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Screen Semi-Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds

S. S. Shukla and Akhilesh Yadav

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Abstract. In this paper, we introduce the notion of screen semi-slant lightlike submanifolds of indefinite Sasakian manifolds giving characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2 and $RadTM$ on screen semi-slant lightlike submanifolds of indefinite Sasakian manifolds have been obtained. Further we obtain necessary and sufficient conditions for foliations determined by above distributions to be totally geodesic. We also study mixed geodesic screen semi-slant lightlike submanifolds of indefinite Sasakian manifolds and obtain a necessary and sufficient condition for induced connection to be metric connection.

Key words: Semi-Riemannian manifold, degenerate metric, radical distribution, screen distribution, screen transversal vector bundle, lightlike transversal vector bundle, Gauss and Weingarten formulae.

1. Introduction

In 1960, S. Sasaki defined a structure on a differentiable manifold with Riemannian metric in his paper "On differentiable manifolds with certain structure which are closely related to almost contact structure I", which is known as Sasakian structure. A differentiable manifold with this structure is called Sasakian manifold. In [10], T. Takahashi introduced the notion of pseudo-Riemannian metric on a Sasakian manifold. A Sasakian manifold with pseudo-Riemannian metric is called indefinite Sasakian manifold. The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu ([4]).

A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be lightlike submanifold if the induced metric g on M is degenerate, i.e. there exists a non-zero $X \in \Gamma(TM)$ such that $g(X,Y) = 0$, $\forall Y \in \Gamma(TM)$. Various classes of lightlike submanifolds of indefinite Sasakian manifolds are defined according to the behaviour of distributions on these submanifolds with respect to the action of (1,1) tensor field ϕ in Sasakian structure of the

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ambient manifolds. Such submanifolds have been studied by Duggal and Sahin in ([5], [6]). In [9], Sahin and Yildirim studied slant and screen-slant lightlike submanifolds of indefinite Sasakian manifolds. In [7], A. Lotta introduced the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold.

The geometry of slant and semi-slant submanifolds of Sasakian manifolds was studied by Cabrerizo, J. L., Carriazo, A., Fernandez, L. M. and Fernandez, M., in ([2], [3]). The theory of invariant, screen slant, contact screen Cauchy-Riemann lightlike submanifolds have been studied in ([6], [8], [9]). Thus motivated sufficiently, we introduce the notion of screen semi-slant lightlike submanifolds of indefinite Sasakian manifolds. This new class of lightlike submanifolds of an indefinite Sasakian manifold includes invariant, screen slant, screen real, contact screen Cauchy-Riemann lightlike submanifolds as its sub-cases. The paper is arranged as follows. There are some basic results in Section 2 . In Section 3, we study screen semi-slant lightlike submanifolds of an indefinite Sasakian manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions on screen semi-slant lightlike submanifolds of indefinite Sasakian manifolds.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [4] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM). \tag{2.1}$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M}|_M = TM \oplus tr(TM), \tag{2.3}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \tag{2.4}$$

Following are four cases of a lightlike submanifold $\big(M, g, S(TM), S(TM^{\perp})\big)$:

Case.1 r-lightlike if $r < \min(m, n)$,
Case.2 co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,
Case.3 isotropic if $r = m < n$, $S(TM) = \{0\}$,
Case.4 totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y, A_V X$ belong to $\Gamma(TM)$ and $h(X, Y), \nabla^t_X V$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.9}$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M respectively.

Now by using (2.5), (2.7)–(2.9) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W)+\overline{g}(Y,D^l(X,W))=g(A_WX,Y), \tag{2.10}$$

$$\overline{g}(D^s(X,N),W)=\overline{g}(N,A_WX). \tag{2.11}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X,Y\in\Gamma(TM)$ and $\xi\in\Gamma(RadTM)$, we have

$$\nabla_X\overline{P}Y=\nabla^*_X\overline{P}Y+h^*(X,\overline{P}Y), \tag{2.12}$$

$$\nabla_X\xi=-A^*_\xi X+\nabla^{*t}_X\xi. \tag{2.13}$$

By using above equations, we obtain

$$\overline{g}(h^l(X,\overline{P}Y),\xi)=g(A^*_\xi X,\overline{P}Y), \tag{2.14}$$

$$\overline{g}(h^*(X,\overline{P}Y),N)=g(A_NX,\overline{P}Y), \tag{2.15}$$

$$\overline{g}(h^l(X,\xi),\xi)=0, \qquad A^*_\xi\xi=0. \tag{2.16}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is metric connection, by using (2.7), we get

$$(\nabla_Xg)(Y,Z)=\overline{g}(h^l(X,Y),Z)+\overline{g}(h^l(X,Z),Y). \tag{2.17}$$

A semi-Riemannian manifold $(\overline{M},\overline{g})$ is called an ϵ-almost contact metric manifold [5] if there exists a (1, 1) tensor field ϕ, a vector field V called characteristic vector field and a 1-form η, satisfying

$$\phi^2X=-X+\eta(X)V, \quad \eta(V)=\epsilon, \quad \eta\circ\phi=0, \quad \phi V=0, \tag{2.18}$$

$$\overline{g}(\phi X,\phi Y)=\overline{g}(X,Y)-\epsilon\eta(X)\eta(Y), \tag{2.19}$$

for all $X,Y\in\Gamma(T\overline{M})$, where $\epsilon=1$ or -1. It follows that

$$\overline{g}(V,V)=\epsilon, \tag{2.20}$$

$$\overline{g}(X,V)=\eta(X), \tag{2.21}$$

$$\overline{g}(X,\phi Y)=-\overline{g}(\phi X,Y). \tag{2.22}$$

Then $(\phi,V,\eta,\overline{g})$ is called an ϵ-almost contact metric structure on $\overline{M}$.

An ϵ-almost contact metric structure $(\phi, V, \eta, \overline{g})$ is called an indefinite Sasakian structure if and only if

$$(\overline{\nabla}_X \phi)Y = \overline{g}(X, Y)V - \epsilon\eta(Y)X, \tag{2.23}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is Levi-Civita connection with respect to $\overline{g}$.

A semi-Riemannian manifold endowed with an indefinite Sasakian structure is called an indefinite Sasakian manifold. From (2.23), for any $X, Y \in \Gamma(T\overline{M})$, we get

$$\overline{\nabla}_X V = -\phi X. \tag{2.24}$$

Let $(\overline{M}, \overline{g}, \phi, V, \eta)$ be an ϵ-almost contact metric manifold. If $\epsilon = 1$, then $\overline{M}$ is said to be a spacelike ϵ-almost contact metric manifold and if $\epsilon = -1$, then $\overline{M}$ is called a timelike ϵ-almost contact metric manifold. In this paper, we consider indefinite Sasakian manifolds with spacelike characteristic vector field V.

3. Screen Semi-Slant Lightlike Submanifolds

In this section, we introduce the notion of screen semi-slant lightlike submanifolds of indefinite Sasakian manifolds. At first, we state the following Lemma for later use:

Lemma 3.1 *Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.*

The proof of above Lemma follows as in Lemma 4.1 of [9], so we omit it.

Definition 3.1 Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then we say that M is a screen semi-slant lightlike submanifold of $\overline{M}$ if following conditions are satisfied:

- (i) $RadTM$ is invariant with respect to ϕ, i.e. $\phi(RadTM) = RadTM$,
- (ii) there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$,
- (iii) the distribution D_1 is an invariant distribution, i.e. $\phi D_1 = D_1$,

(iv) the distribution D_2 is slant with angle $\theta(\neq 0)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between ϕX and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle θ is called slant angle of distribution D_2. A screen semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \pi/2$.

From the above definition, we have the following decomposition

$$TM = RadTM \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}. \tag{3.1}$$

In particular, we have

(i) if $D_1 = 0$, then M is a screen slant lightlike submanifold,
(ii) if $D_2 = 0$, then M is an invariant lightlike submanifold,
(iii) if $D_1 = 0$ and $\theta = \pi/2$, then M is a screen real lightlike submanifold,
(iv) if $D_1 \neq 0$ and $\theta = \pi/2$, then M is a contact SCR-lightlike submanifold.

Thus above new class of lightlike submanifolds of an indefinite Sasakian manifold includes invariant, screen slant, screen real, contact screen Cauchy-Riemann lightlike submanifolds as its sub-cases which have been studied in ([6], [8], [9]).

Let $(\mathbb{R}^{2m+1}_{2q}, \overline{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}^{2m+1}_{2q}$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}\Big(dz - \sum_{i=1}^{m} y^i dx^i\Big), \quad V = 2\partial z,$$

$$\overline{g} = \eta \otimes \eta + \frac{1}{4}\Big(- \sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i \Big),$$

$$\phi\Big(\sum_{i=1}^{m} (X_i \partial x_i + Y_i \partial y_i) + Z\partial z \Big) = \sum_{i=1}^{m} (Y_i \partial x_i - X_i \partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,$$

where (x^i, y^i, z) are the cartesian coordinates on $\mathbb{R}^{2m+1}_{2q}$. Now, we construct some examples of screen semi-slant lightlike submanifolds of an indefinite Sasakian manifold.

Example 1 Let $(\mathbb{R}_2^{13}, \overline{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}_2^{13}$ given by $x^1 = u_1$, $y^1 = u_2$, $x^2 = u_1 \cos\alpha - u_2 \sin\alpha$, $y^2 = u_1 \sin\alpha + u_2 \cos\alpha$, $x^3 = -y^4 = u_3$, $x^4 = y^3 = u_4$, $x^5 = u_5 \sin u_6$, $y^5 = u_5 \cos u_6$, $x^6 = \sin u_5$, $y^6 = \cos u_5$, $z = u_7$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$
\begin{aligned}
Z_1 &= 2(\partial x_1 + \cos\alpha \partial x_2 + \sin\alpha \partial y_2 + y^1 \partial z + \cos\alpha y^2 \partial z),\\
Z_2 &= 2(\partial y_1 - \sin\alpha \partial x_2 + \cos\alpha \partial y_2 - \sin\alpha y^2 \partial z),\\
Z_3 &= 2(\partial x_3 - \partial y_4 + y^3 \partial z), \quad Z_4 = 2(\partial x_4 + \partial y_3 + y^4 \partial z),\\
Z_5 &= 2(\sin u_6 \partial x_5 + \cos u_6 \partial y_5 + \cos u_5 \partial x_6 - \sin u_5 \partial y_6\\
&\qquad + \sin u_6 y^5 \partial z + \cos u_5 y^6 \partial z),\\
Z_6 &= 2(u_5 \cos u_6 \partial x_5 - u_5 \sin u_6 \partial y_5 + u_5 \cos u_6 y^5 \partial z),\\
Z_7 &= V = 2\partial z.
\end{aligned}
$$

Hence $RadTM = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, Z_5, Z_6, V\}$. Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \cos\alpha \partial x_2 + \sin\alpha \partial y_2 - y^1 \partial z + \cos\alpha y^2 \partial z$, $N_2 = -\partial y_1 - \sin\alpha \partial x_2 + \cos\alpha \partial y_2 - \sin\alpha y^2 \partial z$ and $S(TM^{\perp})$ is spanned by

$$
\begin{aligned}
W_1 &= 2(\partial x_3 + \partial y_4 + y^3 \partial z), \quad W_2 = 2(\partial x_4 - \partial y_3 + y^4 \partial z),\\
W_3 &= 2(\sin u_6 \partial x_5 + \cos u_6 \partial y_5 - \cos u_5 \partial x_6 + \sin u_5 \partial y_6\\
&\qquad + \sin u_6 y^5 \partial z + \cos u_5 y^6 \partial z),\\
W_4 &= 2(u_5 \sin u_5 \partial x_6 + u_5 \cos u_5 \partial y_6 + u_5 \sin u_5 y^6 \partial z).
\end{aligned}
$$

It follows that $\phi Z_1 = -Z_2$ and $\phi Z_2 = Z_1$, which implies that $RadTM$ is invariant i.e., $\phi RadTM = RadTM$. On the otherhand, we can see that $D_1 = span\{Z_3, Z_4\}$ such that $\phi Z_3 = -Z_4$ and $\phi Z_4 = Z_3$, which implies that D_1 is invariant with respect to ϕ and $D_2 = span\{Z_5, Z_6\}$ is a slant distribution with slant angle $\pi/4$. Hence M is a screen semi-slant 2-lightlike submanifold of $\mathbb{R}_2^{13}$.

Example 2 Let $(\mathbb{R}^{13}_2, \overline{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^{13}_2$ given by $x^1 = -y^2 = u_1$, $x^2 = y^1 = u_2$, $x^3 = y^4 = u_3$, $x^4 = -y^3 = u_4$, $x^5 = u_5 \cos\theta$, $y^5 = u_6 \cos\theta$, $x^6 = u_6 \sin\theta$, $y^6 = u_5 \sin\theta$, $z = u_7$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$Z_1 = 2(\partial x_1 - \partial y_2 + y^1 \partial z), \quad Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z),$$

$$Z_3 = 2(\partial x_3 + \partial y_4 + y^3 \partial z), \quad Z_4 = 2(\partial x_4 - \partial y_3 + y^4 \partial z),$$

$$Z_5 = 2(\cos\theta \partial x_5 + \sin\theta \partial y_6 + y^5 \cos\theta \partial z),$$

$$Z_6 = 2(\sin\theta \partial x_6 + \cos\theta \partial y_5 + y^6 \sin\theta \partial z),$$

$$Z_7 = V = 2\partial z.$$

Hence $RadTM = span\,\{Z_1, Z_2\}$ and $S(TM) = span\,\{Z_3, Z_4, Z_5, Z_6, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 - \partial y_2 - y^1 \partial z$, $N_2 = -\partial x_2 + \partial y_1 - y^2 \partial z$ and $S(TM^\perp)$ is spanned by

$$W_1 = 2(\partial x_3 - \partial y_4 + y^3 \partial z), \quad W_2 = 2(\partial x_4 + \partial y_3 + y^4 \partial z),$$

$$W_3 = 2(\sin\theta \partial x_5 - \cos\theta \partial y_6 + y^5 \sin\theta \partial z),$$

$$W_4 = 2(\cos\theta \partial x_6 - \sin\theta \partial y_5 + y^6 \cos\theta \partial z).$$

It follows that $\phi Z_1 = -Z_2$ and $\phi Z_2 = Z_1$, which implies that $RadTM$ is invariant i.e., $\phi RadTM = RadTM$. On the otherhand, we can see that $D_1 = span\,\{Z_3, Z_4\}$ such that $\phi Z_3 = Z_4$ and $\phi Z_4 = -Z_3$, which implies that D_1 is invariant with respect to ϕ and $D_2 = span\{Z_5, Z_6\}$ is a slant distribution with slant angle 2θ. Hence M is a screen semi-slant 2-lightlike submanifold of $\mathbb{R}^{13}_2$.

Now, for any vector field X tangent to M, we put $\phi X = PX + FX$, where PX and FX are tangential and transversal parts of ϕX respectively. We denote the projections on $RadTM$, D_1 and D_2 in TM by P_1, P_2 and P_3 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2 respectively. Then, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + \eta(X)V. \tag{3.2}$$

Now applying ϕ to (3.2), we have

$$\phi X = \phi P_1X + \phi P_2X + \phi P_3X, \tag{3.3}$$

which gives

$$\phi X = \phi P_1X + \phi P_2X + fP_3X + FP_3X, \tag{3.4}$$

where fP_3X (resp. FP_3X) denotes the tangential (resp. transversal) component of ϕP_3X. Thus we get $\phi P_1X \in \Gamma(RadTM), \phi P_2X \in \Gamma(D_1), fP_3X \in \Gamma(D_2)$ and $FP_3X \in \Gamma(S(TM^\perp))$. Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = Q_1W + Q_2W. \tag{3.5}$$

Applying ϕ to (3.5), we obtain

$$\phi W = \phi Q_1W + \phi Q_2W, \tag{3.6}$$

which gives

$$\phi W = \phi Q_1W + BQ_2W + CQ_2W, \tag{3.7}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of ϕQ_2W. Thus we get $\phi Q_1W \in \Gamma(ltr(TM))$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(TM^\perp))$.

Now, by using (2.23), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $RadTM$, D_1, D_2, $ltr(TM)$, $S(TM^\perp)$ and $\{V\}$, we obtain

$$\begin{aligned} &P_1(\nabla_X\phi P_1Y) + P_1(\nabla_X\phi P_2Y) + P_1(\nabla_X fP_3Y) \\ &\quad = P_1(A_{FP_3Y}X) + \phi P_1\nabla_XY - \eta(Y)P_1X, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &P_2(\nabla_X\phi P_1Y) + P_2(\nabla_X\phi P_2Y) + P_2(\nabla_X fP_3Y) \\ &\quad = P_2(A_{FP_3Y}X) + \phi P_2\nabla_XY - \eta(Y)P_2X, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &P_3(\nabla_X\phi P_1Y) + P_3(\nabla_X\phi P_2Y) + P_3(\nabla_X fP_3Y) \\ &\quad = P_3(A_{FP_3Y}X) + fP_3\nabla_XY + Bh^s(X,Y) - \eta(Y)P_3X, \end{aligned} \tag{3.10}$$

$$h^l(X,\phi P_1 Y) + h^l(X,\phi P_2 Y) + h^l(X, fP_3 Y)$$
$$= \phi h^l(X,Y) - D^l(X, FP_3 Y), \quad (3.11)$$
$$h^s(X,\phi P_1 Y) + h^s(X,\phi P_2 Y) + h^s(X, fP_3 Y)$$
$$= Ch^s(X,Y) - \nabla^s_X FP_3 Y + FP_3 \nabla_X Y, \quad (3.12)$$
$$\eta(\nabla_X \phi P_1 Y) + \eta(\nabla_X \phi P_2 Y) + \eta(\nabla_X fP_3 Y)$$
$$= \eta(A_{FP_3 Y} X) + \overline{g}(X,Y)V. \quad (3.13)$$

Theorem 3.2 *Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then M is a screen semi-slant lightlike submanifold of $\overline{M}$ if and only if*

(i) *$ltr(TM)$ and D_1 are invariant with respect to ϕ,*
(ii) *there exists a constant $\lambda \in [0,1)$ such that $P^2 X = -\lambda X$.*

Moreover, there also exists a constant $\mu \in (0,1]$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then distributions D_1 and $RadTM$ are invariant with respect to ϕ. Now for any $N \in \Gamma(ltr(TM))$ and $X \in \Gamma(S(TM) - \{V\})$, using (2.22) and (3.4), we obtain $\overline{g}(\phi N, X) = -\overline{g}(N,\phi X) = -\overline{g}(N, \phi P_2 X + fP_3 X + FP_3 X) = 0$. Thus ϕN does not belong to $\Gamma(S(TM) - \{V\})$. For any $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$, from (2.22) and (3.7), we have $\overline{g}(\phi N, W) = -\overline{g}(N,\phi W) = -\overline{g}(N, BW + CW) = 0$. Hence, we conclude that ϕN does not belong to $\Gamma(S(TM^\perp))$. Now suppose that $\phi N \in \Gamma(RadTM)$. Then $\phi(\phi N) = \phi^2 N = -N + \eta(N)V = -N \in \Gamma(ltr(TM))$, which contradicts that $RadTM$ is invariant. Thus $ltr(TM)$ is invariant with respect to ϕ. Now for any $X \in \Gamma(D_2)$ we have $|PX| = |\phi X| \cos\theta$, which implies

$$\cos\theta = \frac{|PX|}{|\phi X|}. \quad (3.14)$$

In view of (3.14), we get $\cos^2\theta = |PX|^2/|\phi X|^2 = g(PX,PX)/g(\phi X,\phi X) = g(X, P^2 X)/g(X, \phi^2 X)$, which gives

$$g(X, P^2X) = \cos^2\theta\, g(X, \phi^2X). \tag{3.15}$$

Since M is a screen semi-slant lightlike submanifold, $\cos^2\theta = \lambda(constant) \in [0, 1)$ and therefore from (3.15), we get $g(X, P^2X) = \lambda g(X, \phi^2X) = g(X, \lambda\phi^2X)$, which implies

$$g(X, (P^2 - \lambda\phi^2)X) = 0. \tag{3.16}$$

Since X is non-null vector, we have $(P^2 - \lambda\phi^2)X = 0$, which implies

$$P^2X = \lambda\phi^2X = -\lambda X. \tag{3.17}$$

For any vector field $X \in \Gamma(D_2)$, we have

$$\phi X = PX + FX, \tag{3.18}$$

where PX and FX are tangential and transversal parts of ϕX respectively. Now, applying ϕ to (3.18) and taking tangential component, we get

$$-X = P^2X + BFX. \tag{3.19}$$

From (3.17) and (3.19), we get

$$BFX = -\sin^2\theta\, X, \tag{3.20}$$

where $\sin^2\theta = 1 - \lambda = \mu(constant) \in (0, 1]$. This proves (ii).

Conversely suppose that conditions (i) and (ii) are satisfied. We can show that $RadTM$ is invariant in similar way that $ltr(TM)$ is invariant. From (3.19), for any $X \in \Gamma(D_2)$, we get

$$-X = P^2X - \mu X, \tag{3.21}$$

which implies

$$P^2X = -\cos^2\theta\, X, \tag{3.22}$$

where $\cos^2\theta = 1 - \mu = \lambda(constant) \in [0, 1)$. Now $\cos\theta = g(\phi X, PX)/|\phi X||PX| = -g(X, \phi PX)/|\phi X||PX| = -g(X, P^2X)/|\phi X||PX| = -\lambda(g(X, \phi^2X)/|\phi X||PX|) = \lambda(g(\phi X, \phi X)/|\phi X||PX|)$. From above equation, we get

$$\cos\theta = \lambda \frac{|\phi X|}{|PX|}. \tag{3.23}$$

Therefore (3.14) and (3.23) give $\cos^2\theta = \lambda(constant)$. Hence M is a screen semi-slant lightlike submanifold. □

Corollary 3.1 *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with slant angle θ, then for any $X, Y \in \Gamma(D_2)$, we have*

(i) $g(PX, PY) = \cos^2\theta\,(g(X,Y) - \eta(X)\eta(Y))$,
(ii) $g(FX, FY) = \sin^2\theta\,(g(X,Y) - \eta(X)\eta(Y))$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [8].

Lemma 3.3 *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then for any $X, Y \in \Gamma(TM - \{V\})$, we have*

(i) $g(\nabla_X Y, V) = \overline{g}(Y, \phi X)$,
(ii) $g([X,Y], V) = 2\overline{g}(X, \phi Y)$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Since $\overline{\nabla}$ is a metric connection, from (2.7) and (2.24), for any $X, Y \in \Gamma(TM - \{V\})$, we have

$$g(\nabla_X Y, V) = \overline{g}(Y, \phi X). \tag{3.24}$$

From (2.22) and (3.24), we have $g([X,Y], V) = 2\overline{g}(X, \phi Y)$. □

Theorem 3.4 *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $RadTM$ is integrable if and only if*

(i) $h^s(Y, \phi P_1 X) = h^s(X, \phi P_1 Y)$ *and* $P_2(\nabla_X \phi P_1 Y) = P_2(\nabla_Y \phi P_1 X)$,
(ii) $P_3(\nabla_X \phi P_1 Y) = P_3(\nabla_Y \phi P_1 X)$, *for all* $X, Y \in \Gamma(RadTM)$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(RadTM)$. From (3.12), we have $h^s(X, \phi P_1 Y) = Ch^s(X, Y) + FP_3\nabla_X Y$, which gives $h^s(X, \phi P_1 Y) - h^s(Y, \phi P_1 X) = FP_3[X, Y]$. In view of (3.9), we obtain $P_2(\nabla_X \phi P_1 Y) = \phi P_2 \nabla_X Y$, which implies $P_2(\nabla_X \phi P_1 Y) - P_2(\nabla_Y \phi P_1 X) = \phi P_2[X, Y]$. Also from (3.10), we get $P_3(\nabla_X \phi P_1 Y) = fP_3\nabla_X Y + Bh^s(X, Y)$, which gives $P_3(\nabla_X \phi P_1 Y) - P_3(\nabla_Y \phi P_1 X) = fP_3[X, Y]$. This completes the proof. □

Theorem 3.5 *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_1 \oplus \{V\}$ is integrable if and only if*

(i) $h^s(Y, \phi P_2 X) = h^s(X, \phi P_2 Y)$, $P_1(\nabla_X \phi P_2 Y) = P_1(\nabla_Y \phi P_2 X)$,
(ii) $P_3(\nabla_X \phi P_2 Y) = P_3(\nabla_Y \phi P_2 X)$, *for all* $X, Y \in \Gamma(D_1 \oplus \{V\})$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1 \oplus \{V\})$. From (3.12), we have $h^s(X, \phi P_2 Y) = Ch^s(X, Y) + FP_3\nabla_X Y$, which gives $h^s(X, \phi P_2 Y) - h^s(Y, \phi P_2 X) = FP_3[X, Y]$. In view of (3.8), we obtain $P_1(\nabla_X \phi P_2 Y) = \phi P_1 \nabla_X Y$, which implies $P_1(\nabla_X \phi P_2 Y) - P_1(\nabla_Y \phi P_2 X) = \phi P_1[X, Y]$. Also from (3.10), we get $P_3(\nabla_X \phi P_2 Y) = fP_3\nabla_X Y + Bh^s(X, Y)$, which gives $P_3(\nabla_X \phi P_2 Y) - P_3(\nabla_Y \phi P_2 X) = fP_3[X, Y]$. This concludes the theorem. □

Theorem 3.6 *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ is integrable if and only if*

(i) $P_1(\nabla_X fP_3 Y - \nabla_Y fP_3 X) = P_1(A_{FP_3Y} X - A_{FP_3X} Y)$,
(ii) $P_2(\nabla_X fP_3 Y - \nabla_Y fP_3 X) = P_2(A_{FP_3Y} X - A_{FP_3X} Y)$,

for all $X, Y \in \Gamma(D_2 \oplus \{V\})$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2 \oplus \{V\})$. From (3.8), we have $P_1(\nabla_X fP_3 Y) = P_1(A_{FP_3Y} X) + \phi P_1 \nabla_X Y$, which gives $P_1(\nabla_X fP_3 Y) - P_1(\nabla_Y fP_3 X) - P_1(A_{FP_3Y} X) + P_1(A_{FP_3X} Y) = \phi P_1[X, Y]$. In view of (3.9), we obtain $P_2(\nabla_X fP_3 Y) = P_2(A_{FP_3Y} X) + \phi P_2 \nabla_X Y$, which implies $P_2(\nabla_X fP_3 Y) - P_2(\nabla_Y fP_3 X) - P_2(A_{FP_3Y} X) + P_2(A_{FP_3X} Y) = \phi P_2[X, Y]$. Thus, we obtain the required results. □

Theorem 3.7 *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then the induced connection ∇ is a metric connection if and only if*

(i) $Bh^s(X,Y) = 0$,
(ii) A^*_Y *vanishes on* $\Gamma(TM)$,

for all $X \in \Gamma(TM)$ *and* $Y \in \Gamma(RadTM)$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $RadTM$ is parallel distribution with respect to ∇ ([6]). From (2.7), (2.13) and (2.23), for any $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$, we have $\overline{\nabla}_X \phi Y = \phi\nabla^{*t}_X Y - \phi A^*_Y X + \phi h^l(X,Y) + \phi h^s(X,Y)$. On comparing tangential components of both sides of above equation, we get $\nabla_X \phi Y = \phi\nabla^{*t}_X Y - \phi A^*_Y X + Bh^s(X,Y)$. This proves the theorem. □

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold to be totally geodesic.

Definition 4.1 A screen semi-slant lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is said to be a mixed geodesic if its second fundamental form h satisfies $h(X,Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is mixed geodesic screen semi-slant lightlike submanifold if $h^l(X,Y) = 0$ and $h^s(X,Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 4.1 *Let* M *be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold* $\overline{M}$ *with structure vector field tangent to* M*. Then* $RadTM$ *defines a totally geodesic foliation if and only if* $\overline{g}(h^l(X,PZ),\phi Y) = -\overline{g}(D^l(X,FZ),\phi Y)$*, for all* $X,Y \in \Gamma(RadTM)$ *and* $Z \in \Gamma(S(TM))$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. It is easy to see that $RadTM$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(RadTM)$, for all $X,Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is metric connection, using (2.7) and (2.19), for any $X,Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X PZ + \overline{\nabla}_X FZ, \phi Y)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(h^l(X,PZ) + D^l(X,FZ), \phi Y)$. Thus, the theorem is completed. □

Theorem 4.2 *Let* M *be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold* $\overline{M}$ *with structure vector field tangent to* M*.*

Then $D_1 \oplus \{V\}$ defines a totally geodesic foliation if and only if

(i) $\overline{g}(\nabla_X fZ, \phi Y) = \overline{g}(A_{FZ}X, \phi Y)$,
(ii) $A_{\phi N}X$ *has no component in* $D_1 \oplus \{V\}$,

for all $X, Y \in \Gamma(D_1 \oplus \{V\})$, $Z \in \Gamma(D_2)$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. To prove the distribution $D_1 \oplus \{V\}$ defines a totally geodesic foliation it is sufficient to show that $\nabla_X Y \in \Gamma(D_1 \oplus \{V\})$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$. Since $\overline{\nabla}$ is metric connection, From (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \phi Z, \phi Y)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{FZ}X - \nabla_X fZ, \phi Y)$. In view of (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we have $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N)$, which implies $\overline{g}(\nabla_X Y, N) = \overline{g}(\phi Y, A_{\phi N}X)$. This completes the proof. □

Theorem 4.3 *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla_X fY, \phi Z) = \overline{g}(A_{FY}X, \phi Z)$,
(ii) $\overline{g}(fY, A_{\phi N}X) = \overline{g}(FY, D^s(X, \phi N))$,

for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. The distribution $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2 \oplus \{V\})$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$. From (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X \phi Y, \phi Z)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla_X fY - A_{FY}X, \phi Z)$. In view of (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N)$, which implies $\overline{g}(\nabla_X Y, N) = \overline{g}(fY, A_{\phi N}X) - \overline{g}(FY, D^s(X, \phi N))$. This concludes the theorem. □

Theorem 4.4 *Let M be a mixed geodesic screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

(i) $\nabla_X \phi Z$ *has no component in* $D_2 \oplus \{V\}$,
(ii) $\overline{g}(fY, A_{\phi N}X) = \overline{g}(FY, D^s(X, \phi N))$,

for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Since M is a mixed geodesic screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$, we have $h^s(X, \phi Z) = 0$, for all $X \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$. The distribution $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2 \oplus \{V\})$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$. Since $\overline{\nabla}$ is metric connection, From (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \phi Z, \phi Y)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X \phi Z, fY) - \overline{g}(h^s(X, \phi Z), FY)$. Now, from (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we have $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N)$, which gives $\overline{g}(\nabla_X Y, N) = \overline{g}(fY, A_{\phi N}X) - \overline{g}(FY, D^s(X, \phi N))$. Thus, we obtain the required results. □

Theorem 4.5 *Let* M *be a mixed geodesic screen semi-slant lightlike submanifold of an indefinite Sasakian manifold* $\overline{M}$ *with structure vector field tangent to* M. *Then the induced connection* ∇ *on* $D_1 \oplus D_2$ *is a metric connection if and only if*

(i) $A^*_\xi X$ *has no component in* D_1,
(ii) $\overline{g}(fW, A^*_{\phi\xi}Z) = \overline{g}(FW, h^s(Z, \phi\xi))$,

for all $X \in \Gamma(D_1)$, $Z, W \in \Gamma(D_2)$ *and* $\xi \in \Gamma(RadTM)$.

Proof. Let M be a mixed geodesic screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $h^l(X, Z) = 0$, for all $X \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. From (2.14), for any $X, Y \in \Gamma(D_1)$ and $\xi \in \Gamma(RadTM)$, we have $\overline{g}(h^l(X, Y), \xi) = g(Y, A^*_\xi X)$. Since $\overline{\nabla}$ is metric connection, From (2.7), (2.19) and (2.23), for any $Z, W \in \Gamma(D_2)$ and $\xi \in \Gamma(RadTM)$, we have $\overline{g}(h^l(Z, W), \xi) = -\overline{g}(fW, \nabla_Z \phi\xi) - \overline{g}(FW, h^s(Z, \phi\xi))$, which implies $\overline{g}(h^l(Z, W), \xi) = \overline{g}(fW, A^*_{\phi\xi}Z) - \overline{g}(FW, h^s(Z, \phi\xi))$. This proves the theorem. □

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S. S. Shukla
Department of Mathematics
University of Allahabad
Allahabad-211002, India
E-mail: ssshukla_au@rediffmail.com

Akhilesh Yadav
Department of Mathematics
University of Allahabad
Allahabad-211002, India
E-mail: akhilesh_mathau@rediffmail.com