_i is principally adjacent to v_{i+j} for $j = \pm 1, \pm 2, \dots, \pm d_1$, and define auxiliary edges so that each vertex v_i is auxiliary adjacent to v_{i+j} for $j = \pm(d_1+1)$, $\pm(d_1+2), \dots, \pm(d_1+d_2)$ and is auxiliary adjacent to v_{i+m} (see Figure 2). We note that the condition $d^{(p)} + d^{(a)} \leq n-1$ guarantees that $d_1 + d_2 \leq m-1$. Hence this graph does not have multiple edges. Since the rotations $f_k : V \to V$ $(k = 1, 2, \dots, n-1)$ are automorphisms of our Kähler graph $(V, E^{(p)} \cup E^{(a)})$, we see it is vertex-transitive.

(3) The case that n (≥ 5) and $d^{(a)}$ (≥ 2) are even and $d^{(p)}$ (≥ 2) is odd.

If we change the roles of the principal and the auxiliary edges in the argument in the case of (2), we can obtain a desirable vertex-transitive Kähler graph. We denote $n, d^{(p)}, d^{(a)}$ as $n = 2m$, $d^{(p)} = 2d_1 + 1$ and $d^{(a)} = 2d_2$ with positive integers m, d_1, d_2. We define principal edges so that each vertex v_i is principally adjacent to v_{i+j} for $j = \pm 1, \pm 2, \dots, \pm d_1$ and is principally adjacent to v_{i+m}, and define auxiliary edges so that each vertex v_i is auxiliary adjacent to v_{i+j} for $j = \pm(d_1+1), \pm(d_1+2), \dots, \pm(d_1+d_2)$ (see Figure 3). Since the rotations $f_k : V \to V$ $(k = 1, 2, \dots, n-1)$ are automorphisms of our Kähler graph $(V, E^{(p)} \cup E^{(a)})$, we see it is vertex-transitive.

(4) The case that n (≥ 5) is even and both $d^{(a)}$ $d^{(p)}$ (≥ 2) are odd.

We denote $n, d^{(p)}, d^{(a)}$ as $n = 2m$, $d^{(p)} = 2d_1+1$ and $d^{(a)} = 2d_2+1$ with positive integers m, d_1, d_2. First, we define principal edges so that $v_{2\ell-2}$ and $v_{2\ell-1}$ with $\ell = 1, \dots, m$ are principally adjacent to each other, and define auxiliary edges so that $v_{2\ell-1}$ and $v_{2\ell}$ are auxiliary adjacent to each other. Next we define principal edges so that each vertex v_i is principally adjacent to v_{i+j} for $j = \pm 2, \pm 3, \dots, \pm(d_1+1)$, and define auxiliary edges so that each vertex v_i is auxiliary adjacent to v_{i+j} for $j = \pm(d_1+2), \pm(d_1+3), \dots, \pm(d_1+d_2+1)$ (see Figure 4). We note that the condition $d^{(p)} + d^{(a)} \leq n-1$ guarantees that $d_1 + d_2 + 1 \leq m - 1$. Hence this graph does not have multiple edges.

When k is even, the rotation $f_k : V \to V$ is an automorphism. When k is odd, we define a map $g_k : V \to V$ by $g_k(v_i) = v_{-i+k}$ which is a composition of a reflection and a rotation. We denote $k = 2a - 1$ with a positive integer a. As we have $g_k(v_{2\ell-2}) = v_{2(a-\ell+1)-1}$, $g_k(v_{2\ell-1}) = v_{2(a-\ell+1)-2}$ the principal and the auxiliary edges we first took are map to principal and auxiliary edges, respectively. Since the sets of principal and auxiliary edges we secondary took are invariant under the action of reflection $v_i \mapsto v_{-i}$, we find that g_k is an automorphism of our Kähler graph $(V, E^{(p)} \cup E^{(a)})$. Hence it is vertex-transitive. This completes the proof. □

Theorem 1 shows that there are many vertex-transitive Kähler graphs. We here give some more examples.

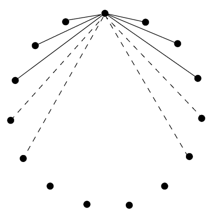

Figure 1.

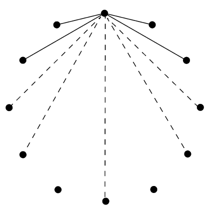

Figure 2.

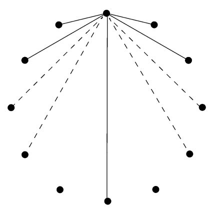

Figure 3.

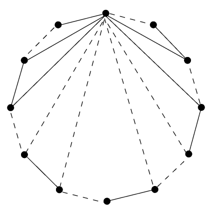

Figure 4.

Example 7 Our way of proof of Theorem 1 shows that the Heawood graph of 14 vertices induces many vertex-transitive Kähler graphs; five kinds of Kähler graphs of auxiliary degree 2, eight kinds of Kähler graphs of auxiliary degree 3, and so on. The original Heawood graph is given as $G = (V, E)$ with $V = \{v_0, v_1, \dots, v_{13}\}$ and

$$E = \left\{ \begin{array}{l} \{v_i, v_{i+1}\}\ (0 \le i \le 13), \\ \{v_0, v_5\},\ \{v_2, v_7\},\ \{v_4, v_9\},\ \{v_6, v_{11}\},\ \{v_8, v_{13}\},\ \{v_{10}, v_1\},\ \{v_{12}, v_3\} \end{array} \right\}$$

(see Figure 5). Here, we consider the index by modulo 13. If we set $E^{(p)} = E$ and

$$E_1^{(a)} = \big\{\{v_i, v_{i+2}\} \mid 0 \le i \le 13\big\}, \qquad E_2^{(a)} = \big\{\{v_i, v_{i+3}\} \mid 0 \le i \le 13\big\},$$

$$E_3^{(a)} = \big\{\{v_i, v_{i+4}\} \mid 0 \le i \le 13\big\}, \qquad E_4^{(a)} = \big\{\{v_i, v_{i+6}\} \mid 0 \le i \le 13\big\},$$

$$E_5^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+7}\}\ (0 \le i \le 6), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

we obtain 5 vertex-transitive Kähler graphs $(V, E^{(p)} \cup E_i^{(a)})$ $(i = 1, 2, 3, 4, 5)$ of $d^{(a)} = 2$ (see Figures 6–10). Also, if we set

$$E_{i+5}^{(a)} = E_i^{(a)} \cup \big\{(v_i, v_{i+7}) \mid 0 \le i \le 6\big\}$$

and

$$E_{i+9}^{(a)} = E_i^{(a)} \bigcup \left\{ \begin{array}{l} \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\}, \\ \{v_9, v_0\},\ \{v_{11}, v_2\}, \{v_{13}, v_4\} \end{array} \right\},$$

for $i = 1, 2, 3, 4$, we obtain 8 vertex-transitive Kähler graphs $(V, E^{(p)} \cup E_{i+5}^{(a)})$ and $(V, E^{(p)} \cup E_{i+9}^{(a)})$ of $d^{(a)} = 3$.

Example 8 We set $Q_k = \big\{(a_1, a_2, \dots, a_k) \mid a_i \in \{0, 1\}\big\}$ $(k \ge 3)$. We define that two vertices $v = (a_1, \dots, a_k)$, $w = (b_1, \dots, b_k) \in Q_k$ are adjacent to each other in the principal graph if and only if there is i_0 $(1 \le i_0 \le k)$ satisfying that $a_{i_0} \ne b_{i_0}$ and $a_i = b_i$ for $i \ne i_0$, and define that they are adjacent to each other in the auxiliary graph if and only if there are i_1, i_2 $(1 \le i_1 < i_2 \le k)$ satisfying that $a_{i_1} \ne b_{i_1}$, $a_{i_2} \ne b_{i_2}$ and $a_i = b_i$ for

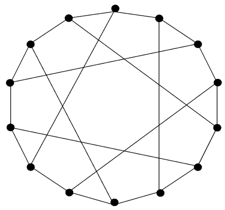

Figure 5. original.

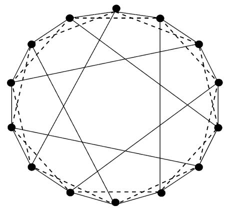

Figure 6.

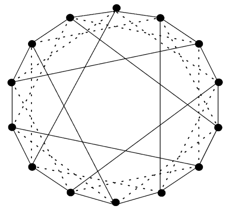

Figure 7.

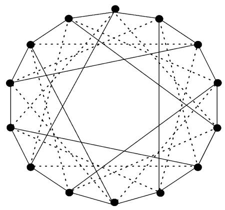

Figure 8.

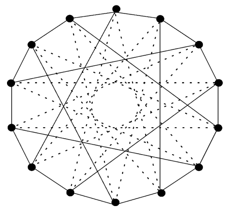

Figure 9.

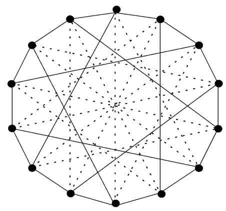

Figure 10.

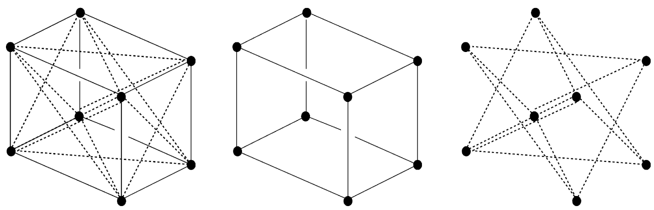

Figure 11. 3-Kähler cube with its principal and its auxiliary.

$i \neq i_1, i_2$, Since the graph $(Q_k, E^{(p)})$ is called k-cube, we shall call the graph $(Q_k, E^{(p)} \cup E^{(a)})$ a k-Kähler cube. It is clear that this k-Kähler cube is vertex-transitive.

It is well known that the Petersen graph on 10 vertices is the smallest example of a vertex-transitive graph which is not a Cayley graph.

Example 9 For a set $V = \{v_{1,0}, v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{2,0}, v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}\}$ of vertices, we set

$$E^{(p)} = \left\{ \begin{array}{l} \{v_{1,0}, v_{1,1}\}, \{v_{1,1}, v_{1,2}\}, \{v_{1,2}, v_{1,3}\}, \{v_{1,3}, v_{1,4}\}, \{v_{1,4}, v_{1,0}\}, \\ \{v_{2,0}, v_{2,2}\}, \{v_{2,2}, v_{2,4}\}, \{v_{2,4}, v_{2,1}\}, \{v_{2,1}, v_{2,3}\}, \{v_{2,3}, v_{2,0}\}, \\ \{v_{1,0}, v_{2,0}\}, \{v_{1,1}, v_{2,1}\}, \{v_{1,2}, v_{2,2}\}, \{v_{1,3}, v_{2,3}\}, \{v_{1,4}, v_{2,4}\}, \end{array} \right\}.$$

The graph $(V, E^{(p)})$ is the Petersen graph. If we set

$$E_1^{(a)} = \left\{ \begin{array}{l} \{v_{1,0}, v_{1,2}\}, \{v_{1,2}, v_{1,4}\}, \{v_{1,4}, v_{1,1}\}, \{v_{1,1}, v_{1,3}\}, \{v_{1,3}, v_{1,0}\}, \\ \{v_{2,0}, v_{2,1}\}, \{v_{2,1}, v_{2,2}\}, \{v_{2,2}, v_{2,3}\}, \{v_{2,3}, v_{2,4}\}, \{v_{2,4}, v_{2,0}\}, \end{array} \right\},$$

$$E_2^{(a)} = \left\{ \begin{array}{l} \{v_{1,0}, v_{2,1}\}, \{v_{1,0}, v_{2,2}\}, \{v_{1,0}, v_{2,3}\}, \{v_{1,0}, v_{2,4}\}, \\ \{v_{1,1}, v_{2,0}\}, \{v_{1,1}, v_{2,2}\}, \{v_{1,1}, v_{2,3}\}, \{v_{1,1}, v_{2,4}\}, \\ \{v_{1,2}, v_{2,0}\}, \{v_{1,2}, v_{2,1}\}, \{v_{1,2}, v_{2,3}\}, \{v_{1,2}, v_{2,4}\}, \\ \{v_{1,3}, v_{2,0}\}, \{v_{1,3}, v_{2,1}\}, \{v_{1,3}, v_{2,2}\}, \{v_{1,3}, v_{2,4}\}, \\ \{v_{1,4}, v_{2,0}\}, \{v_{1,4}, v_{2,1}\}, \{v_{1,4}, v_{2,2}\}, \{v_{1,4}, v_{2,3}\}, \end{array} \right\},$$

then $(V, E^{(p)} \cup E_1^{(a)})$ and $(V, E^{(p)} \cup E_2^{(a)})$ are vertex-transitive Kähler graphs which are not Cayley Kähler graphs (see Figures 12, 13). For these graphs, automorphisms are given as compositions of rotations $f_k(v_{i,j}) = v_{i,j+k}$ $(k = 1, 2, 3, 4, 5)$ and the reversing g of upper and lower which is given by

$$g : \begin{array}{l} v_{1,0} \mapsto v_{2,0},\ v_{1,1} \mapsto v_{2,2},\ v_{1,2} \mapsto v_{2,4},\ v_{1,3} \mapsto v_{2,1},\ v_{1,4} \mapsto v_{2,3}, \\ v_{2,0} \mapsto v_{1,0},\ v_{2,1} \mapsto v_{1,2},\ v_{2,2} \mapsto v_{1,4},\ v_{2,3} \mapsto v_{1,1},\ v_{2,4} \mapsto v_{1,3}, . \end{array}$$

Thus, with the Kähler graph $\big(V, E^{(p)} \cup (E_1^{(a)} \cup E_2^{(a)})\big)$ given by use of the complement graph, the Petersen graph induces three kinds of vertex-transitive Kähler graphs which are not Cayley Kähler graphs.

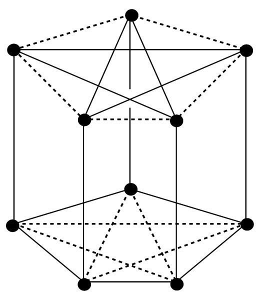

Figure 12.

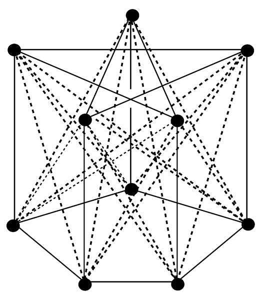

Figure 13.

We say that two Kähler graphs $(V, E^{(p)} \cup E^{(a)})$, $(V, F^{(p)} \cup F^{(a)})$ having the same sets of vertices are *dual* to each other if $F^{(p)} = E^{(a)}$, $F^{(a)} = E^{(p)}$ hold. It is clear that the dual Kähler graph of a vertex-transitive Kähler graph is also vertex-transitive.

4. Complete Kähler graphs

An ordinary graph is called complete if for every pair of its distinct vertices there is an edge joining them. We say a Kähler graph to be *complete* if it is regular as a Kähler graph and is complete as an ordinary graph. It is clear that if G is a regular ordinary graph then the complement-filled Kähler graph G^K is complete. By Theorem 1, we obtain the following:

Proposition 2 *Let n, $d^{(p)}$, $d^{(a)}$ be positive integers satisfying $n \geq 5$, $d^{(p)} \geq 2$, $d^{(a)} \geq 2$ and $d^{(p)} + d^{(a)} = n - 1$. There exists a finite complete Kähler graph whose cardinality of the set of vertices is n and whose principal and auxiliary degrees are $d^{(p)}$ and $d^{(a)}$ if and only if one of the following conditions holds*;

1) *n is odd and both $d^{(p)}$, $d^{(a)}$ are even,*
2) *n is even and one of $d^{(p)}$, $d^{(a)}$ is even and the other is odd.*

Corollary 1 *Let n be a positive integer satisfying $n \geq 5$. There exists a finite complete Kähler graph whose cardinality of the set of vertices is n and whose principal and auxiliary degrees coincide if and only if $n - 1$ can be factored by* 4.

Proof. If we have a complete Kähler graph whose cardinality of the set of vertices is n and whose principal and auxiliary degrees are d, we have $n - 1 = 2d$. Therefore n is odd. By Proposition 2 we find d is even, hence find that $n-1$ is divided by 4. On the other hand, if n satisfies the condition Proposition 2 shows that we have such a complete Kähler graph. □

Theorem 1 guarantees that under the conditions in Proposition 2, hence under the condition in Corollary 1, we have a complete vertex-transitive Kähler graph.

We say that two Kähler graphs are congruent to each other if there is an isometry between them. It is trivial that two ordinary complete finite graphs are congruent to each other if and only if their sets of vertices have the same cardinality. We are hence interested in congruence classes of complete vertex-transitive Kähler graphs. We study complete vertex-transitive Kähler graphs of small cardinality n_G of the set of vertices. When $n_G = 5$, the principal and the auxiliary degrees are $d^{(p)} = d^{(a)} = 2$, hence we have a unique complete vertex-transitive Kähler graph with $n_G = 5$. When $n_G = 7$, we find that a complete vertex-transitive Kähler graph has either $d^{(p)} = 2$,

$d^{(a)} = 4$ or $d^{(p)} = 4$, $d^{(a)} = 2$. We call a graph connected if it is connected as a 1-dimensional CW-complex. Vertex-transitivity guarantees that a regular ordinary graph of degree 2 and of $n_G = 7$ is connected. As a matter of fact, if we suppose it is not connected, then it has two connected components. One of the components consists of 3 vertices and the other consists of 4 vertices. Since automorphisms preserve connected components, we have no automorphisms which map a component to the other, hence this graph is not vertex-transitive. Thus we can conclude that we have two complete vertex-transitive Kähler graphs with $n_G = 7$. They are dual to each other.

Example 10 We have complete vertex-transitive Kähler graphs whose sets of vertices have the same cardinality and which have the same principal and auxiliary degrees.

Figure 14 shows complete vertex-transitive Kähler graphs with $n_G = 6$, $d_G^{(p)} = 2$, $d_G^{(a)} = 3$ which are not isomorphic. One of their principal graph is connected but the other is not. Their auxiliary graphs, which are principal graphs of their dual Kähler graphs, are connected.

Figure 15 shows complete vertex-transitive Kähler graphs with $n_G = 9$, $d_G^{(p)} = d_G^{(a)} = 4$ which are not isomorphic and whose principal and whose auxiliary graphs are connected. For the set of vertices $V = \{v_0, v_1, v_2, \dots, v_8\}$, their sets of principal edges are

$$E_1^{(p)} = \big\{\{v_i, v_{i+1}\},\ \{v_i, v_{i+2}\} \mid 0 \le i \le 8\big\},$$
$$E_2^{(p)} = \big\{\{v_i, v_{i+1}\},\ \{v_i, v_{i+3}\} \mid 0 \le i \le 8\big\}.$$

We can see that they are not isomorphic by observing 3-step closed principal paths.

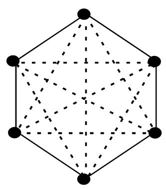

Figure 14. complete vertex-transitive Kähler graphs with $n_G = 6$, $d^{(p)} = 2$.

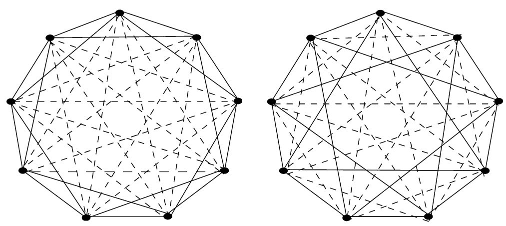

Figure 15. complete vertex-transitive Kähler graphs with $n_G = 9$, $d^{(p)} = 4$.

As a matter of course our problem is equivalent to the problem on congruency of regular ordinary graphs. This example suggests us that our problem is not so easy because principal graphs of Kähler graphs may not be connected.

Lemma (1) *Two complete Kähler graphs are congruent to each other if and only if their principal graphs are congruent to each other.*

(2) *Two complete Kähler graphs are congruent to each other if and only if their auxiliary graphs are congruent to each other.*

We should note that even if a Kähler graph is connected its principal graph and auxiliary graph are not necessarily connected. We here restrict ourselves onto complete Kähler graphs with auxiliary degree 2. We use the partition function $p : \mathbb{N} \to \mathbb{N}$. For a positive integer n, we consider its representation as a sum of positive integers. Here, we are allowed to use same integers in the representation, but the order of summing is irrelevant. The (integer) partition $p(n)$ is the number of such representations of n (see Section 19 of [7] for more detail).

Theorem 2 *For each positive number n (≥ 5) the number of congruence classes of complete Kähler graphs whose sets of vertices have the cardinality n and whose auxiliary degrees are 2 is $p(n) - p(n-1) - p(n-2) + p(n-3)$.*

Proof. Let $G = (V, E^{(p)} \cup E^{(a)})$ be a complete finite Kähler graph with $n_G = n$ and $d_G^{(a)} = 2$. If we consider its auxiliary graph, each of its component is a circuit, which is a circle as a 1-dimensional CW-complex. Since G is obtained by considering the complement graph of $(V, E^{(a)})$, we are enough to consider the number of congruence classes of ordinary regular graphs of degree 2. Since these graphs do not have multiple edges, each component of them contains at least three vertices. As two finite circuit graphs are congruent to each other if and only if they have the same number of vertices, the number of congruence classes coincides with the number of partition of n by use of integers greater than 2. □

Remark 1 Principal graphs of those complete Kähler graphs in Theorem 2 are connected. We take a vertex v of such a complete Kähler graph. Since $n_G - 3$ vertices are adjacent to this vertex in the principal graph, only two vertices are left. As the principal degree is greater than 1, these two vertices are contained in the connected component containing v.

Remark 2 Clearly the same property as in Theorem 2 holds for complete Kähler graphs whose principal degree is 2. For each positive n (≥ 5) the number of congruence classes of complete Kähler graphs whose sets of vertices have the cardinality n and whose principal degree is 2 is $p(n) - p(n-1) - p(n-2) + p(n-3)$.

In view of the proof of Theorem 2 we have the following.

Proposition 3 (1) *Two finite complete Kähler graphs whose auxiliary graphs are connected and are of degree* 2 *are congruent to each other if and only if cardinalities of their sets of vertices coincide.*

(2) *Two finite complete Kähler graphs whose principal graphs are connected and are of degree* 2 *are congruent to each other if and only if cardinalities of their sets of vertices coincide.*

Proposition 4 *For a positive n (≥ 5), the number of congruence classes of complete vertex-transitive Kähler graphs whose sets of vertices have the cardinality n and whose auxiliary degree is* 2 *coincides with the number of divisors of n which are greater than* 2.

Proof. Suppose $n = n_1 n_2$ with some positive integers n_1, n_2 satisfying $n_2 \geq 3$. We prepare n_1 circuit graphs having n_2 vertices. By making them an auxiliary graph we have a complete Kähler graph $(V, E^{(p)} \cup E^{(a)})$ satisfying $\sharp(V) = n$ and $d_G^{(a)} = 2$. Since all the components of $(V, E^{(a)})$ are circuits having the same numbers of vertices, for arbitrary distinct $v, v' \in V$ we have an isomorphism of $(V, E^{(a)})$ which maps v to v' and maps the component containing v to the component containing v'. It is clear that this induces an isomorphism of $(V, E^{(p)} \cup E^{(a)})$. Thus, we find that this Kähler graph is vertex-transitive, and get the conclusion. $\square$

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