Estimates of operator convex and operator monotone functions on bounded intervals

Hamed NAJAFI, Mohammad Sal MOSLEHIAN,
Masatoshi FUJII and Ritsuo NAKAMOTO

Abstract. Recently the behavior of operator monotone functions on unbounded intervals with respect to the relation of strictly positivity has been investigated. In this paper we deeply study such behavior not only for operator monotone functions but also for operator convex functions on bounded intervals. More precisely, we prove that if f is a nonlinear operator convex function on a bounded interval (a, b) and A, B are bounded linear operators acting on a Hilbert space with spectra in (a, b) and $A - B$ is invertible, then $sf(A) + (1 - s)f(B) > f(sA + (1 - s)B)$. A short proof for a similar known result concerning a nonconstant operator monotone function on $[0, \infty)$ is presented. Another purpose is to find a lower bound for $f(A) - f(B)$, where f is a nonconstant operator monotone function, by using a key lemma. We also give an estimation of the Furuta inequality, which is an excellent extension of the Löwner–Heinz inequality.

Key words: Löwner–Heinz inequality, Furuta inequality and operator monotone function.

1. Introduction

Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$ equipped with the operator norm $\| \cdot \|$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called *positive* if $\langle Ax, x \rangle \geq 0$ holds for every $x \in \mathscr{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. Further, we write $A > B$ if $A \geq B$ and $A - B$ is invertible. When $A > 0$, we call A strictly positive.

Let f be a real-valued function defined on an interval J. If for any self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J,

- $A \leq B$ implies $f(A) \leq f(B)$, then f is said to be *operator monotone*;
- $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$ for all $\lambda \in [0, 1]$, then f is said to be *operator convex*.

Let us state our main terminology.

Definition   A continuous real valued function f defined on an interval $J = (a, b)$ is called

  (i) *strictly operator monotone* if $A < B$ implies $f(A) < f(B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J.
  (ii) *strictly operator convex* if $f(\lambda A + (1 - \lambda)B) < \lambda f(A) + (1 - \lambda)f(B)$ for all $0 < \lambda < 1$ and all selfadjoint operators A, B with spectra in J such that $A - B$ is invertible.

Kwong [9] showed that $f(t) = t^r$ is strictly operator monotone whenever $0 < r \leq 1$. In addition, Uchiyama [13] studied strictly operator convex functions.

Recall that if f is an operator monotone function on $[0, \infty)$, then f can be represented as

$$f(t) = f(0) + \beta t + \int_0^\infty \frac{\lambda t}{\lambda + t} d\mu(\lambda),$$

where $\beta \geq 0$ and μ is a positive measure on $[0, \infty)$. If f is an operator convex function on $[0, \infty)$, then f can be represented as

$$f(t) = f(0) + \beta t + \gamma t^2 + \int_0^\infty \frac{\lambda t^2}{\lambda + t} d\mu(\lambda),$$

in which $\gamma \geq 0$, $\beta = f'_+(0) = \lim_{t \to 0^+}((f(t) - f(0))/t)$ and μ is a positive measure on $[0, \infty)$. The integral representations of operator convex and operator monotone functions on bounded intervals are different as we will see later.

The Löwner–Heinz inequality is one of the most important facts in the theory of operator inequalities. It says that the function t^p is operator monotone for $p \in [0, 1]$; cf. [7]. Recently the behavior of operator monotone functions on unbounded intervals with respect to the relation of strictly positivity has been investigated. In [11], an estimation of the Löwner–Heinz inequality was proposed as follows.

Theorem A   *If $A > B \geq 0$ and $0 < r \leq 1$, then $A^r - B^r \geq \|A\|^r - (\|A\| - m)^r > 0$, and $\log A - \log B \geq \log \|A\| - \log(\|A\| - m) > 0$, where $m = \|(A - B)^{-1}\|^{-1}$.*

Very recently, the following generalization of Theorem A is given in [2],

see also [5].

Theorem B   *If $A > B \geq 0$ and f is a non-constant operator monotone function on $[0, \infty)$, then $f(A) - f(B) \geq f(\|B\| + m) - f(\|B\|) > 0$, where $m = \|(A - B)^{-1}\|^{-1}$.*

As a consequence, we have the following improvement of Theorem A.

Theorem C   *If $A > B \geq 0$ and $0 < r \leq 1$, then*

$$A^r - B^r \geq (\|B\| + m)^r - \|B\|^r > 0$$

and $\log A - \log B \geq \log(\|B\| + m) - \log \|B\| > 0$, where $m = \|(A - B)^{-1}\|^{-1}$.

The first aim of this paper is to prove that a nonlinear operator convex function on a bounded interval is strictly operator convex. The second one is to give a precise consideration of Theorem B for operator monotone functions on a finite interval. The third purpose is to extend Theorem C to the Furuta inequality. We recall the unforgettable fact that the Furuta inequality is a beautiful extension of the Löwner–Heinz inequality.

2.   Strictly operator convex functions

In this section, we treat the behavior of operator convex functions on bounded intervals with respect to the relation of strictly positivity. To this end we need some lemmas.

Lemma 2.1   *The function $f(x) = 1/x$ is strictly operator convex on $(0, \infty)$, that is, if A, B are strictly positive operators such that $A - B$ is invertible, then for each $0 < s < 1$,*

$$sA^{-1} + (1 - s)B^{-1} > (sA + (1 - s)B)^{-1}.$$

Proof.   Let $s \in (0, 1)$. Put $H = A^{1/2}B^{-1}A^{1/2}$ and $K = sA^{-1} + (1 - s)B^{-1} - (sA + (1 - s)B)^{-1}$. Then

$$A^{1/2}KA^{1/2} = s + (1 - s)H - (s + (1 - s)H^{-1})^{-1} = f(H),$$

where $f(x) = s + (1 - s)x - (s + (1 - s)x^{-1})^{-1}$ for $x > 0$. It is clear that $f(x) > 0$ for $x > 0$ except $x = 1$.

Now, since $A - B$ is invertible, so is $1 - H$ and thus 1 does not belong

to the spectrum of H. Hence we have $f(H) > 0$, so $K > 0$, which is the desired inequality. □

Lemma 2.2   *For each λ with $|\lambda| \leq 1$, the function $f_\lambda(x) = x^2/(1 - \lambda x)$ is strictly operator convex on $J = (-1, 1)$.*

Proof.   We show that if A, B are selfadjoint operators with spectra in J such that $A - B$ is invertible, then

$$s f_\lambda(A) + (1 - s) f_\lambda(B) > f_\lambda(sA + (1 - s)B)$$

holds for $0 < s < 1$. Fix $\lambda \neq 0$. Since $f_\lambda(x) = (-x)/\lambda + (-1)/\lambda^2 + 1/\lambda^2(1 - \lambda x)$, we have

$$\lambda^2(s f_\lambda(A) + (1 - s) f_\lambda(B) - f_\lambda(sA + (1 - s)B))$$
$$= -s\lambda A - s + s(1 - \lambda A)^{-1} - (1 - s)\lambda B - (1 - s) + (1 - s)(1 - \lambda B)^{-1}$$
$$\quad + \lambda(sA + (1 - s)B) + 1 - (1 - \lambda(sA + (1 - s)B))^{-1}$$
$$= s(1 - \lambda A)^{-1} + (1 - s)(1 - \lambda B)^{-1} - (1 - \lambda(sA + (1 - s)B))^{-1}.$$

Applying Lemma 2.1 to $1 - \lambda A$ and $1 - \lambda B$, we reach the conclusion.

Incidentally, for the case $\lambda = 0$, i.e., $f_0(x) = x^2$, we have

$$sA^2 + (1 - s)B^2 - (sA + (1 - s)B)^2$$
$$= sA^2 + (1 - s)B^2 - s^2 A^2 - (1 - s)^2 B^2 - s(1 - s)(AB + AB)$$
$$= s(1 - s)(A - B)^2 > 0.$$
□

Theorem 2.3   *Any nonlinear operator convex function on a finite interval (a, b) is strictly operator convex on (a, b).*

Proof.   We may assume that $(a, b) = (-1, 1)$. It is known that any operator convex function f on $(-1, 1)$ can be represented as

$$f(x) = f(0) + \alpha x + \int_{-1}^{1} \frac{x^2}{1 - \lambda x} d\mu(\lambda), \tag{1}$$

where $\alpha \in \mathbb{R}$ and μ is a positive measure on $[-1, 1]$, [6, Theorem 4.5]. Since f is nonlinear, we infer that $\mu \neq 0$. By Lemma 2.2 we conclude that

$$sf(A) + (1-s)f(B) > f(sA + (1-s)B)$$

for each $0 < s < 1$ and selfadjoint operators A, B with spectra in $(-1, 1)$ such that $A - B$ is invertible. $\square$

3. Estimates of Operator Monotone Functions

We start this section with the following lemma. The first part is borrowed from [11] and the second is another variant of it.

Lemma 3.1   *Let $A > B > 0$ and $m = \|(A - B)^{-1}\|^{-1}$. Then*

(i) $B^{-1} - A^{-1} \geq 1/(\|A\| - m) - 1/\|A\|$;
(ii) $B^{-1} - A^{-1} \geq m/(\|B\| + m)\|B\|$.

Proof.   (i) See Lemma 2.1 of [11].
(ii) Because of $A - B \geq m$, we have

$$B^{-1} - A^{-1} \geq B^{-1} - (B + m)^{-1} = mB^{-1}(B + m)^{-1}$$

$$\geq \frac{m}{\|B\|(\|B\| + m)}. \qquad \square$$

Now we note that $m = \|(A - B)^{-1}\|^{-1}$ is the maximum among $c \geq 0$ such that $A - B \geq c$.

Let us give an alternative proof of Theorem B by using integral representation of operator monotone functions on $[0, \infty)$.

Proof of Theorem B.   Note that f admits the the following integral representation:

$$f(t) = a + bt + \int_{-\infty}^{0} \frac{1 + ts}{s - t} dm(s) = a + bt + \int_{-\infty}^{0} \left(-s - \frac{1 + s^2}{t - s} \right) dm(s)$$

where $b \geq 0$ and $m(s)$ is a positive measure. Hence it follows from Lemma 3.1 (ii) that

$$f(A) - f(B) = b(A - B) + \int_{-\infty}^{0} (1 + s^2)((B - s)^{-1} - (A - s)^{-1}) dm(s)$$

$$\geq bm + \int_{-\infty}^{0} (1 + s^2) \left(\frac{1}{\|B\| - s} - \frac{1}{\|B\| - s + m} \right) dm(s)$$
$$= f(\|B\| + m) - f(\|B\|) \; (> 0). \qquad \qquad \square$$

Finally we propose an explicit lower bound of the difference $f(A) - f(B)$ for operator monotone functions on a finite interval (a, b) when $A > B$; see [11, Proposition 2.2]. Let us $m_X = \min \operatorname{sp}(X)$ and $M_X = \max \operatorname{sp}(X)$ for a self-adjoint operator X, where $\operatorname{sp}(X)$ denotes the spectrum of X. Note if $-1 < \lambda < 0$ and $-1 \leq X \leq 1$, then $1 - \lambda X \geq 0$. Hence

$$\|1 - \lambda X\| = \max\{1 - \lambda t : t \in \operatorname{sp}(X)\} = 1 - \lambda \max\{t : t \in \operatorname{sp}(X)\} = 1 - \lambda M_X.$$

Similarly if $0 < \lambda < 1$ and $-1 \leq X \leq 1$, then $\|1 - \lambda X\| = 1 - \lambda m_X$.

To achieve our main result we need the following key lemma. For $\lambda \in (-1, 1)$, set

$$f_\lambda(t) := \frac{t}{1 - \lambda t}, \qquad t \in (-1, 1).$$

Lemma 3.2   *If $-1 < B < A < 1$ and $A - B \geq m > 0$, then*

$$f_\lambda(A) - f_\lambda(B) \geq \begin{cases} f_\lambda(M_B + m) - f_\lambda(M_B) & for \; -1 < \lambda \leq 0 \\ f_\lambda(m_A) - f_\lambda(m_A - m) & for \; 0 < \lambda < 1. \end{cases}$$

Proof.   First of all, we have

$$f_0(A) - f_0(B) = A - B \geq m = f_0(M_B + m) - f_0(M_B).$$

Next suppose that $-1 < \lambda < 0$. Since

$$(1 - \lambda A) - (1 - \lambda B) = -\lambda(A - B) \geq -\lambda m > 0,$$

it follows from Lemma 3.1 that

$$(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \geq \|1 - \lambda B\|^{-1} - (\|1 - \lambda B\| - \lambda m)^{-1}$$
$$= (1 - \lambda M_B)^{-1} - (1 - \lambda M_B - \lambda m)^{-1},$$

whence

$$f_\lambda(A) - f_\lambda(B) = \frac{1}{-\lambda}((1 - \lambda B)^{-1} - (1 - \lambda A)^{-1})$$

$$\geq \frac{1}{-\lambda}((1 - \lambda M_B)^{-1} - (1 - \lambda(M_B + m))^{-1})$$

$$= f_\lambda(M_B + m) - f_\lambda(M_B).$$

Finally suppose that $0 < \lambda < 1$. Since

$$(1 - \lambda B) - (1 - \lambda A) = \lambda(A - B) \geq \lambda m > 0,$$

it follows from Lemma 3.1 that

$$(1 - \lambda A)^{-1} - (1 - \lambda B)^{-1} \geq \|1 - \lambda A\|^{-1} - (\|1 - \lambda A\| + \lambda m)^{-1}$$

$$= (1 - \lambda m_A)^{-1} - (1 - \lambda m_A + \lambda m)^{-1}.$$

Hence we have

$$f_\lambda(A) - f_\lambda(B) = \frac{1}{\lambda}((1 - \lambda A)^{-1} - (1 - \lambda B)^{-1})$$

$$\geq \frac{1}{\lambda}((1 - \lambda m_A)^{-1} - (1 - \lambda(m_A - m))^{-1})$$

$$= f_\lambda(m_A) - f_\lambda(m_A - m). \qquad \square$$

Theorem 3.3 *Any non-constant operator monotone function on a finite interval J is strictly operator monotone on J.*

Proof. We may assume that $J = (-1, 1)$. Put $f_\lambda(t) = t/(1 - \lambda t)$ for $\lambda \in (-1, 1)$. An operator monotone function on $(-1, 1)$ is represented as

$$f(t) = f(0) + f'(0) \int_{-1}^{1} f_\lambda(t) d\mu(\lambda),$$

where μ is a nonzero positive measure on $(-1, 1)$. Since f is nonconstant, we have $f'(0) > 0$. We here decompose f as $f = f(0) + g_1 + g_2$, where

$$g_1(t) = f'(0) \int_{-1}^{0} f_\lambda(t) d\mu(\lambda) \quad \text{and} \quad g_2(t) = f'(0) \int_{0}^{1} f_\lambda(t) d\mu(\lambda).$$

Then g_1 (resp., g_2) is an operator concave (resp. operator convex) increasing function. Hence it follows from Lemma 3.2 that

$$f(A) - f(B) = f'(0) \int_{-1}^{1} (f_\lambda(A) - f_\lambda(B)) d\mu(\lambda)$$

$$= g_1(A) - g_1(B) + g_2(A) - g_2(B)$$

$$\geq g_1(M_B + \epsilon) - g_1(M_B) + g_2(m_A) - g_2(m_A - \epsilon) > 0. \quad \square$$

4.   Furuta inequality

First of all, we cite *the Furuta inequality (FI)* established in [3] for reader's convenience, see also [1], [4], [8] and [12] for the best possibility of it. It says that if $A \geq B \geq 0$, then for each $r \geq 0$,

$$A^{(p+r)/q} \geq (A^{r/2} B^p A^{r/2})^{1/q}$$

holds for $p \geq 0$, $q \geq 1$ with

$$(1 + r)q \geq p + r.$$

To extend Theorem B, we remark that the case $r = 0$ in (FI) is just the Löwner–Heinz inequality. Now we introduce a constant $k(b, m, p, q, r)$ for $b, m, p, q, r \geq 0$ by

$$k(b, m, p, q, r) = (b + m)^{(p+r)/q-r} - b^{(p+r)/q-r}.$$

As a matter of fact, we have an extension of Theorem B in the form of Furuta inequality. We denote by $m_A = \|A^{-1}\|^{-1}$, the minimum of the spectrum of A.

Theorem 4.1   *Let A and B be invertible positive operators with $A - B \geq m > 0$. Then for $0 < r \leq 1$,*

$$A^{(p+r)/q} - (A^{r/2} B^p A^{r/2})^{1/q} \geq k(\|B\|, m, p, q, r) m_A{}^r$$

holds for $p \geq 0$, $q \geq 1$ with $(1 + r)q \geq p + r \geq qr$.

Proof.   We note that $q \geq 1$ and $(1 + r)q \geq p + r \geq qr$ assure that the

exponent $(p+r)/q - r$ in the constant $k(b, m, p, q, r)$ belongs to $[0, 1]$. Since $0 \le r \le 1$, it follows from Theorem B that

$$A^{(p+r)/q} - (A^{r/2}B^p A^{r/2})^{1/q}$$

$$= A^{(p+r)/q} - A^{r/2}B^{p/2}(B^{p/2}A^r B^{p/2})^{1/q-1}B^{p/2}A^{r/2}$$

$$= A^{(p+r)/q} - A^{r/2}B^{p/2}(B^{-p/2}A^{-r}B^{-p/2})^{1-1/q}B^{p/2}A^{r/2}$$

$$\ge A^{(p+r)/q} - A^{r/2}B^{p/2}(B^{-p/2}B^{-r}B^{-p/2})^{1-1/q}B^{p/2}A^{r/2}$$

$$= A^{(p+r)/q} - A^{r/2}B^{p-(p+r)(1-1/q)}A^{r/2}$$

$$= A^{r/2}(A^{(p+r)/q-r} - B^{(p+r)/q-r})A^{r/2}$$

$$\ge k(\|B\|, m, p, q, r)A^r$$

$$\ge k(\|B\|, m, p, q, r)m_A{}^r. \qquad \square$$

Next we cosider the optimal case $q = (1+r)/(p+r)$ with $p \ge 1$, which is the most important in the Furuta inequality, by virtue of the Löwner–Heinz inequality. The proof of the following theorem is as same as that of the Furuta inequality.

Theorem 4.2 *Let A and B be invertible positive operators with $A - B \ge m > 0$. Then*

$$A^{1+r} - (A^{r/2}B^p A^{r/2})^{(1+r)/(p+r)} \ge mm_A{}^r$$

holds for $p \ge 1$ and $r \ge 0$.

Proof. The conclusion for $r \in [0, 1]$ is ensured by Theorem 4.1 because $k(b, m, p, (p+r)/(1+r), r) = m$. In particular, taking $r = 1$, we have

$$A^2 - (A^{1/2}B^p A^{1/2})^{2/(p+1)} \ge mm_A := m_1.$$

Since $A_1 = A^2$ and $B_1 = (A^{1/2}B^p A^{1/2})^{2/(p+1)}$ satisfy $A_1 - B_1 \ge m_1 > 0$, we have, for an arbitrary $s \in [0.1]$,

$$A_1^{1+s} - (A_1^{(1+s)/2}B_1^p A_1^{(1+s)/2})^{(1+s)/(p_1+s)} \ge m_1, \text{ where } p_1 = \frac{p+1}{2}.$$

Namely, it is proved that if $A - B \geq m > 0$, then

$$A^{2(1+s)} - \left(A^{(1+2s)/2} B^p A^{(1+2s)/2}\right)^{2(1+s)/(p+1+2s)}$$

$$\geq m_1 m_{A_1}{}^s = m m_A{}^{1+2s}, \tag{2}$$

that is, putting $r = 1 + 2s$, the conclusion holds for $r \in [1, 3]$.

For the next step, if we put $s = 1$ in (2) and $A_2 = A^4$, $B_2 = (A^{3/2} B^p A^{3/2})^{p_2}$, where $p_2 = (p+3)/4$, then $A_2 - B_2 \geq m m_A{}^3 := m_2$ holds and for an arbitrary $s \in [0, 1]$,

$$A_2^{1+s} - \left(A_2^{(1+s)/2} B_2^p A_2^{(1+s)/2}\right)^{(1+s)/(p_2+s)} \geq m_2 m_{A_2}{}^s.$$

Hence, putting $r = 3 + 4s$, we obtain that the conclusion holds for $r \in [3, 7]$. By repeating this method, we can complete the proof of the theorem. $\quad\square$

Corollary 4.3   *Let A and B be invertible positive operators with $A - B \geq m > 0$. Then for each $r \geq 0$*

$$A^{(p+r)/q} - (A^{r/2} B^p A^{r/2})^{1/q} \geq \|A\|^{(p+r)/q} - (\|A\|^{1+r} - m m_A{}^r)^{(p+r)/q(1+r)}$$

holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$.

Proof.   Since $\alpha = (p+r)/q(1+r) \leq 1$, we apply Theorem A to A_1 and A_2 such that

$$A_1 = A^{1+r} > B_1 = (A^{r/2} B^p A^{r/2})^{(1+r)/(p+r)}$$

by Theorem 4.2. Then we have

$$A_1^\alpha - B_1^\alpha \geq \|A_1\|^\alpha - (\|A_1\| - m m_A{}^r)^\alpha.$$

So we get the desired lower bound. $\quad\square$

References

[1] Fujii M., *Furuta's inequality and its mean theoretic approach*. J. Operator Theory **23** (1990), 67–72.

[2] Fujii M., Kim Y. O. and Nakamoto R., *A characterization of convex functions and its application to operator monotone functions*. Banach J. Math. Anal. **8** (2014), 118–123.

[3] Furuta T., $A \geq B \geq 0$ *assures* $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0,\ p \geq 0,\ q \geq 1$ *with* $(1 + 2r)q \geq p + 2r$. Proc. Amer. Math. Soc. **101** (1987), 85–88.

[4] Furuta T., *Elementary proof of an order preserving inequality*. Proc. Japan Acad. **65** (1989), 126.

[5] Furuta T., *Comprehensive survey on an order preserving operator inequality*. Banach J. Math. Anal. **7** (2013), 14–40.

[6] Hansen F. and Pedersen G. K., *Jensen's inequality for operators and Löwner's theorem*. Math. Ann. **258** (1982), 229–241.

[7] Heinz E., *Beiträge zur Störungstheorie der Spektralzerlegung*. Math. Ann. **123** (1951), 415–438.

[8] Kamei E., *A satellite to Furuta's inequality*. Math. Japon. **33** (1988), 883–886.

[9] Kwong M. K., *Inequalities for the powers of nonnegative Hermitian operators*. Proc. Amer. Math. Soc. **51** (1975), 401–406.

[10] Mitrinović D. S., Pečarić J. E. and Fink A. M., *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[11] Moslehian M. S. and Najafi H., *An extension of the Löwner–Heinz inequality*. Linear Algebra Appl. **437** (2012), 2359–2365.

[12] Tanahashi K., *Best possibility of the Furuta inequality*. Proc. Amer. Math. Soc. **124** (1996), 141–146.

[13] Uchiyama M., *Strong monotonicity of operator functions*. Integral Equations Operator Theory **37** (2000), 95–105.

Hamed Najafi
Department of Pure Mathematics
Center of Excellence in Analysis on Algebraic Structures (CEAAS)
Ferdowsi University of Mashhad
P.O.Box 1159, Mashhad 91775, Iran
E-mail: hamednajafi@gmail.com
          h.najafi@um.ac.ir

Mohammad Sal Moslehian
Department of Pure Mathematics
Center of Excellence in Analysis on Algebraic Structures (CEAAS)
Ferdowsi University of Mashhad
P.O.Box 1159, Mashhad 91775, Iran
E-mail: Moslehian@um.ac.ir
          moslehian@member.ams.org

Masatoshi Fujii
Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582-8582, Japan
E-mail: mfujii@cc.osaka-kyoiku.ac.jp

Ritsuo Nakamoto
3-4-13, Daihara-cho
Hitachi 316-0021, Japan
E-mail: r-naka@net1.jway.ne.jp