Hokkaido Mathematical Journal Vol. 45 (2016) p. 181–194

Integral Homology of the Moduli Space of Tropical Curves of Genus 1 with Marked Points

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(Received January 15, 2014; Revised December 11, 2014)

Abstract. Kozlov has studied the topological properties of the moduli space of tropical curves of genus 1 with marked points, such as its mod 2 homology, while the integral homology remained a conjecture. In this paper, we present a complete proof of Kozlov's conjecture concerning the integral homology of this moduli space.

Key words: tropical curve, moduli space, equivariant homology.

1. Introduction

Tropical curves are objects of interest in the field of *tropical geometry*. The moduli spaces of tropical curves with marked points were introduced by Mikhalkin in [6], [7] from the tropical geometric point of view. Kozlov, on the other hand, has investigated the same object from the topological point of view in [3], [4], [5]. In particular, he studied the genus 1 case in depth. Among other things, Kozlov showed the following property.

Theorem 1.1 (Kozlov) *Let n be a positive integer, then the moduli space $TM_{1,n+1}$ of tropical curves of genus 1 with $n+1$ marked points is homotopy equivalent to a quotient space $T^n/\mathbb{Z}_2$ of the n-torus, where $\mathbb{Z}_2$ acts diagonally on $T^n = S^1 \times \cdots \times S^1$ by conjugation of each factor S^1 viewed as the unit circle of the complex plane. Therefore*

$$H_*(TM_{1,n+1}) \cong H_*(T^n/\mathbb{Z}_2).$$

Using Theorem 1.1, Kozlov computed the mod 2 homology of $TM_{1,n+1}$. His result is as follows.

Theorem 1.2 (Kozlov) *The mod 2 homology of $TM_{1,n+1}$ has the form*

$$\tilde{H}_k(TM_{1,n+1}; \mathbb{Z}_2) \cong \mathbb{Z}_2^{\tilde{\beta}_k(T^n/\mathbb{Z}_2;\mathbb{Z}_2)},$$

2000 Mathematics Subject Classification : 14T05, 55N91.

where

$$\tilde{\beta}_k(T^n/\mathbb{Z}_2;\mathbb{Z}_2) = \begin{cases} \binom{n-1}{k-1} + 2\binom{n-2}{k-1} + \cdots + 2^{n-k}\binom{k-1}{k-1}, & 2 \le k \le n; \\ 0, & \textit{otherwise.} \end{cases}$$

Kozlov also suggested a conjecture concerning the integral homology of $TM_{1,n+1}$. Our main result is a proof of his conjecture.

Theorem 1.3 *The integral homology of the moduli space $TM_{1,n+1}$ of tropical curves of genus 1 with $n+1$ marked points has the form*

$$\begin{aligned} \tilde{H}_{2i}(TM_{1,n+1}) &\cong \mathbb{Z}_2^{a(i,n)} \oplus \mathbb{Z}^{b(i,n)}, \quad 2 \le 2i \le n, \\ \tilde{H}_j(TM_{1,n+1}) &= 0, \qquad \textit{otherwise.} \end{aligned}$$

where

$$\begin{aligned} a(i,n) &= \tilde{\beta}_{2i+1}(T^n/\mathbb{Z}_2;\mathbb{Z}_2), \\ a(i,n) + b(i,n) &= \tilde{\beta}_{2i}(T^n/\mathbb{Z}_2;\mathbb{Z}_2). \end{aligned}$$

In Section 2, we present the definition of the space $TM_{1,n+1}$ in study and explain Theorem 1.1. Then we focus on the space $T^n/\mathbb{Z}_2$. Section 3 consists of a description of a cellular structure of $T^n/\mathbb{Z}_2$ that is suitable for our computation. In order to conclude our main theorem from the result of the mod 2 homology and the universal coefficient theorem, it suffices to show the following two claims.

- The homology group $H_{2i+1}(T^n/\mathbb{Z}_2)$ is trivial for all i. (Proposition 5.1)
- The homology group $H_{2i}(T^n/\mathbb{Z}_2)$ has no odd torsion nor higher 2-torsion for $2 \le 2i \le n$. (Proposition 5.2)

Section 4 and 5 are devoted to proving the two claims.

Throughout this paper, homology means integral homology unless otherwise specified, and $\tilde{H}$ means reduced homology.

2. The moduli space of metric graphs of genus 1 with marked points.

Kozlov studied the topological properties of the moduli spaces of tropical curves with marked points in [3], [4], [5]. The contents of this section are taken from [4]. However our definitions and notations here are slightly different.

Definition 2.1 A finite graph G (allowing loops and multiedges) is called a *metric graph* if it is given an edge-length function

$$l_G : E(G) \to (0, \infty),$$

where $E(G)$ denotes the set of edges of G. For a nonnegative integer n, a metric graph G is called a *metric graph with n marked points* if it is given a marking function

$$p_G : [n] \to \Delta(G),$$

where $[n] := \{1, \ldots, n\}$ for $n \geq 1$ and $[0] := \emptyset$, $\Delta(G)$ is the space obtained by viewing G as a 1-dimensional CW complex.

Let MG_n denote the set of isometry classes of finite metric graphs with n marked points. Kozlov introduced a suitable topology for MG_n and called the obtained topological space the *moduli space of metric graphs with n marked points*. Here we describe this topology in brief. The interested reader is referred to Subsection 3.1 of [4] for an explicit definition.

Let G be a metric graph with n marked points. Set $r(G) := \min d(x, y)$, where x, y run over the set of vertices and marked points, d is the standard metric on $\Delta(G)$ induced by l_G. (The explicit definition of this metric is given in Subsection 2.3 of [4].) Now for a number $\varepsilon \in (0, r(G)/2)$, we define a set $N_\varepsilon(G)$ by saying that a metric graph H with n marked points is in $N_\varepsilon(G)$ if and only if

- the edges of H of lengths less than ε form a subforest;
- the graph G can be obtained from H by first shrinking all the edges of lengths less than ε and then varying the lengths of the remaining edges and positions of marked points by up to ε.

For an isometry class $[G]$, we set $N_\varepsilon([G]) := \{[H] \mid H \in N_\varepsilon(G)\}$. This is independent of the choice of representatives. The topology of MG_n can be given as follows: a subset $X \subset MG_n$ is open if and only if for every $[G] \in X$, there exists $\varepsilon \in (0, r(G)/2)$ such that $N_\varepsilon([G]) \subset X$.

Definition 2.2 Let d be a positive real number. We define $TM_n(d)$ to be the subspace of MG_n consisting of the isometry classes of all connected metric graphs G with n marked points, such that

- G has no vertices of valency 2;
- G has exactly n leaves (vertices of valency 1), and these are marked 1 through n;
- the lengths of the edges leading to leaves are equal to d.

Note that for arbitrary $d_1, d_2 \in (0, \infty)$, $TM_n(d_1)$ and $TM_n(d_2)$ are homeomorphic. So we could suppress d and just write TM_n. Furthermore TM_n is homeomorphic to *the moduli space of tropical curves with* n *marked points*. The latter can be "considered" as $TM_n(\infty)$. See Section 3.5 of [4] for details.

Recall that the genus of a graph G is the first Betti number of $\Delta(G)$. It is known that the connected components of TM_n are indexed by the genera of consisting graphs. We denote by $TM_{g,n}$ the connected component of TM_n consisting of isometry classes of graphs of genus g. Remark that $TM_{g,n}$ is homeomorphic to the moduli space of tropical curves of genus g with n marked points.

For the case $g = 1$, $TM_{1,n+1} \simeq S^1 \times \cdots \times S^1/O(2) = T^{n+1}/O(2)$, where $O(2)$ acts on each S^1 as orthogonal transformation and diagonally on T^{n+1}. We see that $T^{n+1}/O(2)$ is homeomorphic to $T^n/\mathbb{Z}_2$ by fixing the last coordinate of a point on T^{n+1} to be $1 \in \mathbb{C}$. Hence we conclude Theorem 1.1 (See Section 4.1 of [4]).

Theorem 2.3 (Kozlov) *We have the following homotopy equivalence,*

$$TM_{1,n+1} \simeq T^n/\mathbb{Z}_2,$$

where the nonidentity t *of* $\mathbb{Z}_2$ *acts on* $T^n = S^1 \times \cdots \times S^1$ *as*

$$t(z_1, \ldots, z_n) = (\bar{z}_1, \ldots, \bar{z}_n)$$

for $z_1, \ldots, z_n$ *on the unit circle of the complex plane. Therefore,*

$$H_*(TM_{1,n+1}) \cong H_*(T^n/\mathbb{Z}_2).$$

From now on, we focus on the computation of $H_*(T^n/\mathbb{Z}_2)$.

3. Cellular structures

We shall give $T^n/\mathbb{Z}_2$ a cellular structure so that we could compute $H_*(T^n/\mathbb{Z}_2)$. First let us give a cellular structure to the unit circle S^1 of $\mathbb{C}$ as follows.

- 0-cells: e_0^+, e_0^- denoting $1, -1 \in S^1$ respectively.
- 1-cells: e_1^+, e_1^- denoting the upper and lower arcs joining -1 with 1 respectively with orientations given by the following boundary maps.
- boudary maps: $\partial e_1^+ = \partial e_1^- = e_0^+ - e_0^-$.

Thus $T^n = S^1 \times \cdots \times S^1$ has been given a cellular structure as follows.

- k-cells ($0 \leq k \leq n$): $a_1 \times \cdots \times a_n$
  where $a_{i_l} \in \{e_1^+, e_1^-\}$ for $l = 1, 2, \ldots, k$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and
  $a_j \in \{e_0^+, e_0^-\}$ for other j.
- boundary maps:

$$\partial(a_1 \times \cdots \times a_n) = \sum_{i=1}^{n} (-1)^{d_0+d_1+\cdots+d_{i-1}} a_1 \times \cdots \times \partial a_i \times \cdots \times a_n,$$

  where $d_l = \dim a_l$ for $l = 1, 2, \ldots, n-1$ and $d_0 = 0$. In particular, for $a_1 \times \cdots \times a_n$ as above,

$$\partial(a_1 \times \cdots \times a_n) = \sum_{l=1}^{k} (-1)^{l-1} a_1 \times \cdots \times \partial a_{i_l} \times \cdots \times a_n,$$

  where $\partial a_{i_l} = e_0^+ - e_0^-$.

Now consider $\mathbb{Z}_2 = \{e, t\}$ acting on S^1 by

- $e(e_0^\pm) = e_0^\pm, e(e_1^\pm) = e_1^\pm$.
- $t(e_0^\pm) = e_0^\pm, t(e_1^\pm) = e_1^\mp$.

The group $\mathbb{Z}_2$ acts on T^n diagonally, therefore we obtain an induced cellular structure of $T^n/\mathbb{Z}_2$.

- k-cells: $\overline{a_1 \times \cdots \times a_n}$, where $a_1 \times \cdots \times a_n$ is a k-cell of T^n as above and overline means $\mathbb{Z}_2$-orbit. Note that $\overline{a_1 \times \cdots \times a_n} = \overline{t(a_1 \times \cdots \times a_n)}$.
- Induced boundary maps:

$$\begin{aligned}&\bar{\partial}\overline{a_1 \times \cdots \times a_n}\\ &\quad = \sum_{l=1}^{k} (-1)^{l-1} (\overline{a_1 \times \cdots \times \underset{i_l}{e_0^+} \times \cdots \times a_n} - \overline{a_1 \times \cdots \times \underset{i_l}{e_0^-} \times \cdots \times a_n}).\end{aligned}$$

Let us denote the complement of $\{i_1, \cdots, i_k\}$ in $[n]$ by $\{j_1, \ldots, j_m\}$ with $j_1 < \cdots < j_m$ $(k + m = n)$. Then we introduce the following sets.

$$\begin{aligned}&A(i_1, \ldots, i_k)\\ &\quad = \left\{\overline{a_1 \times \cdots \times a_n} \;\middle|\; a_{i_1} = \cdots = a_{i_k} = e_1^+, a_{j_1}, \cdots, a_{j_m} \in \{e_0^+, e_0^-\}\right\},\end{aligned}$$

and

$$B(i_1, \ldots, i_k; j_l) = \left\{\overline{a_1 \times \cdots \times a_n} \;\middle|\; \begin{array}{l} a_{i_1} = \cdots = a_{i_k} = e_1^+, a_{j_l} \in \{e_1^+, e_1^-\}, \\ a_{j_1}, \ldots, a_{j_{l-1}}, a_{j_{l+1}}, \ldots, a_{j_m} \in \{e_0^+, e_0^-\} \end{array}\right\}.$$

For later use, we prove the following proposition.

Proposition 3.1 *Let k be a natural number. For $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, any k-chain c of $T^n/\mathbb{Z}_2$ can be expressed by*

$$\begin{aligned} c = &\sum_{\overline{a_1 \times \cdots \times a_n} \in A(i_1, \ldots, i_k)} x(\overline{a_1 \times \cdots \times a_n}) \overline{a_1 \times \cdots \times a_n}\\ &+ \sum_{\overline{a_1 \times \cdots \times a_n} \notin A(i_1, \ldots, i_k)} x(\overline{a_1 \times \cdots \times a_n}) \overline{a_1 \times \cdots \times a_n},\end{aligned}$$

where $x(\overline{a_1 \times \cdots \times a_n}) \in \mathbb{Z}$. If c is a k-boundary, then

$$\sum_{\overline{a_1 \times \cdots \times a_n} \in A(i_1, \ldots, i_k)} x(\overline{a_1 \times \cdots \times a_n}) = 0.$$

Proof. Since $(k + 1)$-cells outside of

$$\bigcup_{l=1}^{m} B(i_1, \dots, i_k; j_l)$$

do not have cells in $A(i_1, \dots, i_k)$ as faces, it suffices to compute the boundaries of cells in $B(i_1, \dots, i_k; j_l)$. Suppose $\overline{a_1 \times \cdots \times a_n} \in B(i_1, \dots, i_k; j_l)$, then

$$\begin{aligned} &\bar{\partial}\overline{a_1 \times \cdots \times a_n} \\ &\quad = (-1)^{\#\{p \mid i_p < j_l\}} (\overline{a_1 \times \cdots \times \underset{j_l}{e_0^+} \times \cdots \times a_n} - \overline{a_1 \times \cdots \times \underset{j_l}{e_0^-} \times \cdots \times a_n}) \\ &\qquad + \cdots \end{aligned}$$

where the omitted term is a linear combination of cells outside of $A(i_1, \dots, i_k)$. Hence the desired result follows. $\square$

4. Further explorations

By exactly the same argument as in Section 4.6 of [4], we have the following exact sequence.

$$\begin{aligned} \cdots \to \tilde{H}_k(T^n) \xrightarrow{(q_*, q_*)} \tilde{H}_k(T^n/\mathbb{Z}_2) \oplus \tilde{H}_k(T^n/\mathbb{Z}_2) \to \tilde{H}_k(T^{n+1}/\mathbb{Z}_2) \\ \to \tilde{H}_{k-1}(T^n) \xrightarrow{(q_*, q_*)} \cdots \end{aligned} \qquad (\star)$$

where $q_* : \tilde{H}_*(T^n) \to \tilde{H}_*(T^n/\mathbb{Z}_2)$ is induced by the quotient map $q : T^n \to T^n/\mathbb{Z}_2$. We denote the induced chain map by $q_\# : C_*(T^n) \to C_*(T^n/\mathbb{Z}_2)$.

Study of q_* requires a detailed discussion on $H_*(T^n)$. For $1 \le i_1 < \cdots < i_k \le n$, define a k-chain $\sigma_{i_1, \dots, i_k}$ of T^n by

$$\sigma_{i_1, \dots, i_k} = c_1 \times \cdots \times c_n \in C_k(T^n),$$

where

- $c_{i_l} = e_1^+ - e_1^-$ for $l = 1, \dots, k$.
- $c_j = e_0^+$ for other j.

Theorem 4.1 *The chain $\sigma_{i_1, \dots, i_k}$ is a k-cycle of T^n. Futhermore $H_k(T^n)$ is free with basis $\{[\sigma_{i_1, \dots, i_k}] \mid 1 \le i_1 < \cdots < i_k \le n\}$.*

Proof. It is immediate from Künneth formula. □

Remark 4.2 The induced $\mathbb{Z}_2$-action on $C_k(T^n)$ is given by

$$e\Big(\sum \lambda_i \tau_i\Big) = \sum \lambda_i e(\tau_i) = \sum \lambda_i \tau_i,$$
$$t\Big(\sum \lambda_i \tau_i\Big) = \sum \lambda_i t(\tau_i).$$

Then we conclude

$$\begin{aligned} q_\#\Big(t\Big(\sum \lambda_i \tau_i\Big)\Big) &= q_\#\Big(\sum \lambda_i t(\tau_i)\Big) = \sum \lambda_i \overline{t(\tau_i)} \\ &= \sum \lambda_i \overline{\tau_i} = q_\#\Big(\sum \lambda_i \tau_i\Big). \end{aligned}$$

Proposition 4.3 *The induced homomorphism*

$$q_* : H_k(T^n) \to H_k(T^n/\mathbb{Z}_2)$$

is the 0*-map if* k *is odd and is injective if* k *is even.*

Proof. It suffices to investigate q_* applied to the basis.

$$\begin{aligned} q_*([\sigma_{i_1,\dots,i_k}]) &= [q_\#(\sigma_{i_1,\dots,i_k})] \\ &= [q_\#(c_1 \times \cdots \times \underset{i_1}{e_1^+} \times \cdots \times c_n) - q_\#(c_1 \times \cdots \times \underset{i_1}{e_1^-} \times \cdots \times c_n)] \\ &= [q_\#(c_1 \times \cdots \times \underset{i_1}{e_1^+} \times \cdots \times c_n) \\ &\quad + (-1)^k q_\#(c_1 \times \cdots \times \underset{i_1}{e_1^-} \times \cdots \times (-c_{i_2}) \\ &\qquad\qquad \times \cdots \times (-c_{i_k}) \times \cdots \times c_n)] \\ &= [q_\#(c_1 \times \cdots \times \underset{i_1}{e_1^+} \times \cdots \times c_n) \\ &\quad + (-1)^k q_\#(t(c_1 \times \cdots \times \underset{i_1}{e_1^+} \times \cdots \times c_n))] \\ &= \begin{cases} 0, & k : \text{odd}; \\ [2q_\#(c_1 \times \cdots \times \underset{i_1}{e_1^+} \times \cdots \times c_n)], & k : \text{even}. \end{cases} \end{aligned}$$

Thus q_* is the 0-map if k is odd. To see the injectivity when k is even, take a chain

$$\sigma = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1,\ldots,i_k} \sigma_{i_1,\ldots,i_k} \in C_k(T^n)$$

such that

$$q_\#(\sigma) = q_\#\Big(\sum x_{i_1,\ldots,i_k} \sigma_{i_1,\ldots,i_k}\Big)$$

is a k-boundary of $T^n/\mathbb{Z}_2$. We show that all the coefficients $x_{i_1,\ldots,i_k}$'s are 0. We have already obtained

$$q_\#(\sigma) = \sum x_{i_1,\ldots,i_k} q_\#(\sigma_{i_1,\ldots,i_k}) = \sum 2x_{i_1,\ldots,i_k} q_\#(c_1 \times \cdots \times \underset{i_1}{e_1^+} \times \cdots \times c_n),$$

where $q_\#(c_1 \times \cdots \times \underset{i_1}{e_1^+} \times \cdots \times c_n)$ is of the form

$$\sum \pm \overline{e_0^+ \times \cdots \times \underset{i_1}{e_1^+} \times \cdots \times \underset{i_l}{e_1^\pm} \times \cdots \times e_0^+} \tag{$*$}$$

which is a linear combination of 2^{k-1} k-cells, with each cell having a coefficient 1 (resp. -1) if it contains even (resp. odd) number of e_1^-'s in its representative with the i_1-th component e_1^+.

Note that the k-cell

$$\alpha_{i_1,\ldots,i_k} = \overline{\cdots \times \underset{i_p}{e_1^+} \times \cdots \times \underset{j_l}{e_0^+} \times \cdots} \in A(i_1,\ldots,i_k)$$

with all j_l-th components e_0^+ is in $(*)$. Then $q_\#(\sigma)$ is of the form

$$q_\#(\sigma) = 2x_{i_1,\ldots,i_k}\alpha_{i_1,\ldots,i_k} + \sum_{\beta \in A(i_1,\ldots,i_k) - \{\alpha_{i_1,\ldots,i_k}\}} y_\beta \beta + \cdots$$

where the omitted term is a linear combination of cells outside of $A_{i_1,\ldots,i_k}$. Since $q_\#(\sigma)$ is a k-boundary of $T^n/\mathbb{Z}_2$, by Proposition 3.1,

$$2x_{i_1,\ldots,i_k} + \sum y_\beta = 0.$$

One also observes that $q_{\#}(\sigma)$ does not contain cells with e_0^-, in particular β as above. Thus

$$y_\beta = 0$$

for all $\beta \in A(i_1, \ldots, i_k) - \{\alpha_{i_1,\ldots,i_k}\}$ and we conclude

$$x_{i_1,\ldots,i_k} = 0.$$

This completes the proof. □

5. The integral homology

We are ready to prove the two claims mentioned in the introduction.

Proposition 5.1 *The homology $H_{2i+1}(T^{n+1}/\mathbb{Z}_2)$ is trivial for non-negative integer i.*

Proof. We prove this by induction on n.

For $n = 1$, it is evident to see that $T^2/\mathbb{Z}_2$ is homeomorphic to sphere S^2, hence the proposition follows. Then suppose it is true for $T^n/\mathbb{Z}_2$. We derive an exact sequence from $(\star)$

$$0 \to \tilde{H}_{2i+1}(T^{n+1}/\mathbb{Z}_2) \to \tilde{H}_{2i}(T^n) \xrightarrow{(q_*,q_*)} \cdots$$

Proposition 4.3 shows that $\mathrm{Ker} q_* = 0$. Therefore,

$$\tilde{H}_{2i+1}(T^{n+1}/\mathbb{Z}_2) \cong \mathrm{Ker}(q_*, q_*) = 0.$$

The induction is complete. □

The homology in even dimension is more complicated. Our aim now is to show the following proposition.

Proposition 5.2 *The homology $H_{2i}(T^n/\mathbb{Z}_2)$ has no odd torsion nor higher 2-torsion for $2 \leq 2i \leq n$.*

Proof. Set $X = T^n$ and $G = \mathbb{Z}_2 = \{e, t\}$. We would like to study the structure of $H_{2i}(X/G)$. Denote by V the 0-skeleton of X. Then G acts trivially on V. Thus we consider $V = V/G$ as a subspace of X/G. By the long exact sequence of the pair $(X/G, V)$, it is immediate that

$$H_{2i}(X/G) \cong H_{2i}(X/G, V).$$

Since G acts freely on $X - V$,

$$H^G_{2i}(X, V) \cong H_{2i}(X/G, V).$$

where H^G_* means G-equivariant homology (cf. [1, Section VII.7]). Recall the long exact sequence of equivariant homology of the pair (X, V) (cf. Section 7 of [2])

$$\cdots \to H^G_k(V) \to H^G_k(X) \to H^G_k(X, V) \to H^G_{k-1}(V) \to \cdots$$

To decide $H^G_{2i}(X, V)$, we have to know $H^G_*(V)$ and $H^G_*(X)$. For $H^G_*(X)$, there is a spectral sequence (cf. [1, Section VII.7])

$$E^2_{pq}(X) = H_p(G; H_q(X)) \Rightarrow H^G_{p+q}(X).$$

We compute $H_p(G; H_q(X))$ now, where $H_q(X)$ is a G-module. Note that the generator t of G acts on $H_q(X)$ as multiplication by $(-1)^q$, in fact it is a consequence of the fact that t acts on $H_1(S^1)$ as multiplication by -1 together with Künneth formula.

Set $N = t + e$. Note that $Ngm = Nm$ $(g \in G, m \in H_q(X))$ and $NH_q(X) \subseteq H_q(X)^G$. Then N induces a map

$$\overline{N} : H_q(X)_G \to H_q(X)^G.$$

where $H_q(X)_G$ and $H_q(X)^G$ denote the group of co-invariants and the group of invariants of $H_q(X)$ respectively. We conclude (cf. [1, Section III.1, Example 2])

$$H_p(G; H_q(X)) \cong \begin{cases} H_q(X)_G, & p = 0; \\ \mathrm{Coker}\overline{N}, & p \geq 1 \text{ odd}; \\ \mathrm{Ker}\overline{N}, & p \geq 2 \text{ even}. \end{cases}$$

To be precise,

- If q is even. The group G acts trivially on $H_q(X)$, hence by definition, both $H_q(X)_G$ and $H_q(X)^G$ are $H_q(X)$ itself.

$$\overline{N} : H_q(X) = H_q(X)_G \xrightarrow{t+e} H_q(X)^G = H_q(X)$$

is multiplication by 2. Thus

$$H_p(G; H_q(X)) \cong \begin{cases} \mathbb{Z}^{\binom{n}{q}}, & p = 0; \\ \mathbb{Z}_2^{\binom{n}{q}}, & p \geq 1 \text{ odd}; \\ 0, & p \geq 2 \text{ even}. \end{cases}$$

- If q is odd. The co-invariants $H_q(X)_G$ is the quotient of $H_q(X) \cong \mathbb{Z}^{\binom{n}{q}}$ with respect to its submodule generated by twice of its each element, then $H_q(X)_G \cong \mathbb{Z}_2^{\binom{n}{q}}$. On the other hand, nothing of $H_q(X)$ is fixed by $t \in G$ except 0, hence $H_q(X)^G = 0$.

$$\overline{N} : \mathbb{Z}_2^{\binom{n}{q}} \cong H_q(X)_G \xrightarrow{t+e} H_q(X)^G = 0$$

is the 0-map. Thus

$$H_p(G; H_q(X)) \cong \begin{cases} \mathbb{Z}_2^{\binom{n}{q}}, & p \geq 0 \text{ even}; \\ 0, & \text{otherwise}. \end{cases}$$

We summarize the results as follows.

$$\begin{aligned} E^2_{pq}(X) &= H_p(G; H_q(X)) \\ &\cong \begin{cases} \mathbb{Z}^{\binom{n}{q}}, & (p,q) = (0, 2j),\ j \geq 0; \\ \mathbb{Z}_2^{\binom{n}{q}}, & (p,q) = (2i, 2j+1) \text{ or } (2i+1, 2j),\ i,j \geq 0; \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

One easily checks that the spectral sequence collapses at the E^2-page. Hence we have computed

$$H^G_{2i}(X) \cong E^\infty_{0,2i}(X) \cong E^2_{0,2i}(X) \cong \mathbb{Z}^{\binom{n}{2i}}.$$

Now for V, we have

$$H_k^G(V) \cong H_k(G)^{\bigoplus \#V} \cong \begin{cases} \mathbb{Z}^{2^n}, & k = 0; \\ \mathbb{Z}_2^{2^n}, & k \geq 1 \text{ odd}; \\ 0, & \text{otherwise}. \end{cases}$$

Recall the long exact sequence of the equivariant homology of the pair (X, V)

$$\cdots \to H_k^G(V) \to H_k^G(X) \to H_k^G(X, V) \xrightarrow{\partial_k} H_{k-1}^G(V) \to \cdots$$

For $k = 2i \geq 2$, we derive the following short exact sequence

$$0 \to \mathbb{Z}^{\binom{n}{2i}} \to H_{2i}^G(X, V) \to \mathbb{Z}_2^{\mu_i} \to 0$$

for some integer $0 \leq \mu_i \leq 2^n$ with $\mathrm{Im}\partial_{2i} \cong \mathbb{Z}_2^{\mu_i}$, being a subgroup of $H_{2i-1}^G(V) \cong \mathbb{Z}_2^{2^n}$. We see that $H_{2i}^G(X, V)$ has no odd torsion nor higher 2-torsion. □

Finally, together with Theorem 1.1, we complete the proof of Theorem 1.3 by showing the following theorem.

Theorem 5.3 (Conjecture 4.4 of [4]) *The integral homology of $T^{n+1}/\mathbb{Z}_2$ is of the form*

$$\tilde{H}_{2i}(T^{n+1}/\mathbb{Z}_2) \cong \mathbb{Z}_2^{a(i,n+1)} \oplus \mathbb{Z}^{b(i,n+1)}, \quad 2 \leq 2i \leq n+1,$$
$$\tilde{H}_j(T^{n+1}/\mathbb{Z}_2) = 0, \qquad \textit{otherwise.}$$

where

$$a(i, n+1) = \tilde{\beta}_{2i+1}(T^{n+1}/\mathbb{Z}_2; \mathbb{Z}_2), \tag{1}$$

$$a(i, n+1) + b(i, n+1) = \tilde{\beta}_{2i}(T^{n+1}/\mathbb{Z}_2; \mathbb{Z}_2). \tag{2}$$

Proof. By Proposition 5.1 and Proposition 5.2, it suffices to show that the two equations hold. They are obtained from the universal coefficient theorem. For $2 \leq 2i \leq n+1$,

$$\tilde{H}_{2i}(T^{n+1}/\mathbb{Z}_2; \mathbb{Z}_2) \cong \tilde{H}_{2i}(T^{n+1}/\mathbb{Z}_2) \otimes \mathbb{Z}_2,$$

$$\tilde{H}_{2i+1}(T^{n+1}/\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathrm{Tor}\big(\tilde{H}_{2i}(T^{n+1}/\mathbb{Z}_2), \mathbb{Z}_2\big).$$

These two isomorphisms imply (2) and (1) respectively. □

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